ANNALI DELLA
SCUOLA NORMALE SUPERIORE DI PISA
*Classe di Scienze*

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*Annali della Scuola Normale Superiore di Pisa, Classe di Scienze 4e série*, tome 28, nº 1 (1999), p. 31-40
<http://www.numdam.org/item?id=ASNSP_1999_4_28_1_31_0>

Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)
Vol. XXVIII (1999), pp. 31-40

# Non Interpolation in Morrey-Campanato and Block Spaces

OSCAR BLASCO – ALBERTO RUIZ – LUIS VEGA

**Abstract.** We prove non interpolation results for the family of Morrey spaces. We introduce a scale of block spaces, which are preduals of Morrey spaces in some range. Negative interpolation results are also obtained in this case.

**Mathematics Subject Classification (1991):** 46A20 (primary), 46B70 (secondary).

## 1. – Introduction

The spaces $\mathcal{L}^{p,\alpha}$, for the range $\alpha \in (0, n/p]$ and $p \in [1, \infty]$, were introduced by Morrey in order to study regularity questions which appear in the Calculus of Variations, later Campanato extended the definition to the range $\alpha \in (-1, n/p]$.

$\mathcal{L}^{p,\alpha}$ is defined as the set of functions $f$ locally in $L^p(\mathbb{R}^n)$ and such that there exists a constant $\sigma$ for which

$$\sup_Q r^\alpha \left( r^{-n} \int_Q |f(x) - \sigma|^p dx \right)^{1/p} < \infty \,, \tag{1.1}$$

where the sup is taken over all the cubes in $\mathbb{R}^n$ and $r$ denotes the side length. The norm $\|\cdot\|_{p,\alpha}$ is defined as the infimum of (1,1) when $\sigma \in \mathbb{R}$.

In the range defined by Morrey, functions in this space have been used as weights, to substitute the Lebesgue spaces $L^p$ by weighted-$L^2$, in Sobolev-Poincaré inequalities, unique continuation, potentials in wave and Schrödinger equations and some other problems in PDE, see [CS], [ChR], [FP], [Sc], [T], [W]. There are still some interesting open problems, for example in unique continuation and in the restriction properties of the Fourier transform, see [K], [RV1], [RV2]. In this range we can, without loss of generality, take $\sigma = 0$, the

The first author is partially supported by the Spanish DGCYT Proyecto PB95-0261. The second is partially supported by DGCYT PB94-0192. The third author is partially supported by the Spanish DGCYT PB94-1365.
Pervenuto alla Redazione il 25 febbraio 1998.

endpoint case $p = \frac{n}{\alpha}$ is just $L^p$, being in the other case, $p < n/\alpha$, $L^p$ strictly included in $\mathcal{L}^{p,\alpha}$.

When $\alpha < 0$ (Campanato's extended range) it has been proved that $\mathcal{L}^{p,\alpha}$ is the space of $(-\alpha)$-Hölder continuous functions, see [C] and [M]. When $\alpha = 0$ we have BMO.

In this work we reduce ourselves to the range $\alpha \in (0, n/p]$, $p \in (1, \infty]$. Therefore (see, for instance [Ku]) we have $\mathcal{L}^{p,\alpha}$ is the set of functions $f$ locally in $L^p(\mathbb{R}^n)$ and such that

$$\|f\|_{p,\alpha} = \sup_Q r^\alpha \left( r^{-n} \int_Q |f(x)|^p dx \right)^{1/p} < \infty , \tag{1.2}$$

where the sup is taken over all the cubes in $\mathbb{R}^n$ and $r$ denotes the side length. Our concerns are duality and interpolation properties of this two-parameters family of spaces. A few more historical comments are in order.

Interpolation properties of $\mathcal{L}^{p,\alpha}$ were the objects of attention in several works during the 60's. Stampacchia [St], and Campananto and Murthy [CM] proved that if $T$ is a linear operator bounded from $L^{q_i}$ to $\mathcal{L}^{p_i,\alpha_i}$, $i = 1, 2$, with operator norm $K_i$, then $T$ is bounded from $L^{q_\theta}$ to $\mathcal{L}^{p_\theta,\alpha_\theta}$ with norm at most $CK_1^{1-\theta}K_2^{\theta}$, where $1/p_\theta = (1-\theta)/p_1 + \theta/p_2$, $1/q_\theta = (1-\theta)/q_1 + \theta/q_2$, $\alpha_\theta = (1-\theta)/\alpha_1 + \theta/\alpha_2$ and $C$ only depends on $\theta$, $\alpha_i$, $p_i$ and $q_i$. A similar property was proved by Peetre, see [P], for a extended family of spaces. Actually as J. Peetre points out — see [P, pg. 77], any interpolation theorem will do, in the sense that one can replace $(L^{p_0}, L^{p_1})$ by an abstract pair $(A_0, A_1)$, and $L^p$ by an abstract interpolation space $A$ constructed from $(A_0, A_1)$ and still have an inequality as before. In particular this gives us that $\mathcal{L}^{p_\theta,\alpha_\theta}$ contains the corresponding interpolated space. The main purpose of this paper is to prove that the inclusion in the other direction does not hold. In the range $\alpha \in (-1, n/p]$, this was proved by Stein and Zygmund [SteZ] by constructing a linear operator bounded from $(-\alpha)$-Hölder continuous functions to $(-\alpha)$-Hölder continuous functions and from $L^2$ to $L^2$ which is not bounded from BMO to BMO.

Recently two of the authors, see [RV3], obtained negative results on interpolation properties in the Morrey range. To be precise, the lack of convexity which characterises interpolation functors of exponent $\theta$, see [BL, page 27], is proved. In that work they need the dimension $n > 1$.

In the present work we go further and give examples, in the one dimensional case, of operators which are bounded from $\mathcal{L}^{p_i,\alpha} \to L^{q_i}$, $i = 1, 2$, $0 < \alpha \le n/p$, $p \in (0, \infty)$ and are not bounded from the intermediate $\mathcal{L}^{p_\theta,\alpha}$ to $L^{q_\theta}$.

We also give a description of the predual spaces of $\mathcal{L}^{p,\alpha}$ in the context of "block spaces" (see [SOS] and [So] for this terminology in other situations). We say that a measurable function $b$ is a $(q, \beta)$-block if it is supported in a cube $Q$ of lenghtside $r$ in such a way that

$$\left( \frac{1}{|Q|} \int_Q |b(x)|^q dx \right)^{1/q} \le \frac{1}{r^\beta} . \tag{1.3}$$

The question of the preduals of Morrey spaces has been already studied — see [Z], and [A]. But in there the corresponding blocks have mean zero. Therefore we prove that this condition can be omitted and still have a description of the predual spaces. This question is of course related to the two possible definitions of Morrey spaces mentioned above.

In Section 2 we prove that the predual of $\mathcal{L}^{p,\alpha}$, when $0 < \alpha < n/p$ is the space

$$\mathcal{B}_{q,\beta} = \left\{ \sum_{k=1}^{\infty} \lambda_k b_k : \sum |\lambda_k| < \infty \text{ and } b_k \text{ is a } (q,\beta) - \text{block} \right\} \tag{1.4}$$

for $\beta = n - \alpha$ and $1/p + 1/q = 1$.

Meanwhile the spaces defined by (1.2) reduce to $\{0\}$ when $\alpha > n/p$, the definition of $\mathcal{B}_{q,\beta}$ gives non trivial spaces in the corresponding range, beyond the preduality exponents, $\beta > n/q$. So it makes sense to study interpolation properties in this range; in Section 3 we also give a negative result in this context which is not covered by preduality and which is a complement to the questions posed in the 60's.

We would like to thank F. Cobos for enlightening conversations.

## 2. – Preduals

Let us start with some elementary properties of block spaces. Let us define

$$\|f\|_{\mathcal{B}_{q,\beta}} = \inf \left\{ \sum |\lambda_k| \text{ such that } f = \sum \lambda_k b_k \right\} \tag{2.1}$$

where the infimum is taken over all possible decompositions of $f$ into $(q,\beta)$-blocks.

LEMMA 1.
(a) $\mathcal{B}_{q,\beta} \subset L^{n/\beta}$ *if* $n/\beta \leq q$.
(b) $L^q \subset \mathcal{B}_{q,\beta}$ *if* $q \leq n/\beta$.
(c) *For any cube* $Q$ *and* $f \in L^q_{\mathrm{loc}}$ *we have*

$$\|\chi_Q f\|_{\mathcal{B}_{q,\beta}} \leq |Q|^{\beta/n - 1/q} \|\chi_Q f\|_q .$$

PROOF. (a) follows from Hölder and Minkowsky inequalities and the condition $\sum |\lambda_k| < \infty$.
(b) Denote by $Q_k = \{x : |x_i| \leq 2^k,\ i = 1, \dots, n\}$. Assume first that $q < n/\beta$. Then

$$f = \sum_1^{\infty} |Q_k|^{\beta/n - 1/q} \|f\|_q b_k ,$$

where $b_k = \frac{f(\chi_{Q_k} - \chi_{Q_{k-1}})}{|Q_k|^{\beta/n-1/q}\|f\|_q}$ is a $(q,\beta)$-block.

Assume now that $q = n/\beta$ and $f \in L^q$. Since $f\chi_{Q_k}$ is a Cauchy sequence in $L^q$ then we can find $n_k$ such that $\|f\chi_{Q_{n_{k+1}}} - f\chi_{Q_{n_k}}\|_q < 2^{-k}$. Now write

$$f = f\chi_{Q_{n_1}} + \sum_{k=1}^{\infty} f\chi_{Q_{n_{k+1}}} - f\chi_{Q_{n_k}}.$$

This clearly shows that $f \in \mathcal{B}_{q,\beta}$ and $\|f\|_{\mathcal{B}_{q,\beta}} \leq 2\|f\|_q$.

(c) Just observe that

$$(f\chi_Q)(x) = |Q|^{\beta/n-1/q}\|\chi_Q f\|_q b$$

with $b$ a $(q,\beta)$-block.

REMARKS. 1. $\mathcal{B}_{q,\beta} = L^q$ for $\beta = n/q$. Then its dual is $L^p = \mathcal{L}^{p,\alpha}$ for $\alpha = n/p$.

2. As we observe in the introduction $\mathcal{B}_{q,\beta}$ is a meaningful space in the case (b) of Lemma 1 and it contains $L^q$.

THEOREM 1. *Let $1 < p < \infty$ and $\alpha \in (0, n/p)$, then if we take $\beta$ and $q$ such that $\alpha + \beta = n$ and $1/p + 1/q = 1$ we have*

$$(\mathcal{B}_{q,\beta})^* = \mathcal{L}^{p,\alpha}.$$

PROOF. Assume $f \in \mathcal{L}^{p,\alpha}$ and take $b$ a $(q,\beta)$-block supported in a cube $Q$ of side $r$, then

$$\begin{aligned}\left|\int_{\mathbb{R}^n} fb\right| &\leq \left(\int_Q |f|^p\right)^{1/p}\left(\int_Q |b|^q\right)^{1/q} \\ &= r^\alpha\left(\frac{1}{|Q|}\int_Q |f|^p\right)^{1/p} r^\beta\left(\frac{1}{|Q|}\int_Q |b|^q\right)^{1/q} \leq \|f\|_{p,\alpha}.\end{aligned}$$

Take now $g = \sum \lambda_k b_k$, then

$$\left|\int_{\mathbb{R}^n} fg\right| \leq \sum|\lambda_k|\left|\int_{\mathbb{R}^n} fb_k\right| \leq \sum|\lambda_k|\|f\|_{p,\alpha} \leq \|g\|_{\mathcal{B}_{q,\beta}}\|f\|_{p,\alpha}.$$

This proves that $\mathcal{L}^{p,\alpha} \subset (\mathcal{B}_{q,\beta})^*$.

To prove the other inclusion take $\Phi \in (\mathcal{B}_{q,\beta})^*$ and a cube $Q$, from (c) of Lemma 1 we have that $\Phi$ restricted to the subset $L^q(Q)$ is in $L^q(Q)^*$, and hence there exists a $f_Q \in L^q(Q)$ such that

$$\int f_Q g = \Phi(g) \text{ for any } g \in L^q(Q).$$

Write $\mathbb{R}^n = \cup_1^\infty Q_k$, $Q_k$ increasing, define $f(x) = f_{Q_k}(x)$ if $x \in Q_k$, which makes sense since $\int_E f_{Q_k} = \int_E f_{Q_{k+1}}$ for any Borel subset of $Q_k$ and hence $f_{Q_k}(x) = f_{Q_{k+1}}(x)$ a.e. $x \in Q_k$.

Only remains to prove that $f \in \mathcal{L}^{p,\alpha}$. Take a cube $Q$ and $j$ such that $Q \subset Q_j$, then

$$\begin{aligned} |Q|^{\alpha/n}\left(\frac{1}{|Q|}\int_Q |f|^p\right)^{1/p} &= |Q|^{\alpha/n-1/p} \sup_{\|h\|_q=1} \int_Q fh\,dx \\ &\le \sup_{\|h\|_q=1} \int_{Q_j} f_{Q_j}\left(h\chi_Q |Q|^{\alpha/n-1/p}\right) dx \\ &\le \|\Phi\|\, \|h\chi_Q|Q|^{\alpha/n-1/p}\|_{\mathcal{B}_{q,\beta}} \le \|\Phi\|\,, \end{aligned}$$

since $h\chi_Q|Q|^{\alpha/n-1/p} = h\chi_Q|Q|^{\beta/n-1/q}$ is a $(q,\beta)$-block.

## 3. – Interpolation

In [RV3] it was proved the lack of logarithmic convexity of the operator norm of an operator bounded from $\mathcal{L}^{p_i,\alpha}$ to $L^1$, $i = 1, 2$ with $1 \le p_2 \le \frac{n-1}{2} \le p_1 < \infty$, $0 < \alpha < n$. This operator requires the dimension $> 1$. We start by exhibiting an example in dimension 1 of non-boundedness in an intermediate space. Similar examples can be constructed in higher dimension.

Theorem 2. *Take $p_1$, $p_2$, and $p_3$ such that $1 < p_2 < p_3 < p_1 \le 1$, $\alpha = \frac{1}{p_1}$. Then there exists $q_1, q_2 \in (1, \infty)$ and a linear operator $T$ such that, we have*

$$T : \mathcal{L}^{p_i,\alpha} \to L^{q_i}, \quad i = 1, 2, \tag{3.1}$$

*and*

$$T : \mathcal{L}^{p_3,\alpha} \not\to L^{q_3}, \tag{3.2}$$

*where $\frac{1}{p_3} = \frac{1-\theta}{p_1} + \frac{\theta}{p_2}$ and $\frac{1}{q_3} = \frac{1-\theta}{q_1} + \frac{\theta}{q_2}$.*

Lemma 2. *Let $0 < p_3$, $\alpha$, $\beta$ be positive numbers such that*

$$\max\{p_3, 1\} \le \frac{\beta}{(\beta+1)\alpha} \tag{3.3}$$

*and let $N_0$ such that $(\beta + 1) < \frac{N_0}{\log N_0}$. Define*

$$I_j^N = \left[2^N + jN^\beta, 2^N + jN^\beta + 1\right]$$

*for $N > N_0$, $N \in \mathbb{N}$, and $j = 0, 1, \ldots, N-1$. Then*

$$\left\| \sum_{N>N_0} \sum_{j=0}^{N-1} \chi_{I_j^N} \right\|_{p_3,\alpha} \le C\,,$$

*where $C$ is a universal constant.*

PROOF. By inspection one can see that the biggest value of

$$|I|^{\alpha}(\frac{1}{|I|}|I \cap (\cup I_j^N)|)^{\frac{1}{p_3}}$$

is achieved when the $\inf I$ is at a point $2^N$ and $N^{\beta+1} \approx |I|$.

PROOF OF THEOREM 2. Choose $\beta$ and $q_1$ such that

$$p_3 < \frac{\beta}{\alpha(\beta+1)} < p_1\,, \tag{3.4}$$

$$\frac{2}{q_1} = \min\left\{\frac{2}{p_2} + \alpha(1+\beta) - \frac{\beta}{p_1}, 2\right\}\,, \tag{3.5}$$

and $q_2 = p_2$.

Hence $q_1 < q_2 = p_2$.

Define $E_N = \cup_{j=0}^{N-1} I_j^N$ with $I_j^N$ as in Lemma 2 and

$$Tf(x) = \sum_{N>N_0} \lambda_N \chi_{E_N}(x) f(x)\,, \tag{3.6}$$

with $\lambda_N = \frac{1}{N^{\gamma}}$, $\gamma$ such that

$$\frac{2}{p_2} < \gamma < \frac{2}{q_3} = \frac{2(1-\theta)}{q_1} + \frac{2\theta}{q_2}\,. \tag{3.7}$$

Notice that from (3.5) $\frac{1}{q_3} > \frac{1}{p_2}$. Hence we trivially have

$$\begin{aligned} \|Tf\|_{q_1} &= \left(\sum_N \lambda_N^{q_1} \int_{E_N} |f|^{q_1}\right)^{\frac{1}{q_1}} \\ &\le \left(\sum_N \lambda_N^{q_1} N^{q_1(\frac{1}{q_1}-\frac{1}{p_1})} \|f\|^{q_1}_{L^{p_1}(E_N)}\right)^{\frac{1}{q_1}} \\ &\le \left(\sum_N \lambda_N^{q_1} N^{1-q_1(\alpha(1+\beta)-\frac{\beta}{p_1})}\right)^{\frac{1}{q_1}} \|f\|_{p_1,\alpha}\,, \end{aligned} \tag{3.8}$$

since

$$\|f\|_{L^{p_1}(E_N)} \le N^{\frac{(\beta+1)}{p_1}-\alpha(\beta+1)} \|f\|_{p_1,\alpha}\,.$$

Then $\|Tf\|_{q_1} \le C\|f\|_{p_1,\alpha}$, follows from (3.5) and (3.7).

We have from (3.7) that $1-\gamma q_2 < -1$, and for $q_2 = p_2$:

$$\|Tf\|_{q_2} \le \left(\sum_N \lambda_N^{q_2} \sum_{j=1}^{N-1} \|f\|^{p_2}_{L^{p_2}(I_j^N)}\right)^{1/p_2}$$
$$\le \left(\sum_N \lambda_N^{q_2} N\right)^{\frac{1}{q_2}} \|f\|_{p_2,\alpha} \le C\|f\|_{p_2,\alpha}\,.$$

On the other hand we know from Lemma 2 that $f = \sum_N \chi_{E_N} \in \mathcal{L}^{p_3,\alpha}$ and from (3.7) $1-\gamma q_3 > -1$, hence

$$\|Tf\|_{q_3} = \left(\sum_N \lambda_N^{q_3} N\right)^{\frac{1}{q_3}} = \left(\sum N^{1-\gamma q_3}\right)^{\frac{1}{q_3}} = \infty\,.$$

The proof is over.

The next theorem states that interpolation for the blocks spaces $\mathcal{B}_{q,\beta}$ does not hold between points at both sides of the line $q = \frac{1}{\beta}$. In fact we give an operator bounded $L^{p_i} \to \mathcal{B}_{q_i,\beta}$, $i = 1, 2$ which is not bounded from $L^{p_\theta} \to \mathcal{B}_{q_\theta,\beta}$, and such that $\mathcal{B}_{q_2,\beta}$ is on the predual range and $\mathcal{B}_{q_1,\beta}$ is out of it.

THEOREM 3. *Let* $1 \le q_1 < q_2 \le 2$. *There exist* $\beta \in (\frac{1}{q_2}, \frac{1}{q_1})$, $\theta \in (0,1)$, $p_1$, $p_2$ *and a linear operator* $T$ *such that we have*

$$T : L^{p_1} \to \mathcal{B}_{q_1,\beta} \tag{3.9}$$
$$T : L^{p_2} \to \mathcal{B}_{q_2,\beta} \tag{3.10}$$

*and*

$$T : L^{p_\theta} \not\to \mathcal{B}_{q_\theta,\beta} \tag{3.11}$$

*where* $\frac{1}{p_\theta} = \frac{1-\theta}{p_1} + \frac{\theta}{p_2}$ *and* $\frac{1}{q_\theta} = \frac{1-\theta}{q_1} + \frac{\theta}{q_2}$.

In the proof of Theorem 3 the following lemma will be used:

LEMMA 3. *Let* $\{E_k\} \subset \mathbb{R}$ *and let* $B_k = \mathrm{co}(E_k)$ *denotes its convex hull, let* $q_1 < p_1$, $\beta > 0$ *and* $f \in L^{p_1}(\mathbb{R})$ *and* $\|f\|_{p_1} = 1$. *If* $\{B_k\}$ *are disjoint, and*

$$\lambda_k |E_k|^{\frac{1}{q_1}-\frac{1}{p_1}} |B_k|^{\beta-\frac{1}{q_1}} \in l^1\,, \tag{3.12}$$

*then* $\sum \lambda_k f \chi_{E_k} \in \mathcal{B}_{q_1,\beta}$.

PROOF. Just observe that from Hölder $|E_k|^{\frac{1}{p_1}-\frac{1}{q_1}} |B_k|^{\frac{1}{q_1}-\beta} f\chi_{E_k}$ is a $(q_1,\beta)$-block.

PROOF OF THEOREM 3. Observe that $\frac{1}{q_2'} < \frac{1}{q_1}$. Take $p_2 = q_2$ and $p_2 < p_1$ such that

$$\frac{1}{q_2'} < \frac{1}{q_1} - \frac{1}{p_1}. \tag{3.13}$$

Now choose $\theta$ such that

$$\frac{1}{q_2'} < (1-\theta)\left(\frac{1}{q_1} - \frac{1}{p_1}\right), \tag{3.14}$$

and now $\beta$ such that

$$\frac{1}{q_\theta} < \beta < \frac{1}{q_1} \tag{3.15}$$

where $\frac{1}{q_\theta} = \frac{1-\theta}{q_1} + \frac{\theta}{q_2}$.

Consider now $E_k = [2^k, 2^k + 1) \cup [2^{k+1}, 1, 2^{k+1})$ and define

$$T(f) = \left(\sum_{k=1}^{\infty} k^{-\gamma} \chi_{E_k}\right) f$$

for $\gamma$ chosen such that

$$\frac{1}{p_2'} = \frac{1}{q_2'} < \gamma < (1-\theta)\left(\frac{1}{q_1} - \frac{1}{p_1}\right). \tag{3.16}$$

Since $\beta < \frac{1}{q_1}$, $|E_k| = 2$ and $|B_k| = 2^k$ then Lemma 3 implies (3.9).

Now from condition (3.15) we have that $\frac{1}{q_2} < \beta$ what allows us to use Theorem 1 and write (remember that $q_2 = p_2$) for $\alpha = 1 - \beta$:

$$\begin{aligned}
\left\|\sum k^{-\gamma} f \chi_{E_k}\right\|_{\mathcal{B}_{q_2,\beta}} &= \sup_{\|g\|_{q_2',\alpha}\le 1} \int \left(\sum k^{-\gamma} f \chi_{E_k}\right) g \\
&= \sup_{\|g\|_{p_2',\alpha}\le 1} \int \left(\sum k^{-\gamma} g \chi_{E_k}\right) f \\
&\le \|f\|_{p_2} \sup_{\|g\|_{p_2',\alpha}\le 1} \left\|\sum k^{-\gamma} g \chi_{E_k}\right\|_{p_2'} \\
&= \|f\|_{p_2} \sup_{\|g\|_{p_2',\alpha}\le 1} \left(\sum k^{-\gamma p_2'} \int_{E_k} |g|^{p_2'}\right)^{\frac{1}{p_2'}} \\
&\le \|f\|_{p_2} 2^{\frac{1}{p_2'}} \left(\sum k^{-\gamma p_2'}\right)^{\frac{1}{p_2'}}.
\end{aligned}$$

The last inequality follows from the fact that $\int_{E_k} |g|^{p_2'} \leq 2\|g\|_{p_2',\alpha}^{p_2'}$. The above series converges from (3.16). This proves (3.10).

Finally, since $\frac{1}{q_\theta} < \beta$, then (3.11) is equivalent to see that $T^* = T$ is not bounded from $\mathcal{L}^{q_\theta',1-\beta}$ to $L^{p_\theta'}$.

Take now $A_N = \cup_{k=1}^N E_k$ and $f_N = |A_N|^{-\frac{1}{q_\theta'}} \chi_{A_N}$.
It is elementary to see that $\|f_N\|_{q_\theta',1-\beta} \leq 1$.
On the other hand

$$\|T(f_N)\|_{p_\theta'} = 2^{\frac{1}{p_\theta'}} (2N)^{-\frac{1}{q_\theta'}} \left( \sum_{k=1}^N k^{-\gamma p_\theta'} \right)^{\frac{1}{p_\theta'}} .$$

Note that

$$\frac{1}{q_\theta} - \frac{1}{p_\theta} = (1-\theta)\left(\frac{1}{q_1} - \frac{1}{p_1}\right) \tag{3.17}$$

and then (3.16) gives that $\gamma p_\theta' < 1$ what allows to write

$$\sum_{k=1}^N k^{-\gamma p_\theta'} \geq C N^{-\gamma p_\theta' + 1} . \tag{3.18}$$

Using (3.17) and (3.18) we get that

$$\|T(f_N)\|_{p_\theta'} \geq C N^{-\gamma + (1-\theta)(\frac{1}{q_1} - \frac{1}{p_1})} .$$

Now (3.16) gives that $\sup_N \|T(f_N)\|_{p_\theta'} = \infty$ and the proof is completed.

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