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José Francisco Rodrigues
Lisa Santos

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A Parabolic Quasi-Variational Inequality Arising in a Superconductivity Model

JOSÉ FRANCISCO RODRIGUES – LISA SANTOS

Abstract. We consider the existence of solutions for a parabolic quasilinear problem with a gradient constraint which threshold depends on the solution itself. The problem may be considered as a quasi-variational inequality and the existence of solution is shown by considering a suitable family of approximating quasilinear equations of p-Laplacian type. *A priori* estimates on the time derivative of the approximating solutions and on the nonlinear diffusion coefficients are used in the passage to the limit, as well as a suitable sequence of convex sets with variable gradient constraint. The asymptotic behaviour as $t \to \infty$ is also considered, and the solutions of the quasi-variational inequality are shown to converge, at least for subsequences, to a solution of a stationary quasi-variational inequality. These results can be applied to the critical-state model of type-II superconductors in longitudinal geometry.

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1. – Introduction

In a critical-state model of type-II superconductors with a longitudinal geometry, the main unknown is the magnetic field $H = (0, 0, u(x, t))$, where $x = (x_1, x_2) \in \Omega \subseteq \mathbb{R}^2$. By Maxwell's equations,

$$\begin{cases} \mu H_t + \nabla \times E = 0, \\ \nabla \times H = J, \end{cases}$$

where E denotes the electric field, the unknown density of the induced current J is given by

$$J = \left(\frac{\partial u}{\partial x_2}, -\frac{\partial u}{\partial x_1}, 0\right) = \nabla^\perp u \ .$$

In some superconductivity models (see [3]), it is assumed that the electric field E inside the superconductor depends on the current density J through

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a power law (which generalizes the classical Ohm's law $E = \rho J$) with the scalar resistivity given by $\rho = \rho(|J|) = \rho_0|J|^{p-2} = \rho_0|\nabla u|^{p-2}$, where $\rho_0 > 0$ is a constant and $p \geq 2$. Hence, the Maxwell's equation for the longitudinal component of the magnetic field implies that $u = u(x,t)$ satisfies the two-dimensional quasilinear parabolic equation

$$\mu u_t - \rho_0 \nabla . \left(|\nabla u|^{p-2}\nabla u\right) = 0.$$

In Bean's critical-state model (see [11]), the current density cannot exceed the critical value $j_c > 0$ and it has been suggested that this threshold may also depend on $|H| = |u|$ (see Kim *et al.* [6]).

The constitutive relation for E is then modified to

$$E = \begin{cases} \rho_0 \, |\nabla u|^{p-2} \, \nabla^{\perp} u & \text{if } |\nabla u| < j_c(|u|), \\ (\rho_0 \, j_c^{p-2} + \lambda) \, \nabla^{\perp} u & \text{if } |\nabla u| = j_c(|u|) \ , \end{cases}$$

being $\lambda \geq 0$ an unknown Lagrange multiplier.

Imposing initial and boundary conditions (in a bounded domain Ω in $\mathbb{R}^2$), it may be easily shown that this problem is equivalent to the following quasi-variational inequality (for details about this derivation see [11], where only the degenerate case $\rho_0 = 0$ was considered) to a new variable $h = u - h_e$,

$$\begin{cases} h(t) \in \mathbb{K}_{h(t)} \quad \text{for a.e. } t \in [0,T] \ , \qquad h(0) = h_0 \in \mathbb{K}_{h_0} \ , \\ \displaystyle\int_\Omega h_t(t)\,(v - h(t)) + \frac{\rho_0}{\mu}\int_\Omega |\nabla h(t)|^{p-2}\,\nabla h(t)\,.\,\nabla(v - h(t)) \\ \qquad\qquad \displaystyle\geq \int_\Omega f(t)\,(v - h(t)) \ , \qquad \forall\, v \in \mathbb{K}_{h(t)} \quad \text{for a.e. } t \in [0,T] \ , \end{cases}$$

being

$$\mathbb{K}_{h(t)} = \left\{ v \in W_0^{1,p}(\Omega) \colon \ |\nabla v| \leq j_c\Big(|h(t) + h_e(t)|\Big) \ \text{a.e.} \right\}$$

where $h_e = h_e(t)$ is related to the density of external currents, as well as f.

This model is the main motivation for the study of this new type of quasi-variational inequalities. This kind of problems seem not well studied in the literature and are not considered, for instance, in the classical references on quasi-variational inequalities [1] or [2]. Recent works in this area are, for example, the modelling works [10], [11] or a very special case of a one-dimensional quasi-variational inequality considered in [7].

In Section 2 we present the main results of this paper, a result on the existence of solutions for the quasi-variational inequality and a result about the asymptotic behaviour in time of these solutions.

In Section 3 we consider a family of approximating solutions and we establish *a priori* estimates. Section 4 includes the passage to the limit in the family of solutions of the approximated problems, concluding with the proof of the existence of at least a solution for the quasi-variational inequality in the N-dimensional case.

Section 5 studies the asymptotic behaviour in time of the solutions, which, in particular, also yields the existence of a solution to the corresponding elliptic quasi-variational inequality, extending a result of [8].

2. – Main results

Let Ω be a bounded, open subset of $\mathbb{R}^N$, with smooth boundary $\partial\Omega$, and let $T \in \mathbb{R}^+$, $I =]0, T[$, $Q_T = \Omega \times I$, $\Sigma = \partial\Omega \times I$, $\Omega_0 = \overline{\Omega} \times \{0\}$. Let $\nabla = (\frac{\partial}{\partial x_1}, ..., \frac{\partial}{\partial x_N})$ denote the spatial gradient and suppose that F, f and h are given functions such that

$$(1) \qquad F \in C^0(\mathbb{R}) \ , \qquad \exists\, m > 0: \qquad F \geq m \ ,$$

$$(2) \qquad f \in L^\infty(Q_T); \qquad f_t \in M(Q_T) \ ,$$

$$(3) \qquad h \in W_0^{1,\infty}(\Omega) \ , \quad |\nabla h| \leq F(h) \ \text{ a.e. in } \Omega \ , \quad \Delta_p h \in M(\Omega) \ ,$$

where $\Delta_p h = \nabla \, . \, \big(|\nabla h|^{p-2} \nabla h\big)$ denotes the p-Laplacian, $1 < p < \infty$ and $M(\Omega)$, $M(Q_T)$ denote the spaces of bounded measures in Ω and Q_T, respectively, $M(\Omega) = \big[C^0(\overline{\Omega})\big]'$ and $M(Q_T) = \big[C^0(\overline{Q_T})\big]'$. We also denote by $W_0^{1,\infty}(\Omega)$ the space of Lipschitz functions that vanish on $\partial\Omega$.

Given $u \in L^\infty(\Omega)$, we can define a convex set

$$(4) \qquad \mathbb{K}_u = \Big\{ v \in W_0^{1,p}(\Omega) \colon \ |\nabla v| \leq F(u) \ \text{ a.e. in } \Omega \Big\}$$

and the quasi-variational problem as follows:

To find u in an appropriate class of functions, such that:

$$(5) \qquad \begin{cases} u(t) \in \mathbb{K}_{u(t)} \cap L^\infty(\Omega) \quad \text{for a.e. } t \in I \ , \qquad u(0) = h \ , \\ \displaystyle\int_\Omega u_t(t)\,(v - u(t)) + \int_\Omega |\nabla u(t)|^{p-2}\, \nabla u(t) \, . \, \nabla(v - u(t)) \\ \qquad \qquad \displaystyle \geq \int_\Omega f(t)\,(v - u(t)) \ , \ \ \forall\, v \in \mathbb{K}_{u(t)}, \quad \text{for a.e. } t \in I \ . \end{cases}$$

Our first aim is to prove existence of a solution to the problem (5). For that, we consider, for $\varepsilon > 0$, f_ε, h_ε and F_ε smooth functions approximating, respectively, f, h and F in the norms of $L^q(Q_T)$, $W_0^{1,q}(\Omega)$, for any $q < \infty$ and in $C^0(\mathbb{R})$, with $h^\varepsilon\big|_{\partial\Omega} = 0$, $\|\Delta_p h^\varepsilon + \varepsilon \Delta h^\varepsilon\|_{L^1(\Omega)} \leq C$ (C independent of ε) and a C^2 function $k_\varepsilon \colon \mathbb{R} \to \mathbb{R}$, non decreasing and such that

$$(6) \qquad k_\varepsilon = 1 \ \text{ if } \ s \leq 0, \qquad \qquad k_\varepsilon = e^{\frac{s}{\varepsilon}} \ \text{ if } \ s \geq \varepsilon.$$

We consider a sequence of the following approximated problems

$$(7) \qquad \begin{cases} u_t^\varepsilon - \nabla \, . \, \Big[k_\varepsilon\Big(|\nabla u^\varepsilon|^2 - F_\varepsilon^2(u^\varepsilon)\Big) \Big(|\nabla u^\varepsilon|^2 + \varepsilon\Big)^{\frac{p-2}{2}} \nabla u^\varepsilon + \varepsilon\, \nabla u^\varepsilon \Big] = f_\varepsilon \ \text{ in } Q_T, \\ u^\varepsilon(0) = h^\varepsilon \ , \\ u^\varepsilon\big|_\Sigma = 0 \ . \end{cases}$$

We noticed that a similar approximation was introduced by Gerhardt (see [5]) in the treatment of the elastic-plastic torsion problem with multiply connected cross-section, which is an elliptic variational inequality and it was also used for studying a parabolic variational inequality with gradient constraint in [12].

THEOREM 1. *With the assumptions* (1), (2) *and* (3), *for any* $1 < p < \infty$, *the problem* (5) *has at least a solution* u *belonging to* $L^\infty(0,T;W_0^{1,\infty}(\Omega)) \cap C^0(\overline{Q}_T)$, *such that* $u_t \in L^\infty(0,T;M(\Omega))$. *In addition,* u *is the weak limit in* $L^q(0,T;W_0^{1,q}(\Omega))$ *(for any* $q < \infty$*) of a subsequence* $\{u^{\varepsilon_n}\}_n$ *of solutions of the family of approximated problems* (7), $u^{\varepsilon_n} \underset{n}{\rightarrow} u$ *in* $C^0(\overline{Q}_T)$ *and also* $u_t^{\varepsilon_n} \rightharpoonup u_t$ *in* $L^\infty(0,T;M(\Omega))$ *weak-*$*$.

REMARK 1. Since u is continuous and bounded, for each $t \in [0,T]$, $\mathbb{K}_{u(t)} \subset W_0^{1,\infty}(\Omega) \subset C^0(\overline{\Omega})$ and the first integral of the left hand side of (5) should be understood in the duality sense between $M(\Omega)$ and $C^0(\overline{\Omega})$, taking into account that $u_t(t)$ is a.e. a bounded measure.

Consider now the corresponding stationary quasi-variational inequality:

To find

$$(8)\qquad \begin{cases} u^\infty \in \mathbb{K}_{u^\infty} \cap L^\infty(\Omega)\ , \\ \displaystyle\int_\Omega |\nabla u^\infty|^{p-2}\nabla u^\infty \,.\, \nabla(v-u^\infty) \geq \int_\Omega f^\infty(v-u^\infty)\ , \quad \forall\, v \in \mathbb{K}_{u^\infty}\ , \end{cases}$$

where we assume $f^\infty \in L^1(\Omega)$, F satisfies (1) and

$$\mathbb{K}_{u^\infty} = \left\{ v \in W_0^{1,p}(\Omega) \colon |\nabla v| \leq F(u^\infty) \ \text{ a.e. in } \Omega \right\}.$$

Existence of a solution of the problem (8) may be proved as in [3].

We may also consider the problem (5) with $T = +\infty$. Then we have the following

THEOREM 2. *Suppose that the assumption* (2) *is satisfied for* $T = +\infty$, *i.e., there exists* $C > 0$ *such that*

$$(9)\qquad \sup_{0<t<\infty} \|f(t)\|_{L^\infty(\Omega)} + \int_0^{+\infty}\!\!\int_\Omega |f_t| \leq C\,,$$

and, in addition,

$$(10)\qquad \frac{1}{t}\int_0^t\!\!\int_\Omega |f_t(\tau)|\tau\, d\tau \longrightarrow 0 \quad \text{when } t \to \infty\,,$$

$$(11)\qquad f(t) \rightharpoonup f_\infty \quad \text{in } L^1(\Omega)\text{-weak}, \quad \text{when } t \to +\infty\,.$$

Let u *denote a solution of the problem* (5). *Then there exists a sequence* $\{t_n\}_n$, $t_n \underset{n}{\rightarrow} +\infty$, *and a function* $u^\infty \in W_0^{1,\infty}(\Omega)$ *such that*

$$\begin{cases} u(t_n) \underset{n}{\rightarrow} u^\infty \quad \text{in } W_0^{1,\infty}(\Omega) \text{ weak-}* \text{ and uniformly in } \overline{\Omega}, \\ u^\infty \text{ is a solution of the problem (8).} \end{cases}$$

3. – The approximating problems

Let us consider the family of approximated problems defined by (7).

PROPOSITION 1. *With the assumptions* (1), (2) *and* (3), *the problem* (7) *has a unique solution* $u^\varepsilon \in C^{2,1}_{\alpha,\alpha/2}(Q_T) \cap C^0(\overline{Q}_T)$, $0 < \alpha < 1$.

PROOF. This is a direct consequence of well known results for quasilinear parabolic equations (see [9]). □

We shall establish some a priori estimates for the solutions of the approximated problems.

LEMMA 1. *Suppose that* (1), (2) *and* (3) *are satisfied and that* u^ε *is the solution of the problem* (7). *Then there exists a constant* $M > 0$ *independent of* $\varepsilon \in]0, 1[$ *such that*

$$|u^\varepsilon(x,t)| \leq M \qquad \text{for all } (x,t) \in \overline{Q}_T\ . \tag{12}$$

PROOF. This estimate is a consequence of the assumptions and global boundedness for weak solutions of the Dirichlet problem for quasilinear degenerate parabolic equations (see [4] and its references). We need only to remark that in (7) $k_\varepsilon \geq 1$ and this is the crucial fact for deriving the weak maximum principle estimate. □

REMARK 2. We remark that the constant M that appears in (12) depends on T only through $\|f(t)\|_{L^\infty(\Omega)}$, and therefore it will be independent of T if $f \in L^\infty(0,\infty; L^\infty(\Omega)) = L^\infty(Q_\infty)$, i.e., under the assumption (9).

LEMMA 2. *Suppose that* (1), (2) *and* (3) *are satisfied and* u^ε *is the solution of the problem* (7). *Then there exists a constant* $C > 0$ *independent of* $\varepsilon \in]0, 1[$ *such that*

$$\|u^\varepsilon_t\|_{L^\infty(0,T;L^1(\Omega))} \leq C\ .$$

PROOF. We differentiate the equation of the approximated problem (7) in order to t and set $v = u^\varepsilon_t$. Then

$$\begin{aligned}
v_t - \nabla\,.\,\Bigg[& k'_\varepsilon\Big(|\nabla u^\varepsilon|^2 - F^2_\varepsilon(u^\varepsilon)\Big)\Big(2\,\nabla u^\varepsilon\,.\,\nabla v - 2\,F_\varepsilon\,F'_\varepsilon\,v\Big)\Big(|\nabla u^\varepsilon|^2 + \varepsilon\Big)^{\frac{p-2}{2}}\,\nabla u^\varepsilon \\
& + k_\varepsilon\Big(|\nabla u^\varepsilon|^2 - F^2_\varepsilon(u^\varepsilon)\Big)\,\frac{p-2}{2}\,\Big(|\nabla u^\varepsilon|^2 + \varepsilon\Big)^{\frac{p-4}{2}}\,2\,\nabla u^\varepsilon\,.\,\nabla v\,\nabla u^\varepsilon \\
& + k_\varepsilon\Big(|\nabla u^\varepsilon|^2 - F^2_\varepsilon(u^\varepsilon)\Big)\,\Big(|\nabla u^\varepsilon|^2 + \varepsilon\Big)^{\frac{p-2}{2}}\,\nabla v + \varepsilon\,\nabla v\Bigg] = f^\varepsilon_t\ .
\end{aligned} \tag{13}$$

Consider the sign function, denoted by sgn^0 and defined as follows:

$$\mathrm{sgn}^0(\tau) = \begin{cases} -1 & \text{if } \tau < 0, \\ 0 & \text{if } \tau = 0, \\ 1 & \text{if } \tau > 0. \end{cases} \qquad (\tau \in \mathbb{R})\ ,$$

Let $s_\delta\colon \mathbb{R} \to \mathbb{R}$, $\delta > 0$, be a sequence of C^2 functions, approximating pointwise sgn^0 when $\delta \to 0$, such that

$$s_\delta\big|_{\{0\}\cup\mathbb{R}\setminus]-\delta,\delta[} = \operatorname{sgn}^0\big|_{\{0\}\cup\mathbb{R}\setminus]-\delta,\delta[}\ ,\quad s'_\delta \geq 0\ .$$

Notice that, if $\zeta_\delta(\tau) = \displaystyle\int_0^\tau s_\delta(\sigma)\,d\sigma$, then

$$\forall\,\tau \in \mathbb{R}\ \ \lim_{\delta\to 0} \tau\, s'_\delta(\tau) = 0,\ \ \lim_{\delta\to 0} \zeta_\delta(\tau) = |\tau|\ .$$

Multiply the equation (13) by $s_\delta(v)$ and integrate over Q_t. Then, since $v\big|_{\partial\Omega\times]0,t[} = 0$ and $s_\delta(v)\big|_{\partial\Omega\times]0,t[} = 0$, we have

$$\begin{aligned}
&\int_\Omega \zeta_\delta(v(t)) - \int_\Omega \zeta_\delta(v(0)) + \int_{Q_t} 2\,k'_\varepsilon\Big(|\nabla u^\varepsilon|^2 - F^2_\varepsilon(u^\varepsilon)\Big)(\nabla u^\varepsilon.\nabla v)^2\Big(|\nabla u^\varepsilon|^2 + \varepsilon\Big)^{\frac{p-2}{2}} s'_\delta(v)\\
&+ \int_{Q_t} -2\,k'_\varepsilon\Big(|\nabla u^\varepsilon|^2 - F^2_\varepsilon(u^\varepsilon)\Big) F_\varepsilon F'_\varepsilon\Big(|\nabla u^\varepsilon|^2 + \varepsilon\Big)^{\frac{p-2}{2}} \nabla u^\varepsilon\,.\,\nabla v\ \ v\,s'_\delta(v)\\
(14)\quad &+ \int_{Q_t} k_\varepsilon\Big(|\nabla u^\varepsilon|^2 - F^2_\varepsilon(u^\varepsilon)\Big)\,(p-2)\,\Big(|\nabla u^\varepsilon|^2 + \varepsilon\Big)^{\frac{p-4}{2}} (\nabla u^\varepsilon\,.\,\nabla v)^2\, s'_\delta(v)\\
&+ \int_{Q_t} k_\varepsilon\Big(|\nabla u^\varepsilon|^2 - F^2_\varepsilon(u^\varepsilon)\Big)\,\Big(|\nabla u^\varepsilon|^2 + \varepsilon\Big)^{\frac{p-2}{2}} |\nabla v|^2\, s'_\delta(v)\\
&+ \int_{Q_t} k_\varepsilon\Big(|\nabla u^\varepsilon|^2 - F^2_\varepsilon(u^\varepsilon)\Big)\,\varepsilon\,|\nabla v|^2\, s'_\delta(v) \;=\; \int_{Q_t} f^\varepsilon_t\, s_\delta(v)\ .
\end{aligned}$$

Notice that, since k_ε is monotone nondecreasing,

$$\int_{Q_t} 2\,k'_\varepsilon\Big(|\nabla u^\varepsilon|^2 - F^2_\varepsilon(u^\varepsilon)\Big)\,(\nabla u^\varepsilon\,.\,\nabla v)^2\,\Big(|\nabla u^\varepsilon|^2 + \varepsilon\Big)^{\frac{p-2}{2}} s'_\delta(v) \;\geq\; 0$$

$$\int_{Q_t} k_\varepsilon\Big(|\nabla u^\varepsilon|^2 - F^2_\varepsilon(u^\varepsilon)\Big)\,\varepsilon\,|\nabla v|^2\, s'_\delta(v) \;\geq\; 0,$$

and

$$\int_{Q_t} -2\,k'_\varepsilon\Big(|\nabla u^\varepsilon|^2 - F^2_\varepsilon(u^\varepsilon)\Big)\,F_\varepsilon\,F'_\varepsilon\Big(|\nabla u^\varepsilon|^2 + \varepsilon\Big)^{\frac{p-2}{2}} \nabla u^\varepsilon\,.\,\nabla v\ \ v\,s'_\delta(v) \;\underset{\delta\to 0}{\longrightarrow}\; 0\ .$$

On the other hand,

$$\int_{Q_t} k_\varepsilon\Big(|\nabla u^\varepsilon|^2 - F_\varepsilon^2(u^\varepsilon)\Big)\,(p-2)\,\Big(|\nabla u^\varepsilon|^2+\varepsilon\Big)^{\frac{p-4}{2}}\,(\nabla u^\varepsilon \,.\, \nabla v)^2\, s_\delta'(v)$$

$$+\int_{Q_t} k_\varepsilon\Big(|\nabla u^\varepsilon|^2 - F_\varepsilon^2(u^\varepsilon)\Big)\,\Big(|\nabla u^\varepsilon|^2+\varepsilon\Big)^{\frac{p-2}{2}}\,|\nabla v|^2\, s_\delta'(v)$$

$$=\int_{Q_t} k_\varepsilon\Big(|\nabla u^\varepsilon|^2 - F_\varepsilon^2(u^\varepsilon)\Big)\,\Big(|\nabla u^\varepsilon|^2+\varepsilon\Big)^{\frac{p-4}{2}}$$

$$\times\, s_\delta'(v)\,\Big[(p-2)\,(\nabla u^\varepsilon \,.\, \nabla v)^2 + \Big(|\nabla u^\varepsilon|^2+\varepsilon\Big)\,|\nabla v|^2\Big].$$

For $p \geq 2$, obviously, the second member of this equation is nonnegative. For $1 < p < 2$, using Cauchy–Schwartz inequality, we have

$$\int_{Q_t} k_\varepsilon\Big(|\nabla u^\varepsilon|^2 - F_\varepsilon^2(u^\varepsilon)\Big)\Big(|\nabla u^\varepsilon|^2+\varepsilon\Big)^{\frac{p-4}{2}} s_\delta'(v)\Big[(p-2)(\nabla u^\varepsilon.\nabla v)^2 + \Big(|\nabla u^\varepsilon|^2+\varepsilon\Big)|\nabla v|^2\Big]$$

$$\geq \int_{Q_t} k_\varepsilon\Big(|\nabla u^\varepsilon|^2 - F_\varepsilon^2(u^\varepsilon)\Big)\Big(|\nabla u^\varepsilon|^2+\varepsilon\Big)^{\frac{p-4}{2}} s_\delta'(v)|\nabla u^\varepsilon|^2\,|\nabla v|^2(1-(2-p)) \geq 0.$$

So, since $\int_{Q_t} f_t^\varepsilon s_\delta(v) \leq \int_{Q_t} |f_t^\varepsilon|$, neglecting the nonnegative terms of (14) and letting $\delta \to 0$ in (14), we obtain

$$\int_\Omega |v(t)| - \int_\Omega |v(0)| \leq \int_{Q_t} |f_t^\varepsilon| \,, \tag{15}$$

and so

$$\|u_t^\varepsilon(t)\|_{L^1(\Omega)} \leq \|\Delta_p h^\varepsilon + \varepsilon\,\Delta h^\varepsilon\|_{L^1(\Omega)} + \|f_t^\varepsilon\|_{L^1(Q_T)} \leq C \,,$$

where $C > 0$ may be chosen independent of ε. □

LEMMA 3. *Suppose that assumptions* (1), (2) *and* (3) *are satisfied and* u^ε *is the solution to the problem* (7). *Then there is a constant* $C > 0$ *independent of* $\varepsilon \in]0,1[$ *such that*

$$\Big\|k_\varepsilon\Big(|\nabla u^\varepsilon|^2 - F_\varepsilon^2(u^\varepsilon)\Big)\Big\|_{L^1(Q_T)} \leq C \,. \tag{16}$$

PROOF. We consider the first equation of the problem (7), multiply it by u^ε and integrate over Q_T. Then

$$\int_{Q_T} u_t^\varepsilon u^\varepsilon + \int_{Q_T} k_\varepsilon\Big(|\nabla u^\varepsilon|^2 - F_\varepsilon^2(u^\varepsilon)\Big)\Big[\Big(|\nabla u^\varepsilon|^2+\varepsilon\Big)^{p-2}|\nabla u^\varepsilon|^2 + \varepsilon|\nabla u^\varepsilon|^2\Big] = \int_{Q_T} f^\varepsilon u^\varepsilon.$$

Since f^ε and u^ε are uniformly bounded, independently of ε, in $L^\infty(Q_T)$,

$$\left|\int_{Q_T} f^\varepsilon\, u^\varepsilon\right| \le C \ , \ \ C \text{ independent of } \varepsilon$$

and, since u_t^ε is uniformly bounded in $L^\infty(0, T; L^1(\Omega))$, by Lemma 2,

$$\left|\int_{Q_T} u_t^\varepsilon\, u^\varepsilon\right| \le C \ , \ \ C \text{ independent of } \varepsilon$$

and so, it follows that

$$\int_{Q_T} k_\varepsilon\Big(|\nabla u^\varepsilon|^2 - F_\varepsilon^2(u^\varepsilon)\Big)\, |\nabla u^\varepsilon|^p \le C' \ , \ \ C' \text{ constant independent of } \varepsilon \ .$$

But, on the other hand,

$$\begin{aligned}\int_{Q_T} k_\varepsilon\Big(|\nabla u^\varepsilon|^2 - F_\varepsilon^2(u^\varepsilon)\Big) &= \int_{\left\{|\nabla u^\varepsilon|^2 \le F_\varepsilon^2(u^\varepsilon)+\varepsilon\right\}} k_\varepsilon\Big(|\nabla u^\varepsilon|^2 - F_\varepsilon^2(u^\varepsilon)\Big)\\ &\quad + \int_{\left\{|\nabla u^\varepsilon|^2 > F_\varepsilon^2(u^\varepsilon)+\varepsilon\right\}} k_\varepsilon\Big(|\nabla u^\varepsilon|^2 - F_\varepsilon^2(u^\varepsilon)\Big)\\ &\le |Q_T|\, e + \int_{Q_T} k_\varepsilon\Big(|\nabla u^\varepsilon|^2 - F^2(u^\varepsilon)\Big)\, \frac{|\nabla u^\varepsilon|^p}{m^p}\end{aligned}$$

because in $\{|\nabla u^\varepsilon|^2 \le F_\varepsilon^2(u^\varepsilon) + \varepsilon\}$ we have $k_\varepsilon(|\nabla u^\varepsilon|^2 - F_\varepsilon^2(u^\varepsilon)) \le e$ and in $\{|\nabla u^\varepsilon|^2 \ge F_\varepsilon^2(u^\varepsilon) + \varepsilon\}$ we have $\frac{|\nabla u^\varepsilon|}{m} \ge 1$.

So, we conclude

$$\begin{aligned}\int_{Q_T} k_\varepsilon\Big(|\nabla u^\varepsilon|^2 - F_\varepsilon^2(u^\varepsilon)\Big) &\le |Q_T|\, e + \frac{1}{m^p}\int_{Q_T} k_\varepsilon\Big(|\nabla u^\varepsilon|^2 - F_\varepsilon^2(u^\varepsilon)\Big)\, |\nabla u^\varepsilon|^p\\ &\le |Q_T|\, e + \frac{C'}{m^p} \ , \ \ \text{independently of } \varepsilon \ . \qquad \square\end{aligned}$$

Lemma 4. *Suppose that the assumptions* (1), (2) *and* (3) *are satisfied and* u^ε *is the solution of the problem* (7). *Then, for any* $q \in [1, +\infty[$ *there exists a constant* $C_q > 0$ *independent of* $\varepsilon \in]0, 1[$ *such that*

$$\|\nabla u^\varepsilon\|_{L^q(Q_T)} \le C_q \ .$$

Proof. We know, from (16), that there exists a constant C, independent of ε, such that, for any $\varepsilon \in]0, 1[$,

$$\int_{Q_T} k_\varepsilon\Big(|\nabla u^\varepsilon|^2 - F_\varepsilon^2(u^\varepsilon)\Big) \le C \ .$$

So, we have

$$C \geq \int_{\left\{|\nabla u^\varepsilon|^2 > F_\varepsilon^2(u^\varepsilon)+\varepsilon\right\}} k_\varepsilon\left(|\nabla u^\varepsilon|^2 - F_\varepsilon^2(u^\varepsilon)\right) = \int_{\left\{|\nabla u^\varepsilon|^2 > F_\varepsilon^2(u^\varepsilon)+\varepsilon\right\}} e^{\frac{|\nabla u^\varepsilon|^2 - F_\varepsilon^2(u^\varepsilon)}{\varepsilon}} .$$

Recalling that

$$\forall s \in \mathbb{R}^+ \ \forall j \in \mathbb{N} \ e^s \geq \frac{s^j}{j!}$$

we obtain, for each $j \in \mathbb{N}$,

$$\text{(17)} \quad \begin{aligned} \int_{\left\{|\nabla u^\varepsilon|^2 > F_\varepsilon^2(u^\varepsilon)+\varepsilon\right\}} \left[|\nabla u^\varepsilon|^2 - F_\varepsilon^2(u^\varepsilon)\right]^j &\leq j!\, \varepsilon^j \int_{\left\{|\nabla u^\varepsilon|^2 > F_\varepsilon^2(u^\varepsilon)+\varepsilon\right\}} e^{\frac{|\nabla u^\varepsilon|^2 - F_\varepsilon^2(u^\varepsilon)}{\varepsilon}} \\ &\leq j!\, \varepsilon^j\, C . \end{aligned}$$

Given $q \in \mathbb{N}$, we have

$$\text{(18)} \quad \int_{Q_T} |\nabla u^\varepsilon|^{2q} = \int_{\left\{|\nabla u^\varepsilon|^2 \leq F_\varepsilon^2(u^\varepsilon)+\varepsilon\right\}} |\nabla u^\varepsilon|^{2q} + \int_{\left\{|\nabla u^\varepsilon|^2 > F_\varepsilon^2(u^\varepsilon)+\varepsilon\right\}} |\nabla u^\varepsilon|^{2q} ,$$

Since there exists a constant $M > 0$, not depending on ε, such that $\|u^\varepsilon\|_{L^\infty(Q_T)} \leq M$, we can estimate the first integral in the second member of (18) as follows:

$$\int_{\left\{|\nabla u^\varepsilon|^2 \leq F_\varepsilon^2(u^\varepsilon)+\varepsilon\right\}} |\nabla u^\varepsilon|^{2q} \leq \int_{Q_T} \left[F_\varepsilon^2(u^\varepsilon) + \varepsilon\right]^q \leq \max_{s \in [-M,M]} \left\{F^2(s) + 1\right\}^q |Q_T|.$$

On the other hand the second integral of the second member of (18) satisfies

$$\begin{aligned} &\int_{\left\{|\nabla u^\varepsilon|^2 > F_\varepsilon^2(u^\varepsilon)+\varepsilon\right\}} |\nabla u^\varepsilon|^{2q} \\ &= \int_{\left\{|\nabla u^\varepsilon|^2 > F_\varepsilon^2(u^\varepsilon)+\varepsilon\right\}} \left[|\nabla u^\varepsilon|^2 - F_\varepsilon^2(u^\varepsilon) + F_\varepsilon^2(u^\varepsilon)\right]^q \\ &\leq \int_{\left\{|\nabla u^\varepsilon|^2 > F_\varepsilon^2(u^\varepsilon)+\varepsilon\right\}} \sum_{j=0}^{q} \binom{q}{j} \max_{x \in [-M,M]} \left[F_\varepsilon^2(x)\right]^{q-j} \left[|\nabla u^\varepsilon|^2 - F_\varepsilon^2(u^\varepsilon)\right]^j \\ &\leq \sum_{j=0}^{p} D_j \int_{\left\{|\nabla u^\varepsilon|^2 > F_\varepsilon^2(u^\varepsilon)+\varepsilon\right\}} \left[|\nabla u^\varepsilon|^2 - F_\varepsilon^2(u^\varepsilon)\right]^j , \end{aligned}$$

where D_j is a constant independent of ε.

So, using (17), we easily conclude that

$$\|\nabla u^\varepsilon\|_{L^q(Q_T)} \leq C_q \ , \ C_q \text{ depending on } q, \text{ but not on } \varepsilon . \qquad \square$$

4. – The limiting process and further remarks

In this section we will let $\varepsilon \to 0$ and prove that if we define

$$\mathbb{K}_{u^\varepsilon(t)} = \left\{ v \in W_0^{1,p}(\Omega) : |\nabla v| \le F(u^\varepsilon(t)) \quad \text{a.e. in } \Omega \right\} \tag{19}$$

we have, at least for a subsequence $\varepsilon \to 0$, for any given $v(t) \in \mathbb{K}_{u(t)}$ for every $t \in I$, the convergence of a sequence of functions v^ε, with $v^\varepsilon(t) \in \mathbb{K}_{u^\varepsilon(t)}$, to the function v, when $\varepsilon \to 0$.

Recalling Lemma 4, let u be the weak limit in $L^q(0, T; W_0^{1,q}(\Omega))$, $\forall q < \infty$, of a subsequence of $\{u^\varepsilon\}_{\varepsilon \in]0,1[}$ as $\varepsilon \longrightarrow 0$, i.e.,

$$u^\varepsilon \underset{\varepsilon}{\rightharpoonup} u \text{ in } L^q(0, T; W_0^{1,q}(\Omega)) \; . \tag{20}$$

Then

LEMMA 5. *Suppose that the assumptions* (1), (2) *and* (3) *are satisfied and that u^ε is the solution of the problem* (7). *Let $\mathbb{K}_{u^\varepsilon(t)}$ be the convex set defined in* (19) *and u the function defined in* (20), *weak limit in $L^q(0, T; W_0^{1,q}(\Omega))$ of the subsequence $\{u^\varepsilon\}_\varepsilon$.*

Then for any $v \in L^\infty(0, T; W_0^{1,p}(\Omega))$, $1 \le p \le \infty$ such that $v(t) \in \mathbb{K}_{u(t)}$ for a.e. $t \in [0, T]$, there exists a $v^\varepsilon \in L^\infty(0, T; W_0^{1,p}(\Omega))$ with $v^\varepsilon(t) \in \mathbb{K}_{u^\varepsilon(t)}$ for a.e. $t \in [0, T]$ such that

$$v^\varepsilon \underset{\varepsilon}{\to} v \qquad \textit{in } L^\infty(0, T; W_0^{1,p}(\Omega)), \qquad 1 \le p \le \infty.$$

PROOF. Since there are constants C_1 and C_q such that

$$\|u_t^\varepsilon\|_{L^\infty(0,T;L^1(\Omega))} \le C_1 \qquad \text{and} \qquad \|u^\varepsilon\|_{L^q(0,T;W_0^{1,q}(\Omega))} \le C_q$$

(for $q > N$), by a well known compactness theorem ([13], page 84), $\{u^\varepsilon\}_\varepsilon$ is relatively compact in $C([0, T]; C(\overline{\Omega}))$ and so, at least for another subsequence of the subsequence in (20), still denoted by $\varepsilon \to 0$, we have

$$u^\varepsilon \underset{\varepsilon}{\to} u \qquad \text{uniformly in} \quad C^0(\overline{Q}_T) \tag{21}$$

and therefore also

$$F(u^\varepsilon) \underset{\varepsilon}{\to} F(u) \qquad \text{uniformly in} \quad C^0(\overline{Q}_T) \; .$$

Let $\alpha_\varepsilon(t) = \|F(u^\varepsilon(t)) - F(u(t))\|_{L^\infty(\Omega)}$ and notice that

$$\alpha_\varepsilon(t) \underset{\varepsilon}{\to} 0 \qquad \text{uniformly in} \quad t \in [0, T] \; .$$

Define $\psi_\varepsilon(t) = 1 + \frac{\alpha_\varepsilon(t)}{m}$ and, given $v \in L^\infty(0,T;W_0^{1,p}(\Omega))$ such that $v(t) \in \mathbb{K}_{u(t)}$ for a.e. $t \in [0,T]$, define $v^\varepsilon(t) = \frac{1}{\psi_\varepsilon(t)}\, v(t) \in L^\infty(0,T;W_0^{1,p}(\Omega))$. Then,

$$|\nabla v^\varepsilon(t)| = \frac{1}{\psi_\varepsilon(t)}|\nabla v(t)| \leq \frac{1}{\psi_\varepsilon(t)} F(u(t)) \leq F(u^\varepsilon(t))$$

because

$$\begin{aligned}\psi_\varepsilon(t) &= 1 + \frac{\alpha_\varepsilon(t)}{m} \geq 1 + \frac{\alpha_\varepsilon(t)}{F(u^\varepsilon(t))} \\ &\geq \frac{F(u^\varepsilon(t)) + F(u(t)) - F(u^\varepsilon(t))}{F(u^\varepsilon(t))} = \frac{F(u(t))}{F(u^\varepsilon(t))}\ .\end{aligned}$$

So, we have $v^\varepsilon(t) \in \mathbb{K}_{u^\varepsilon(t)}$ for a.e. $t \in [0,T]$ and $v^\varepsilon \underset{\varepsilon}{\rightarrow} v$ in $L^\infty(0,T;W_0^{1,p}))$ since

$$\|v^\varepsilon(t) - v(t)\|_{W_0^{1,p}(\Omega)} = \left|\frac{1}{\psi_\varepsilon(t)} - 1\right| \|v\|_{L^\infty(0,T;W_0^{1,p}(\Omega))} \underset{\varepsilon}{\longrightarrow} 0$$

uniformly in $t \in [0,T]$. □

Proof of Theorem 1. Recalling (20) and (21), for some subsequence $\varepsilon \to 0$ we have

$$\begin{aligned} &u^\varepsilon \underset{\varepsilon}{\rightarrow} u && \text{uniformly in } C^0(\overline{Q}_T), \\ &u^\varepsilon \underset{\varepsilon}{\rightharpoonup} u && \text{weakly in } L^q(0,T;W^{1,q}(\Omega)), \quad \forall q < \infty, \end{aligned}$$

and, besides that, this subsequence may be chosen in such a way that

$$u_t^\varepsilon \underset{\varepsilon}{\rightharpoonup} u_t \qquad \text{in } L^\infty(0,T;M(\Omega)) \text{ weak-}* \ .$$

Now, given $w \in \mathbb{K}_{u^\varepsilon(t)}$ for a.e. $t \in [0,T]$, we have

$$\begin{aligned} &\int_\Omega \Big(|\nabla w|^2 + \varepsilon\Big)^{\frac{p-2}{2}} \nabla w \,.\, \nabla(w - u^\varepsilon(t)) \\ &= \int_\Omega k_\varepsilon\Big(|\nabla w|^2 - F^2(u^\varepsilon(t))\Big)\Big(|\nabla w|^2 + \varepsilon\Big)^{\frac{p-2}{2}} \nabla w \,.\, \nabla(w - u^\varepsilon(t)) \\ &\geq \int_\Omega k_\varepsilon\Big(|\nabla u^\varepsilon(t)|^2 - F^2(u^\varepsilon(t))\Big)\Big(|\nabla u^\varepsilon(t)|^2 + \varepsilon\Big)^{\frac{p-2}{2}} \nabla u^\varepsilon(t) \,.\, \nabla(w - u^\varepsilon(t)) \end{aligned}$$

since, by monotonicity,

$$\int_\Omega k_\varepsilon\Big(|\nabla w|^2 - F^2(u^\varepsilon(t))\Big)\Big(|\nabla w|^2+\varepsilon\Big)^{\frac{p-2}{2}} \nabla w\,.\,\nabla(w-u^\varepsilon(t))$$
$$-\int_\Omega k_\varepsilon\Big(|\nabla u^\varepsilon(t)|^2 - F^2(u^\varepsilon(t))\Big)\Big(|\nabla u^\varepsilon(t)|^2+\varepsilon\Big)^{\frac{p-2}{2}} \nabla u^\varepsilon(t)\,.\,\nabla(w-u^\varepsilon(t)) \geq 0.$$

Given $v \in L^\infty(0,T; W_0^{1,p}(\Omega))$ such that $v(t) \in \mathbb{K}_{u(t)}$, for a.e. $t \in [0,T]$, we consider $v^\varepsilon \in L^\infty(0,T; W_0^{1,p}(\Omega))$ as in Lemma 5. Since $v^\varepsilon(t) \in \mathbb{K}_{u^\varepsilon(t)}$, multiplying the equation (7) by $v^\varepsilon(t) - u^\varepsilon(t)$ and integrating over $\Omega\times]s,t[$, $0 < s < t < T$, we have

$$\int_s^t\int_\Omega u_t^\varepsilon(v^\varepsilon - u^\varepsilon) + \int_s^t\int_\Omega \Big(|\nabla v^\varepsilon|^2+\varepsilon\Big)^{\frac{p-2}{2}} \nabla v^\varepsilon\,.\,\nabla(v^\varepsilon - u^\varepsilon) \geq \int_s^t\int_\Omega f^\varepsilon(v^\varepsilon - u^\varepsilon)$$

and letting $\varepsilon \to 0$, since $v^\varepsilon \to v$ in $L^\infty(0,T; W_0^{1,p}(\Omega))$ strongly, we obtain

$$\int_s^t\int_\Omega u_t(v-u) + \int_s^t\int_\Omega |\nabla v|^{p-2}\nabla v\,.\,\nabla(v-u) \geq \int_s^t\int_\Omega f(v-u).$$

Since s and t are arbitrary, we may conclude

$$\int_\Omega u_t(t)\,(v-u(t)) + \int_\Omega |\nabla v|^{p-2}\,\nabla v\,.\,\nabla(v-u(t))$$
$$\geq \int_\Omega f(t)\,(v-u(t))\ , \quad \forall\, v \in \mathbb{K}_{u(t)} \text{ for a.e. } t \in I. \tag{22}$$

It still remains to prove that $|\nabla u(t)| \leq F(u(t))$ a.e. in Ω, for a.e. $t \in I$. If we prove this, a variant of Minty's lemma will show that (22) implies (5).

To prove that $|\nabla u(t)| \leq F(u(t))$ a.e. in Ω, for a.e. $t \in I$ is equivalent to prove that the set $\{(x,t) \in Q_T\colon |\nabla u(x,t)| > F(u(x,t))\}$ has measure zero.

We recall from Lemma 3 that

$$\exists\, C>0\ \forall\, \varepsilon \in]0,1[\ \int_{Q_T} k_\varepsilon\Big(|\nabla u^\varepsilon|^2 - F_\varepsilon^2(u^\varepsilon)\Big) \leq C\ .$$

Given $\varepsilon \in]0,1[$ and $M \in \mathbb{R}^+$, $M \geq \varepsilon$, we define

$$A_{M,\varepsilon} = \Big\{(x,t) \in Q_T\colon |\nabla u^\varepsilon(x,t)|^2 - F_\varepsilon^2(u^\varepsilon(x,t)) \geq M\Big\}$$

and then, since $k_\varepsilon(|\nabla u^\varepsilon|^2 - F^2(u^\varepsilon)) \geq e^{\frac{M}{\varepsilon}}$ in $A_{M,\varepsilon}$, we have

$$\forall\, \varepsilon \in]0,1[\ \forall\, M \in [\varepsilon,+\infty[\ |A_{M,\varepsilon}|\, e^{\frac{M}{\varepsilon}} \leq \int_{A_{M,\varepsilon}} k_\varepsilon\Big(|\nabla u^\varepsilon|^2 - F_\varepsilon^2(u^\varepsilon)\Big) \leq C\ ,$$

where $|A|$ denotes the Lebesgue measure of the set A in $\mathbb{R}^{N+1}$.

So we have

$$|A_{M,\varepsilon}| \leq C\, e^{-\frac{M}{\varepsilon}}$$

and, choosing $M = \sqrt{\varepsilon}$, we get

$$|A_{\sqrt{\varepsilon},\varepsilon}| \leq C\, e^{-\frac{1}{\sqrt{\varepsilon}}}$$

and

$$\begin{aligned}
\int_{Q_T} \Big(|\nabla u|^2 - F_\varepsilon^2(u)\Big)^+ &\leq \liminf_{\varepsilon\to 0} \int_{Q_T} \Big(|\nabla u^\varepsilon|^2 - F_\varepsilon^2(u^\varepsilon) - \sqrt{\varepsilon}\Big)^+ \\
&= \liminf_{\varepsilon\to 0} \int_{A_{\sqrt{\varepsilon},\varepsilon}} \Big(|\nabla u^\varepsilon|^2 - F_\varepsilon^2(u^\varepsilon) - \sqrt{\varepsilon}\Big) \\
&\leq \lim_{\varepsilon\to 0} D\, |A_{\sqrt{\varepsilon},\varepsilon}|^{\frac{1}{2}} = 0\ ,
\end{aligned}$$

being D a constant that bounds from above $\left[\int_{Q_T} (|\nabla u^\varepsilon|^2 - F_\varepsilon^2(u^\varepsilon) - \sqrt{\varepsilon})^2\right]^{1/2}$, independently of ε.

So, $(|\nabla u|^2 - F^2(u))^+ = 0$ a.e. in Q_T or, equivalently,

$$|\nabla u| \leq F(u) \qquad \text{a.e. in} \quad Q_T\ .$$

Substitute now in (22), for a.e. $t \in I$, v by $u(t) + \theta(w - u(t))$, where w belongs to $\mathbb{K}_{u(t)}$, $\theta \in\,]0, 1]$ and divide both members by θ. We obtain

$$\begin{aligned}
&\int_\Omega u_t(t)(w-u(t)) + \int_\Omega \Big|\nabla\Big(u(t)+\theta(w-u(t))\Big)\Big|^{p-2} \nabla\Big(u(t)+\theta(w-u(t))\Big).\nabla(w-u(t)) \\
&\quad \geq \int_\Omega f(t)\,(w - u(t))\ ,\quad \forall\, w \in \mathbb{K}_{u(t)} \quad \text{for a.e. } t \in I\ .
\end{aligned}$$

Letting $\theta \to 0$ we conclude that u is a solution to the quasivariational inequality (5). $\square$

5. – Asymptotic behaviour in time

Consider $T = +\infty$. We first show that there exists a global solution of the quasivariational inequality (5) when $T = +\infty$ and then, in some sense, there exists a limit function, when $t \to +\infty$, solution of a stationary quasivariational inequality.

LEMMA 6. *Suppose that* (9) *is satisfied. Then the problem* (5) *has a solution* u *such that*

$$u \in L^{\infty}(0,+\infty; W_0^{1,\infty}(\Omega)) \cap C^0(\mathbb{R}_0^+ \times \overline{\Omega}), \quad u_t \in L^{\infty}(0,+\infty; M(\Omega)),$$

being u *the weak limit in* $L^q(0,+\infty; W^{1,q}(\Omega))$ *(for any* $q < \infty$*) of a subsequence* $\{u^{\varepsilon}\}_{\varepsilon}$ *of solutions of the family of approximated problems (7),* $u_t^{\varepsilon} \rightharpoonup u_t$ *in* $L^{\infty}(0,+\infty; M(\Omega))$ *weak-$*$ and, for all* $T > 0$*, there exists a subsequence of* $\varepsilon \to 0$*, called* $\varepsilon_T \to 0$*, such that* $u^{\varepsilon_T} \underset{\varepsilon_T}{\to} u$ *in* $C^0(\overline{Q}_T)$*. In addition, we have the estimate*

$$t\int_{\Omega}|u_t(t)| \leq \int_0^t \left(\int_{\Omega}|f_t(\tau)|\right)(1+\tau)d\tau, \ \ for\, 0<t<\infty\,. \tag{23}$$

PROOF. The assumption (9) implies that the estimates of Lemmas 1 and 2 may be obtained when $T = +\infty$. The L^{∞} boundedness of $\{u^{\varepsilon}\}_{\varepsilon}$, independently of ε and T and its boundedness in $L^{\infty}(0,+\infty; W^{1,q}(\Omega))$ can also be obtained easily from the boundedness of $k_{\varepsilon}(|\nabla u^{\varepsilon}|^2 - F_{\varepsilon}^2(u^{\varepsilon}))$ in $L^{\infty}(0,+\infty; L^1(\Omega))$, which can be obtained as in Lemma 3, integrating only on Ω, for a.e. $t \in [0,T]$. So, we can pass to the limit in a subsequence $\varepsilon \to 0$ and get a function $u \in L^{\infty}(0,+\infty; W^{1,\infty}(\Omega))$, $u_t \in L^{\infty}(0,+\infty; M(\Omega))$ which is the weak-limit of u^{ε} in these spaces.

The uniform convergence of a subsequence of u^{ε} is obtained only in $C^0(\overline{Q}_T)$, for each T, which is enough to pass to the limit in the approximated problem and to obtain that u is a solution of the quasivariational inequality for almost every $t \in \mathbb{R}^+$

In order to show (23) we multiply the approximated equation (7) by $ts_{\delta}(u_t^{\varepsilon})$ and, arguing as in the proof of Lemma 2, we obtain, after letting $\delta \to 0$,

$$\begin{aligned}\int_{\Omega}(t|v(t)| - |v(0)| + |v(t)|) &= \int_0^t \left(\int_{\Omega}\frac{d|v|}{d\tau}\right)\tau d\tau \\ &\leq \int_0^t \left(\int_{\Omega}|f_t^{\varepsilon}|\right)\tau d\tau, \quad \text{for any } 0<t<\infty.\end{aligned}$$

Using (15) we obtain

$$t\int_{\Omega}|v(t)| \leq \int_0^t \left(\int_{\Omega}|f_t^{\varepsilon}|\right)d\tau + \int_0^t \left(\int_{\Omega}|f_t^{\varepsilon}|\right)\tau d\tau,$$

and letting $\varepsilon \to 0$ we conclude (23). □

PROOF OF THEOREM 2. Let u be a solution of the problem (5), with $T = +\infty$.

Since

$$\exists\, M>0 \ \ \|u\|_{L^{\infty}(\mathbb{R}^+\times\Omega)} \leq M$$

and

$$|\nabla u(x,t)| \leq F(u(x,t)) \leq \max_{|s|\leq M}\{F(s)\} \qquad \text{for a.e.} \quad (x,t) \in \mathbb{R}^+ \times \Omega$$

we have

$$\exists\, C>0 \ \ \|u\|_{L^\infty(\mathbb{R}^+;W^{1,\infty}(\Omega))} \leq C$$

and so, for a subsequence $(t_n)_n$, $t_n \nearrow_n +\infty$ and some function $u^\infty \in W_0^{1,\infty}(\Omega)$, we have

$$u(t_n) \underset{n}{\rightharpoonup} u^\infty \quad \text{in } W^{1,\infty}(\Omega) \text{ weak-}* \ ,$$
$$u(t_n) \underset{n}{\rightarrow} u^\infty \quad \text{in } C^0(\overline{\Omega}) \ .$$

Since for every measurable subset $\omega \subset \Omega$ we have $(1 < q < \infty)$

$$\int_\omega |\nabla u^\infty|^q \leq \liminf_n \int_\omega |\nabla u(t_n)|^q \leq \lim_n \int_\omega F^q(u(t_n)) = \int_\omega F^q(u^\infty),$$

we find that

$$|\nabla u^\infty| \leq F(u^\infty) \qquad \text{a.e. in } \Omega,$$

and, therefore, we have

$$u^\infty \in \mathbb{K}_{u^\infty} \ . \tag{24}$$

It remains to show that u^∞ satisfies the inequality in (8).
By monotonicity, we rewrite (5) for a.e. $t \in \mathbb{R}^+$ in the form

$$\int_\Omega u_t(v-u(t))+\int_\Omega |\nabla v|^{p-2}\nabla v\,.\,\nabla(v-u(t)) \geq \int_\Omega f(t)(v-u(t)), \quad \forall v \in \mathbb{K}_{u(t)}. \tag{25}$$

We consider a given function φ satisfying

$$\varphi \in C^1(\mathbb{R}_0^+),\ 0 \leq \varphi(t) \leq 1,\ \varphi(0)=1,\ \varphi(t) \underset{t\to+\infty}{\longrightarrow} 0,\ \varphi_t(t) \underset{t\to+\infty}{\longrightarrow} 0\ ,$$

and we define, for $0 < \delta < 1$,

$$v^\delta(x,t) = \varphi(t)\,u(x,t) + (1-\delta)\,(1-\varphi(t))\,w^\infty$$

with w^∞ arbitrarily given in $\mathbb{K}_{u^\infty}$.
We have

$$|\nabla v^\delta(t)| \leq \varphi(t)\,F(u(t)) + (1-\delta)\,(1-\varphi(t))\,F(u^\infty)\ ,$$

and, since $F(u(t_n)) \underset{n}{\rightarrow} F(u^\infty)$ uniformly, there exists $n_0 \in \mathbb{N}$ such that

$$\frac{F(u^\infty)}{F(u(t_n))} \leq \frac{1}{1-\delta} \qquad \text{for } n \geq n_0$$

and so,

$$|\nabla v^\delta(t_n)| \le \varphi(t_n)\, F(u(t_n)) + (1-\varphi(t_n))\, F(u(t_n)) = F(u(t_n)) \qquad \text{for } n \ge n_0.$$

Substituting in (25), at time t_n, v by $v^\delta(t_n) \in \mathbb{K}_{u(t_n)}$ we obtain

$$\int_\Omega u_t(t_n)\,(v^\delta(t_n)-u(t_n)) + \int_\Omega |\nabla v^\delta(t_n)|^{p-2}\,\nabla v^\delta(t_n)\,.\,\nabla(v^\delta(t_n)-u(t_n)) \ge \int_\Omega f(t_n)\,(v^\delta(t_n)-u(t_n))$$

and, letting $t_n \to +\infty$, we obtain

$$\int_\Omega |(1-\delta)\,\nabla w^\infty|^{p-2}\,(1-\delta)\,\nabla w^\infty\,.\,\nabla((1-\delta)\,w^\infty - u^\infty) \ge \int_\Omega f^\infty((1-\delta)\,w^\infty - u^\infty)\ ,$$

since $f(t_n) \underset{n}{\rightharpoonup} f^\infty$ in $L^1(\Omega)$-weak and $\int_\Omega |u_t(t_n)| \underset{n}{\to} 0$ by (23) and the assumptions (9) and (10). Letting now $\delta \to 0$, we get

$$\int_\Omega |\nabla w^\infty|^{p-2}\,\nabla w^\infty\,.\,\nabla(w^\infty - u^\infty) \ge \int_\Omega f^\infty(w^\infty - u^\infty)\ ,\quad \forall\, w^\infty \in \mathbb{K}_{u^\infty}$$

and, applying Minty's lemma, we see that u^∞ is a solution to the problem (8), since we have already proved that $u^\infty \in \mathbb{K}_{u^\infty}$. □

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CMAF/Univ. de Lisboa
Av. Prof. Gama Pinto, 2
1649-003 Lisboa, Portugal

Centro de Matemática
University of Minho
Campus de Gualtar
4700-030 Braga, Portugal