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Reinhardt Domains and the Gleason Problem

OSCAR LEMMERS – JAN WIEGERINCK

Abstract. As usual, let $A(\Omega)$ be the uniform algebra consisting of the functions which are holomorphic on Ω, and continuous on $\overline{\Omega}$, and let $H^\infty(\Omega)$ be the set of bounded holomorphic functions on Ω. Throughout this paper Ω will be a bounded Reinhardt domain in $\mathbb{C}^2$ with C^2-boundary.

We show that the maximal ideal (both in $A(\Omega)$ and $H^\infty(\Omega)$), consisting of functions vanishing at $p \in \Omega$, is generated by the functions $(z_1 - p_1)$, $(z_2 - p_2)$, at first for the case that Ω is pseudoconvex, then without this condition.

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1. – Introduction

Let Ω be a bounded domain in $\mathbb{C}^n$. Let $R(\Omega)$ (usually $A(\Omega)$ or $H^\infty(\Omega)$) be a ring of holomorphic functions that contains the polynomials, and let $p = (p_1, \dots, p_n)$ a point of Ω. Recall the Gleason problem, cf. [7]: is the maximal ideal in $R(\Omega)$ consisting of functions vanishing at p, generated by the coordinate functions $(z_1 - p_1), \dots, (z_n - p_n)$?

One says that a domain Ω has the *Gleason R-property* if this is the case for all points $p \in \Omega$. We also say that it has the Gleason-property with respect to $R(\Omega)$.

Gleason mentioned the difficulty of solving this problem even for such a simple domain as the unit ball $B(0,1)$ in $\mathbb{C}^2$, $p = (0,0)$, $R(\Omega) = A(\Omega)$. That case was solved by Leibenzon ([10]), who proved that the Gleason problem can be solved on any convex domain in $\mathbb{C}^n$ having a C^2-boundary. Using different techniques, this result was sharpened by Grangé ([8], for $H^\infty(\Omega)$), and by Backlund and Fällström ([1] and [2], for $H^\infty(\Omega)$ and $A(\Omega)$ respectively), for convex domains in $\mathbb{C}^n$ having only a $C^{1+\epsilon}$-boundary.

Using his theorem on solvability of the $\overline{\partial}$-problem ([13]), Øvrelid proved in [14] that a strictly pseudoconvex domain in $\mathbb{C}^n$ with C^2-boundary has the Gleason A-property. The following results also use this important theorem.

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Weakly pseudoconvex domains Ω in $\mathbb{C}^2$ with C^∞-boundary, having the property that through every point $p \in \Omega$ there is a complex line which intersects $\partial\Omega$ only in strictly pseudoconvex points, have the Gleason A-property, as Beatrous Jr. proved in [5]. Fornæss and Øvrelid proved in [6] that a pseudoconvex domain in $\mathbb{C}^2$ with real analytic boundary has the Gleason A-property. This was extended by Noell ([12]) to pseudoconvex domains in $\mathbb{C}^2$ having a boundary of finite type.

Expanding ideas of [5], Backlund and Fällström proved in [4] that a bounded, pseudoconvex Reinhardt domain in $\mathbb{C}^2$ with C^2-boundary and containing the origin, has the Gleason A-property. For every $p \in \Omega$ they constructed a finite open covering of $\overline{\Omega}$, such that the Gleason-problem can be solved easily on each of its open sets; moreover the pairwise intersections of its open sets intersect the boundary only in strictly pseudoconvex points. Then a global solution is obtained by formulating an additive Cousin-problem and again using Øvrelids theorem. By using similar techniques, we prove that the result of Backlund and Fällström also holds without the assumption that the domain Ω contains the origin. In the second part of this paper we show that the condition that Ω needs to be pseudoconvex can be dropped.

Note that the Gleason problem is not always solvable; in fact, Backlund and Fällström showed ([3]) that there even exists an H^∞-domain of holomorphy on which the problem is not solvable.

MAIN RESULT

Let Ω a bounded Reinhardt domain in $\mathbb{C}^2$ with C^2-boundary. Then Ω has the Gleason-property with respect to both $A(\Omega)$ and $H^\infty(\Omega)$. In other words: given a function f in $A(\Omega)$ that vanishes at $p \in \Omega$, there exist functions f_1, $f_2 \in A(\Omega)$, such that $f = f_1(z_1 - p_1) + f_2(z_2 - p_2)$, and similarly for $H^\infty(\Omega)$.

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2. – Some definitions, notations and lemmas

We denote by $S(\Omega)$ the set of strictly pseudoconvex points in the boundary of Ω. For a function h, let $Z_h := \{z : h(z) = 0\}$ be its zero-set. We denote the boundary of a set D by ∂D, and $Co(D)$ will stand for the convex hull of D.

We recall that a domain in $\mathbb{C}^n$ is Reinhardt if it is invariant under the standard $\mathbb{T}^n$ action on $\mathbb{C}^n$ given by $(\Theta_1,\dots,\Theta_n)\cdot(z_1,\dots,z_n)\mapsto(e^{i\Theta_1}z_1,\dots,e^{i\Theta_n}z_n)$.

The map L: $z=(z_1,\dots,z_n)\mapsto(\log|z_1|,\dots,\log|z_n|)$ sends Ω to its logarithmic image $\omega=L(\Omega)$. The logarithmic image of Z_f is denoted by $L(Z_f)$, and $S(\omega)$ will stand for the C^2 strictly convex points of ω.

We recall some basic facts (cf. [11]) about the relation between Ω and ω: Ω is pseudoconvex $\Leftrightarrow$ ω is convex. If Ω has a C^2-boundary, ω will also have a C^2-boundary. Also note that a point z (having no zero coordinate) in $\partial\Omega$ is strictly pseudoconvex $\Leftrightarrow$ $L(z)$ is a strictly convex point of $\partial\omega$.

LEMMA 1. *Let Ω be a bounded pseudoconvex Reinhardt domain in $\mathbb{C}^2$. Let λ be a smooth $\bar\partial$-closed* (0, 1)*-form on Ω, whose coefficients are bounded on Ω. If $\partial\omega\cap\overline{L(\operatorname{supp}\lambda)}$ consists of a finite number of bounded sets containing only C^2 strictly convex points, then there exists a function u in $C(\overline{\Omega})\cap C^\infty(\Omega)$ such that $\bar\partial u=\lambda$.*

PROOF. First suppose that Ω does not meet the axes $z_1=0$ and $z_2=0$. Then the logarithmic image ω of Ω is bounded. The logarithmic image X of $\overline{\operatorname{supp}\lambda}\cap\partial\Omega$ is a closed subset of the strictly convex points of $\partial\omega$. Hence there exists a bounded strictly convex domain $\tilde\omega\subset\mathbb{R}^2$ such that $\omega\subset\tilde\omega$ and $X\subset\partial\tilde\omega$.

Then $\tilde\Omega:=\{(z_1,z_2)\in\mathbb{C}^2 : L(z)\in\tilde\omega\}$ is a strictly pseudoconvex domain. The form λ can be trivially extended (by defining it to be 0 at $\tilde\Omega\setminus\Omega$) to a C^∞-form $\tilde\lambda$ on $\tilde\Omega$. Since $\tilde\Omega$ is strictly pseudoconvex, there exists a function $\tilde u\in C(\overline{\tilde\Omega})\cap C^\infty(\tilde\Omega)$ such that $\bar\partial\tilde u=\tilde\lambda$ ([13], p. 158-159). The restriction $u=\tilde u|_{\overline{\Omega}}$ has the desired properties.

Next suppose that Ω meets at least one of the axes. Keeping in mind that Ω is C^2 and pseudoconvex, there are two possibilities:

(1) $0\in\Omega$. Then Ω meets each axis in a disk about 0.

(2) $0\notin\Omega$. Then Ω meets only one of the axes, say the z_2-axis, in an annulus.

We will show how to deal with the second case, the first one being completely similar. Observe that

$$\Omega_0=\{(z_1,z_2):\log|z_2|+c|z_1|^2<k,\log|z_2|-c|z_1|^2>-k,|z_1|<\epsilon\},\quad \epsilon,c,k>0$$

is strictly pseudoconvex at the intersection of its boundary with the z_2-axis. Its logarithmic image is

$$\omega_0=\{(x,y):y+ce^{2x}<k,y-ce^{2x}>-k,e^x<\epsilon\}.$$

The logarithmic image ω of Ω is contained in a half-strip: $|y|<N$, $x<N$. Let $Y\subset\partial\omega$ be a (relative) neighborhood of X, contained in the strictly convex boundary points of ω. Let ω' be the intersection of the half-planes that contain ω and are tangent to ω at some point of Y. Then $\omega\subset\omega'$. Now we take $k>N$ and ϵ so small that $\omega_0\subset\omega'$. As in the case where ω is bounded, we can find a strictly convex domain $\tilde\omega$ (with C^2-boundary), the boundary of which contains Y and the part of the boundary of ω_0 where x is sufficiently small. Now $\tilde\Omega:=(\overline{L^{-1}(\tilde\omega)})^\circ$ has C^2-boundary, is strictly pseudoconvex, and we proceed as in the previous case. □

LEMMA 2. *Let Ω be a bounded pseudoconvex Reinhardt domain in $\mathbb{C}^2$ with C^2-boundary. Let $p \in \Omega$. Then there exist analytic polynomials g, h, open sets U_0, U_1, U_2 and a constant $\epsilon > 0$ such that:*

- $Z_g \cap Z_h \cap \overline{\Omega} = \{p\}$,
- *U_0 is strictly pseudoconvex, and $p \in U_0 \subset\subset \Omega$,*
- $|g| > \epsilon$ *on* U_1, $|h| > \epsilon$ *on* U_2,
- $\overline{\Omega} \subset \cup_i U_i$,
- $U_1 \cap U_2 \cap \partial\Omega \subseteq S(\Omega)$.

PROOF. First, we will construct the analytic polynomials g and h, then we construct the open sets U_i. We start with the case that Ω does not contain points with a zero coordinate, using the following elementary fact:
let ω a bounded, convex domain in $\mathbb{R}^2$, having C^2-boundary. Let $q \in \omega$. Then $\partial\omega$ contains 3 strictly convex points, u, v and w, such that q lies in the interior of the triangle uvw. Of course one can choose u, v and w such that the slope of the lines qu and vw are rational numbers.

Given a line l in $\mathbb{R}^2$ passing through $q = L(p_1, p_2)$ with rational slope $\pm\frac{m}{n}$, we construct a polynomial f in $\mathbb{C}^2$ such that $L(Z_f) = l$:
If $\frac{-m}{n} < 0$, m, $n > 0$, we take $f(z) = z_1^m z_2^n - p_1^m p_2^n$.
If $\frac{m}{n} > 0$, m, $n > 0$, we take $f(z) = z_2^n p_1^m - z_1^m p_2^n$.
(Just as in [4].) Choose u, v, $w \in \omega$ as above. Let g be a polynomial on $\mathbb{C}^2$ such that g vanishes at p and the logarithmic image of Z_g is a line in $\mathbb{R}^2$ passing through u. Similarly, let h be a polynomial on $\mathbb{C}^2$ such that h vanishes at p and the logarithmic image of Z_h is a line in $\mathbb{R}^2$ parallel to vw.

Now we are ready to construct the open covering $U_0 \cup U_1 \cup U_2$ of $\overline{\Omega}$.

$\partial\omega$ consists of 3 arcs, namely J_1 (from u to v), J_2 (from v to w), and J_3 (from w to u). Let S_1, S_2, S_3, S_4 be open (in the usual topology on $\partial\omega$) neighborhoods of u, u, v and w respectively, consisting only of strictly convex points, such that $\overline{S_1} \subset S_2$.

It is then possible to choose open sets $V_i \subset \mathbb{R}^2$ as follows: let V_1 such that $d(V_1, L(Z_h)) > \epsilon$, and $V_1 \cap \partial\omega = S_2 \cup (S_3 \cup J_2 \cup S_4)$. V_2 is chosen such that $d(V_2, L(Z_g)) > \epsilon$, and $V_2 \cap \partial\omega = (S_3 \cup J_1 \setminus \overline{S_1}) \cup (S_4 \cup J_3 \setminus \overline{S_1})$.

For sufficiently small ϵ there is a strictly convex set $V_0 \subset\subset \omega$ such that $\overline{\omega} \subset \cup V_i$ and $\partial\omega \subset V_1 \cup V_2$. Then $V_1 \cap V_2 \cap \partial\omega$ contains only strictly convex points. The sets $U_i := (\overline{L^{-1}(V_i)})^\circ$ fulfill the requirements of Lemma 2.

Suppose Ω meets only one of the axes, say $z_1 = 0$ (see [4] for the case $0 \in \Omega$). Let $p = (p_1, p_2) \in \Omega$. If $p_1 = 0$ one defines $g(z) := z_1$, $h(z) := z_2 - p_2$, and the rest is easy. Otherwise, the logarithmic image of $z_2 = p_2$ intersects $\partial\omega$ in only one point, a. Now draw a line through $L(p)$ parallel to the tangent line in a. It intersects $\partial\omega$ at two points, say b and c. Since the boundary of $\partial\omega$ between a and b, a and c are not straight lines, and ω is convex, there must be an extreme point d on the arc ab, and one, e, on the arc ac. These points d and e can be chosen such that they have neighborhoods of strictly convex points in $\partial\omega$, and that the line de has rational slope (since ω is convex with

C^2-boundary). Now we choose $g(z) = z_2 - p_2$, and h a polynomial such that h vanishes at p and the logarithmic image of Z_h is parallel to de. The sets U_i can be constructed as above. □

3. – The Gleason problem for pseudoconvex Reinhardt domains

The following result was obtained by Backlund and Fällström ([4]) under the extra assumption that Ω contains the origin.

THEOREM 3. *Let Ω be a bounded pseudoconvex Reinhardt domain in $\mathbb{C}^2$, having C^2-boundary. Then Ω has the Gleason A-property.*

PROOF. We solve the Gleason-problem locally and patch the solutions together to a global solution using Lemma 1. Let $p \in \Omega$. Choose g, h, U_0, U_1 and U_2 as in Lemma 2. Choose functions $\phi_k \in C_0^\infty(U_k)$, $k = 0, 1, 2$, such that $0 \le \phi_k \le 1$ and $\sum_{k=0}^2 \phi_k \equiv 1$ on $\overline{\Omega}$. Let f be a function in $A(\Omega)$, vanishing at p. Since f is holomorphic on Ω, $U_0 \subset\subset \Omega$, the lemma of Oka-Hefer (cf. [11]) implies that there exist functions f_1^0, f_2^0 in $A(U_0)$ such that

$$f = f_1^0(z_1 - p_1) + f_2^0(z_2 - p_2) \quad \text{on} \quad U_0 .$$

Let $F_1^1 = \frac{f}{g}$, $F_2^1 = 0$, $F_1^2 = 0$, $F_2^2 = \frac{f}{h}$. Then $F_i^k \in A(U_k \cap \Omega)$, and

$$(*) \qquad f = F_1^k g + F_2^k h \quad \text{on} \quad \overline{U_k} \cap \overline{\Omega} .$$

Since g is an analytic polynomial, vanishing at p, there are functions g_1, $g_2 \in H(\mathbb{C}^2)$ such that $g = g_1(z_1 - p_1) + g_2(z_2 - p_2)$ on $\mathbb{C}^2$. A similar formula holds for h. Substituting this in $(*)$, we obtain the existence of functions $f_i^k \in A(U_k \cap \Omega)$, $k = 1, 2$, such that

$$f = f_1^k(z_1 - p_1) + f_2^k(z_2 - p_2) \quad \text{on} \quad \overline{U_k} \cap \overline{\Omega}, \quad k = 0, 1, 2 ,$$

(with $f_i^k = F_1^k g_i + F_2^k h_i$.) So

$$j_1 := \sum \phi_k f_1^k \quad \text{and} \quad j_2 := \sum \phi_k f_2^k$$

give a smooth solution of our problem. We want to find u such that

$$(**) \qquad f_1 = j_1 + u(z_2 - p_2) \quad \text{and} \quad f_2 = j_2 - u(z_1 - p_1)$$

are in $A(\Omega)$. Define a form λ as follows:

$$\lambda := \frac{-\overline{\partial} j_1}{z_2 - p_2} = \frac{\overline{\partial} j_2}{z_1 - p_1} .$$

This form λ is a $\bar{\partial}$-closed (0, 1)-form on Ω, and can be continuously continued to $\overline{\Omega}$. Hence its coefficients are bounded on Ω. The support of λ is contained in $\overline{U_i} \cap \overline{U_j}$, $i \neq j$. Hence we have that $\overline{\text{supp}\lambda} \cap \partial\Omega \subseteq S(\Omega)$. Lemma 1 gives the existence of a function $u \in C(\overline{\Omega}) \cap C^\infty(\Omega)$ such that $\bar{\partial} u = \lambda$. With this u, f_1, f_2 as defined at (**),

$$f = f_1(z_1 - p_1) + f_2(z_2 - p_2) \text{ on } \overline{\Omega},$$

and f_1, f_2 both belong to $A(\Omega)$. This proves that the maximal ideal consisting of functions vanishing at p is generated by $(z_1 - p_1)$ and $(z_2 - p_2)$. □

4. – The Gleason problem for non pseudoconvex Reinhardt domains

To prove that a bounded Reinhardt domain in $\mathbb{C}^2$, with C^2-boundary, has the Gleason-property with respect to $A(\Omega)$ and H^∞, even if it is not pseudoconvex, we need some more machinery. This is developed in the following propositions and corollaries.

DEFINITION. Given a set $V \subseteq \mathbb{R}^n$ and a point $v \in \partial V$, we say v is an extreme point of V if there do not exist r, $s \in \partial V$, distinct from v, and $\lambda \in (0, 1)$ such that $v = \lambda r + (1 - \lambda)s$. In other words: if v is an extreme point of $Co(\omega)$.

Note that V may be strictly convex at a point v without v being an extreme point of V.

LEMMA 4. *Let g be a convex C^1-function such that $g(x) = g(0) + xg'(0) + o(x^2)$ at 0. Then g'' exists at 0 and equals 0.*

PROOF. Without loss of generality, we take $g(0) = g'(0) = 0$. Since g' is increasing, we find for $y > 0$:

$$g(2y) = \int_0^{2y} g'(x)dx \geq \int_y^{2y} g'(x)dx \geq yg'(y),$$

so that $\frac{g'(y)}{y} \leq \frac{g(2y)}{y^2} = o(1)$. Therefore the second right derivative of g at 0 exists and equals 0. Similarly for the second left derivative. □

PROPOSITION 5. *Let ω a domain in $\mathbb{R}^2$, with C^2-boundary. Denote by E the set of extreme points of ω. Then $\overline{E^\circ} = E$.*

PROOF. We endow $\partial Co(\omega)$ with the relative topology. As $E^c \subseteq \partial Co(\omega)$ is clearly open, E is closed in $\partial Co(\omega)$ and $\overline{E^\circ} \subseteq E$. The complement of $\overline{E^\circ}$ in $\partial Co(\omega)$ is a union of disjoint open arcs. We will show that these arcs are in fact straight line segments. Take p in such an arc $U \subseteq \partial Co(\omega)$. If $p \notin E$, then p obviously lies on a straight line segment. So let $p \in E$. Then

$p \in \partial Co(\omega) \cap \partial\omega$. Since ω has C^2-boundary, $Co(\omega)$ has C^1-boundary (in fact it even has $C^{1,1}$-boundary, cf. [9]). After rotating and scaling we can assume that there exists $f \in C^2[-1, 1]$ and $g \in C^1[-1, 1]$ with the following properties:

- $p = (0, f(0)) = (0, g(0))$
- $X = \{(x, f(x)) : x \in [-1, 1]\} \subseteq \partial\omega \cap [-1, 1] \times [\min f, \max f]$
- $Y = \{(x, g(x)) : x \in [-1, 1]\} \subseteq \partial Co(\omega) \cap [-1, 1] \times [\min g, \max g]$
- g is convex
- $f \geq g$ on $[-1, 1]$.

Note that $p \in X \cap Y$ and therefore the tangent to $\partial\omega$ at p equals the tangent to $\partial Co(\omega)$ at p: $g'(0) = f'(0)$. Furthermore, since $p \in E$, $f''(0) \geq 0$. But if $f''(0) > 0$, then $p \in \overline{E^\circ}$. Hence $f''(0) = 0$. It follows that

$$g(0) + xg'(0) \leq g(x) \leq f(x) = f(0) + xf'(0) + o(x^2) = g(0) + xg'(0) + o(x^2).$$

So $g(x) = g(0) + xg'(0) + o(x^2)$. Application of the previous lemma gives that $g''(0) = 0$. Since we can repeat the argument for every point of Y, it follows that g'' is identically 0 on $[-1, 1]$, meaning that Y is a straight line segment. Of course U is a straight line segment too. This yields that U (and any other subset of $\overline{E^\circ}^c$) does not contain extreme points, hence $E \subseteq \overline{E^\circ}$. □

LEMMA 6. *Let ω and E be as above, let $e \in E$. There exists a point $b \in E$ arbitrary close to e such that $\partial\omega$ and $\partial Co(\omega)$ coincide on a neighborhood B of b. Furthermore, this neighborhood can be chosen such that it (as part of $\partial Co(\omega)$) consists only of strictly convex points.*

PROOF. $\overline{E^\circ} = E$, hence one can choose a point $a \in E$ arbitrary close to e, such that there is a neighborhood A of a containing only extreme points of ω. Since the extreme points of ω and $Co(\omega)$ are the same, $\partial\omega$ and $\partial Co(\omega)$ coincide on A. Hence the defining function ρ for $\partial Co(\omega)$ can be chosen such that it is a C^2-function around A. There is a point $b \in A$ for which $\rho''(b) > 0$. Then there is a neighborhood $B \subseteq A$ of b on which ρ'' is strictly positive. □

THEOREM 7. *A bounded Reinhardt domain Ω in $\mathbb{C}^2$, with C^2-boundary, has the Gleason-property with respect to both $A(\Omega)$ and $H^\infty(\Omega)$.*

PROOF. First let $f \in A(\Omega)$, $p \in \Omega$ such that $f(p) = 0$. Note that f extends to $\tilde{\Omega}$, the holomorphic hull of Ω, and that $L(\tilde{\Omega}) = Co(\omega)$.

Suppose ω is bounded. There are extreme points e_1, e_2, $e_3 \in \partial\omega$ with the property $L(p) \in Co(e_1, e_2, e_3)$. According to the previous lemma there exist points a, b, c, arbitrarily close to e_1, e_2, e_3, having neighborhoods A, B, C respectively, containing only strictly convex points such that A, B, $C \subset \partial\omega \cap \partial Co(\omega)$. These a, b and c could have been chosen such that the slopes of the lines ab and $L(p)c$ are rational. Just as in Lemma 2, we construct polynomials g and h that vanish at p, such that $L(Z_g)$ is a line through $L(p)$ and c, and $L(Z_h)$ is a line through $L(p)$ parallel to ab. Then one can construct the appropriate covering of $Co(\omega)$, and simply copy the proof of Theorem 3.

Now suppose ω is not bounded. We only consider the case that Ω contains points of the form $(0, a)$; the other cases can be solved similarly. Applying the ideas of the second part of Lemma 2 (to $Co(\omega)$ instead of to ω) yields the appropriate polynomials g, h and sets U_i. Repeating the proof of Theorem 3 proves the assertion.

Next let $f \in H^\infty(\Omega)$, $p \in \Omega$ such that $f(p) = 0$. Like above, we obtain an open covering $\{U_i\}$ of $\overline{\Omega}$, and matching functions ϕ_i. As in the proof of Theorem 3, we obtain a (0, 1)-form λ:

$$\frac{-\overline{\partial}\left(\sum \phi_k f_1^k\right)}{z_2 - p_2} = \lambda = \frac{\overline{\partial}\left(\sum \phi_k f_2^k\right)}{z_1 - p_1}.$$

The functions f_i^k are bounded and holomorphic. $\phi_k \in C_0^\infty(U_k)$, so $\overline{\partial}\phi_k$ is bounded. The function $\min(|\frac{1}{z_1 - p_1}|, |\frac{1}{z_2 - p_2}|)$ is bounded on suppλ, since $d(p, U_i \cap U_j \cap \Omega) > \delta$. Hence the form λ is bounded on Ω, and we can apply lemma 1 to find a function $u \in C(\overline{\Omega}) \cap C^\infty(\Omega)$ with $\overline{\partial} u = \lambda$. Now copy the proof of Theorem 3. □

Remark. The crux of this approach is to formulate a $\overline{\partial}$-problem ($\overline{\partial} u = \lambda$) on Ω suitable for solving the Gleason problem, in such a way that λ can be extended by 0 to a larger domain $\widehat{\Omega}$ where $\overline{\partial} u = \lambda$ has a good solution. While $\widehat{\Omega}$ is strictly pseudoconvex in our situation, the method will give results in some other cases, e.g. when $\widehat{\Omega}$ is an analytic polyhedron in $\mathbb{C}^2$.

Proposition 8. *Let $\omega \subset \mathbb{R}^2$. If the set of C^2-boundary points of ω contains a dense subset of E, then $\overline{E^\circ} = E$.*

Proof. We endow $\partial Co(\omega)$ and $\partial\omega$ with the relative topology. As $E^c \subseteq \partial Co(\omega)$ is clearly open, E is closed in $\partial Co(\omega)$, thus $\overline{E^\circ} \subseteq E$. To prove the other inclusion, suppose $e \in \overline{E^\circ}^c \cap E$. This point e cannot be an isolated point of E; then it would be in this dense subset of E. But such points have a neighborhood consisting of C^2-boundary points in $\partial\omega$, thus a neighborhood consisting of C^1-boundary points in $\partial Co(\omega)$.

Therefore there would be a sequence $\{e_n\}$ of C^2-boundary points in $\overline{E^\circ}^c \cap E$ that converges to e. However, the proof of proposition 5 shows that such points e_n do not exist. This is a contradiction, hence $\overline{E^\circ}^c \cap E = \emptyset$, and $E \subseteq \overline{E^\circ}$. □

Theorem 9. *Let $\Omega \subset \mathbb{C}^2$ be a bounded Reinhardt domain. Suppose ω is bounded as well, and that the set of C^2-boundary points of ω contains a dense subset of E. Then one can solve the Gleason-problem for both $A(\Omega)$ and $H^\infty(\Omega)$.*

Proof. Using proposition 8 we can repeat the proof of Theorem 7. □

Remark. The only thing that matters is that there are enough strictly pseudoconvex points in the boundary of Ω to make a "good" cover of Ω. This can e.g. be done in the setting of theorem 9 if we merely assume that $0 \notin \overline{\Omega}$ instead of ω being bounded. In that case, given a point $p \in \Omega$, one takes

$g(z) = z_2 - p_2$ (if Ω contains points of the form $(0, a)$) or $g(z) = z_1 - p_1$ (if Ω contains points of the form $(a, 0)$), and proceeds like, e.g., in Lemma 2.

We do not know if the Gleason problem can be solved for a bounded domain $\Omega \subset \mathbb{C}^2$ of the form $|z_1|^2 < |z_2|^3 < 2|z_1|^2$ for $|z_1| \leq 1$, that is rounded off in a strictly pseudoconvex way for larger z_1.

5. – An example

Let Ω be a bounded convex domain in $\mathbb{C}^n$. For every $f \in H^\infty(\Omega)$, vanishing at p, the Leibenzon-divisors ψ_i are defined in the following way:

$$\psi_i(z_1, \ldots, z_n) := \int_0^1 \frac{\partial f}{\partial z_i}(p + t(z - p))dt \quad i = 1, \ldots, n.$$

If in addition Ω has C^2-boundary, then

$$\psi_i \in H^\infty(\Omega), \quad f(z) = \sum_{i=1}^{n}(z_i - p_i)\psi_i(z) \quad \forall z \in \Omega,$$

as Leibenzon proved in [10]. In [8] Grangé was able to show that the functions ψ_i remain in $H^\infty(\Omega)$ if Ω only has $C^{1+\epsilon}$-boundary. There he also gave the following exampl: let $h(x) := \frac{-x}{\log x}$ for $x > 0$, $h(0) := 0$. Let

$$\Omega := \{(z_1, z_2) \in \mathbb{C}^2 : |z_2| < 1, |z_1|^2 + h(|z_2|) - 1 < 0\}.$$

Ω is convex, $\partial\Omega$ is C^1, even C^∞ and strictly pseudoconvex at the points $(z_1, z_2) \in \Omega$, $z_2 \neq 0$. Then a function $\phi \in H^\infty(\Omega)$ was given for which the Leibenzon-divisor $\psi_2 \notin H^\infty(\Omega)$.

However, Ω satisfies the conditions as described in the remark after Theorem 9 and hence there exist functions f_1 and f_2 in $H^\infty(\Omega)$ such that $\phi(z) = z_1 f_1(z) + z_2 f_2(z)$.

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