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Families of Differential Forms on Complex Spaces

VINCENZO ANCONA – BERNARD GAVEAU

Abstract. On every reduced complex space X we construct a family of complexes of soft sheaves Λ_X; each of them is a resolution of the constant sheaf $\mathbb{C}_X$ and induces the ordinary De Rham complex of differential forms on a dense open analytic subset of X. The construction is functorial (in a suitable sense). Moreover each of the above complexes can fully describe the mixed Hodge structure of Deligne on a compact algebraic variety.

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Introduction

By a complex space we mean a reduced, not necessarily irreducible, complex analytic space.

Let X be a complex space. We denote by $\mathbb{C}_X$ the constant sheaf on X. When X is smooth we denote by $\mathcal{E}_X^{\cdot}$ the De Rham complex of differential forms on X.

The complex $D_X^{\cdot}$ of differential forms in sense of Grauert and Grothendieck (see [BH] for precise definitions) shares with the classical De Rham complex $\mathcal{E}_X^{\cdot}$ on a manifold many important properties, namely:
– $D_X^{\cdot}$ is a complex of fine sheaves, provided with an augmentation $\mathbb{C}_X \to D_X^{\cdot}$;
– for $p > 2\dim X$, $D_X^p = 0$;
– the restriction $D_X^{\cdot} \mid U$ to the open subset U of smooth points of X is the ordinary De Rham complex $\mathcal{E}_U^{\cdot}$.
– if $f : X \to Y$ is a morphism of complex spaces, the pullback $f^* : D_Y^{\cdot} \to D_X^{\cdot}$ is defined; it commutes to differentials, and is functorial.

On the other hand, the complex $D_X^{\cdot}$ is not, in general, a resolution of $\mathbb{C}_X$, so it cannot play in the singular case the same role as $\mathcal{E}_X^{\cdot}$ in the most important applications: De Rham theory and Hodge theory.

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In a series of papers ([AG1]-[AG6]) we proposed a different definition of differential forms on a complex space, in order to obtain a resolution of $\mathbb{C}_X$ by means of forms, and De Rham and Hodge type results. The definition was based on a construction called "tower of desingularizations". This construction has essentially one disadvantage: it is not functorial; in particular the pullback of forms cannot be defined. We came to the conclusion that in order to obtain all the properties of the classical forms, including the existence of the pullback and the functoriality, it is necessary to define for any space X a family of complexes, instead of a single one. This is the subject of the present paper.

The forms we introduce here can fully describe also the mixed Hodge structure of Deligne [D] on a compact algebraic variety X. In order to deal with the Hodge-Deligne theory we use our notion of hypercovering (X_l) of a complex space X, instead of the simplicial resolutions of Deligne; our definition has the property that the dimensions of the manifolds X_l are less or equal to $\dim X$ (which is not the case for simplicial resolutions); moreover our approach avoids the use of the cohomological descent theory. Note also that the family of hypercoverings of X is filtered, i.e. two of them are dominated by a third one (this is not true for the cubic hyperresolutions of [GNPP1], [GNPP2]).

As the referee has pointed out, in [C] already appears the idea of iteratively solving singularities using a cone construction to produce an object which computes the cohomology.

To be more precise, for every X we define a family of complexes $\mathcal{R}(X) = \{\Lambda^{\cdot}_X\}$ and for every morphism $f : X \to Y$ a family $\mathcal{R}(\mathrm{Y}, \mathrm{X})$ of morphisms of complexes between the $\Lambda^{\cdot}_Y \in \mathcal{R}(Y)$ and some of the $\Lambda^{\cdot}_X \in \mathcal{R}(X)$, more precisely morphisms $\Lambda^{\cdot}_Y \to f_*\Lambda^{\cdot}_X$ which we simply denote $\Lambda^{\cdot}_Y \to \Lambda^{\cdot}_X$ and call (*admissible*) *pullback* with the following properties.

(I) $\Lambda^{\cdot}_X$ is a fine resolution of $\mathbb{C}_X$.

(II) For $p > 2\dim X$, $\Lambda^p_X = 0$.

(III) If X is smooth, the ordinary De Rham complex $\mathcal{E}^{\cdot}_X$ belongs to $\mathcal{R}(X)$, and for every morphism $f : X \to Y$ between smooth complex manifolds the ordinary De Rham pullback $f^* : \mathcal{E}^{\cdot}_Y \to f_*\mathcal{E}^{\cdot}_X$ is an admissible pullback.

(IV) There exists a smooth, open, dense analytic subset $U \subset X$ such that the restriction $\Lambda^{\cdot}_X \mid U$ is the ordinary De Rham complex $\mathcal{E}^{\cdot}_U$. Here analytic means that the complement of U in X is an analytic subspace of X.

The family of pullback will satisfy the following properties.

(C) (Composition). Let $g : Z \to X$, $f : X \to Y$ be two morphisms, $\alpha : \Lambda^{\cdot}_Y \to \Lambda^{\cdot}_X$, $\beta : \Lambda^{\cdot}_X \to \Lambda^{\cdot}_Z$ two pullback; then the composition $\beta \circ \alpha : \Lambda^{\cdot}_Y \to \Lambda^{\cdot}_Z$ is again a pullback.
For future induction procedure we denote by $(\mathrm{C})_{k,m,n}$ the property (C) when $\dim Z \leq k$, $\dim X \leq m$, $\dim Y \leq n$.

(EP) (Existence of pullback). Let $f : X \to Y$ be a morphism, and fix $\Lambda^{\cdot}_Y \in \mathcal{R}(Y)$; then there exists a $\Lambda^{\cdot}_X \in \mathcal{R}(X)$ and a pullback $\Lambda^{\cdot}_Y \to \Lambda^{\cdot}_X$.

As above we denote by $(\mathrm{EP})_{m,n}$ the property (EP) when $\dim X \le m$, $\dim Y \le n$.

(U) (Uniqueness of pullback). Let $f : X \to Y$ be a morphism, and $\alpha : \Lambda^{\cdot}_Y \to \Lambda^{\cdot}_X$, $\beta : \Lambda^{\cdot}_Y \to \Lambda^{\cdot}_X$ two pullback corresponding to f; then $\alpha = \beta$.
We denote by $(\mathrm{U})_{m,n}$ the property (U) when $\dim X \le m$, $\dim Y \le n$.

(F) (Filtering). If $\Lambda^{\cdot,1}_X$, $\Lambda^{\cdot,2}_X \in \mathcal{R}(X)$, there exists a third $\Lambda^{\cdot}_X \in \mathcal{R}(X)$ and two pullback $\Lambda^{\cdot,1}_X \to \Lambda^{\cdot}_X$, $\Lambda^{\cdot,2}_X \to \Lambda^{\cdot}_X$ corresponding to the identity.
We denote by $(\mathrm{F})_m$ the property (F) when $\dim X \le m$.

In this article we construct the complexes $\Lambda^{\cdot}_X$ together with the pullback. Because the definition of the complexes $\Lambda^{\cdot}_X$ uses the definition and the existence of the pullback, we shall be forced to construct both at the same time. This will be done by a recursion on the dimensions of the complex spaces involved.

In order to explain the definition of $\Lambda^{\cdot}_X$, we consider a resolution of singularities of X, i.e. a commutative diagram:

$$\begin{array}{ccc} \widetilde{E} & \xrightarrow{i} & \widetilde{X} \\ q \downarrow & & \pi \downarrow \\ E & \xrightarrow{j} & X \end{array} \tag{1}$$

where $E \subset X$ is a nowhere dense closed subspace, containing the singularities of X, $j : E \to X$ is the natural inclusion, $\widetilde{X}$ is a smooth manifold and π is a proper modification inducing an isomorphism $\widetilde{X} \setminus \widetilde{E} \simeq X \setminus E$. Let us consider the particular case where E and $\widetilde{E}$ are smooth. The above diagram is formally like the Mayer-Vietoris diagram of a space $X = U_1 \cup U_2$, where $E = U_1$, $\widetilde{X} = U_2$, and $\widetilde{E} = U_1 \cap U_2$. Indeed, consider the mapping cone C obtained by adjoining $\widetilde{E} \times [0, 1]$ to E and X using q and i respectively. Then C is homotopy equivalent to X. Given a diagram of spaces of this kind one can construct a corresponding mapping cone of complexes

$$\Lambda^{\cdot}_X = \pi_* \mathcal{E}^{\cdot}_{\widetilde{X}} \oplus j_* \Lambda^{\cdot}_E \oplus (j \circ q)_* \Lambda^{\cdot}_{\widetilde{E}}(-1)$$

as we do in Definition 2. In this simple example, the spaces in question are smooth, and so we may take the complexes $\Lambda^{\cdot}$ to be the usual De Rham complexes. This complex lives on X and computes the cohomology of X. In the general case an iterative construction is required to define $\Lambda^{\cdot}_X$. Since there are many ways of building up mapping cones of the required kinds using resolutions of singularities, there is no unique complex $\Lambda^{\cdot}_X$, though the complex restricted to an open dense subset of the smooth locus agrees with the usual De Rham complex there.

In Section 1 we define the family $\mathcal{R}(X)$ and the family $\mathcal{R}(\mathrm{Y}, \mathrm{X})$. One basic notion is the hypercovering of a complex space. We also give the main statements of the paper.

In Section 2 we explain the logic of the proof, in particular the recursion scheme.

In Section 3 we give the proofs.

In Section 4 we show how our complexes of forms can describe the mixed Hodge structure of Deligne [D] in the case of a compact algebraic variety X (see [AG7] for detailed proofs).

The Section 5 is devoted to the extension of the previous results to the case of logarithmic complexes in the sense of Griffiths and Schmid [GS].

In the Section 6 we see that the complex $D^{\cdot}_X$ of differential forms of Grauert and Grothendieck is in a natural way a subcomplex of any $\Lambda^{\cdot}_X$, so that in some sense $\Lambda^{\cdot}_X$ appears as the smallest complex of forms containing $D^{\cdot}_X$ and satisfying functorially both De Rham and Hodge-Deligne theory. Further applications are given (details will appear elsewhere).

1. – Definitions and statements

Let X be a complex space, $E \subset X$ a nowhere dense closed subspace. Let us consider a diagram (1) where $j : E \to X$ is the natural inclusion, $\widetilde{X}$ is a smooth manifold and π is a proper modification inducing an isomorphism $\widetilde{X} \setminus \widetilde{E} \simeq X \setminus E$. By a theorem of Łojasiewicz, there exist a fundamental system of neighborhoods W of $\widetilde{E}$ in $\widetilde{X}$ and retractions $r : W \to \widetilde{E}$; it follows that the restriction morphisms $H^k(W, \mathbb{C}) \to H^k(\widetilde{E}, \mathbb{C})$ are isomorphisms. As a consequence we obtain:

PROPOSITION 1. *Let $U \subset X$ be an open neighborhood of a point $x \in E$.*

i) *Let $\eta_1, \eta_2 \in H^k(\pi^{-1}(U), \mathbb{C})$ two cohomology classes whose restrictions to $H^k(\pi^{-1}(U) \cap \widetilde{E}, \mathbb{C})$ coincide. There exists an open neighborhood $V \subset U$ of x such that the restrictions coincide in $H^k(\pi^{-1}(V), \mathbb{C})$.*

ii) *Let $\theta \in H^k(\pi^{-1}(U) \cap \widetilde{E}, \mathbb{C})$. There exists an open neighborhood $V \subset U$ of x and $\eta \in H^k(\pi^{-1}(V), \mathbb{C})$ inducing θ.*

REMARK. We could use the Mayer-Vietoris sequence instead of the theorem of Łojasiewicz.

DEFINITION OF THE FAMILY $\mathcal{R}(X)$. We denote by $\Lambda^{\cdot}_X(-1)$ the complex obtained by shifting the degree in $\Lambda^{\cdot}_X$: more precisely $\Lambda^{p}_X(-1) = \Lambda^{p-1}_X$.

We define the family $\mathcal{R}(X)$ by induction on $n = \dim(X)$. If $\dim(X) = 0$, $\mathcal{R}(X)$ contains only the complex $\mathbb{C}^{\cdot}_X$ with $\mathbb{C}^0_X = \mathbb{C}_X$ and $\mathbb{C}^p_X = 0$ for $p > 0$. We suppose $\mathcal{R}(Y)$ to be known for complex spaces Y of dimension $< n$; then:

DEFINITION 2 $(\mathrm{D})_n$. Let X be a complex space of dimension n. An element $\Lambda^{\cdot}_X \in \mathcal{R}(X)$ is the assignement of the following data:

i) a nowhere dense closed subspace $E \subset X$, $\mathrm{Sing}(X) \subset E$ and a proper modification

$$\begin{array}{ccc} \widetilde{E} & \stackrel{i}{\to} & \widetilde{X} \\ q\downarrow & & \pi\downarrow \\ E & \stackrel{j}{\to} & X \end{array}$$

where $j : E \to X$ is the natural inclusion, $\widetilde{X}$ is a smooth manifold, $\widetilde{E} = \pi^{-1}(E)$ and π induces an isomorphism $\widetilde{X} \setminus \widetilde{E} \simeq X \setminus E$;

ii) there exist $\Lambda^{\cdot}_E \in \mathcal{R}(E)$, $\Lambda^{\cdot}_{\widetilde{E}} \in \mathcal{R}(\widetilde{E})$, and two pullback

$$\phi : \Lambda^{\cdot}_E \to \Lambda^{\cdot}_{\widetilde{E}}$$
$$\psi : \mathcal{E}^{\cdot}_{\widetilde{X}} \to \Lambda^{\cdot}_{\widetilde{E}}$$

(corresponding respectively to q and i);

iii) the complex $\Lambda^{\cdot}_X$ is defined by

$$\Lambda^{\cdot}_X = \pi_*\mathcal{E}^{\cdot}_{\widetilde{X}} \oplus j_*\Lambda^{\cdot}_E \oplus (j \circ q)_*\Lambda^{\cdot}_{\widetilde{E}}(-1)$$

with differential given by

$$\begin{aligned} d : \Lambda^p_X = \pi_*\mathcal{E}^p_{\widetilde{X}} \oplus j_*\Lambda^p_E \oplus (j \circ q)_*\Lambda^{p-1}_{\widetilde{E}} &\to \Lambda^{p+1}_X \\ = \pi_*\mathcal{E}^{p+1}_{\widetilde{X}} \oplus j_*\Lambda^{p+1}_E \oplus (j \circ q)_*\Lambda^p_{\widetilde{E}} & \\ d(\omega, \sigma, \theta) = (d\omega, d\sigma, d\theta + (-1)^p(\psi(\omega) - \phi(\sigma)) & \end{aligned}$$

iv) the augmentation

$$\begin{aligned} \mathbb{C}_X &\to \Lambda^0_X \\ c &\to (c, c, 0) \end{aligned}$$

makes $\Lambda^{\cdot}_X$ a resolution of $\mathbb{C}_X$;

v) there is a uniquely determined family $(X_l, h_l)_{l\in L}$ of smooth manifolds X_l and proper maps $h_l : X_l \to X$ such that

$$\Lambda^p_X = \oplus_l h_{l*}\mathcal{E}^{p-q(l)}_{X_l}$$

where $q(l)$ is a nonnegative integer; moreover, there exist mappings $h_{lm} : X_l \to X_m$, commuting with h_l and h_m, such that the differential $\Lambda^p_X \to \Lambda^{p+1}_X$ is given by

$$d(\omega_l) = \left(d\omega_l + \sum_m \epsilon^{(p)}_{lm} h^*_{lm}\omega_m\right)$$

where $\epsilon^{(p)}_{lm}$ can take the values $0, \pm 1$.

The pullback $\phi : \Lambda^{\cdot}_E \to \Lambda^{\cdot}_{\widetilde{E}}$ and $\psi : \mathcal{E}^{\cdot}_{\widetilde{X}} \to \Lambda^{\cdot}_{\widetilde{E}}$ in (ii) are called **inner pullback** of the complex $\Lambda^{\cdot}_X$.
The family $(X_l, h_l)_{l\in L}$ will be called the **hypercovering** of X associated to $\Lambda^{\cdot}_X$, and $q(l)$ will be the **rank** of X_l.

To simplify the notations, in the sequel we will write $\Lambda^p_X = \oplus_l \mathcal{E}^{p-q(l)}_{X_l}$ instead of $\oplus_l h_{l*}\mathcal{E}^{p-q(l)}_{X_l}$.

1.1. – Construction-existence theorem

THEOREM 3 $(\mathrm{E})_n$. *Let X be a complex space of dimension $\leq n$. Let $E \subset X$ be a nowhere dense closed subspace with* $\mathrm{Sing}(X) \subset E$*, $j : E \to X$ the natural inclusion, and*

$$\begin{array}{ccc} \widetilde{E} & \xrightarrow{i} & \widetilde{X} \\ q\downarrow & & \pi\downarrow \\ E & \xrightarrow{j} & X \end{array}$$

be a proper modification. Let $\Lambda^{\cdot}_E \in \mathcal{R}(E)$. There exists $\Lambda^{\cdot}_{\widetilde{E}} \in \mathcal{R}(\widetilde{E})$, a pullback $\phi : \Lambda^{\cdot}_E \to \Lambda^{\cdot}_{\widetilde{E}}$ (corresponding to q), a pullback $\psi : \mathcal{E}^{\cdot}_{\widetilde{X}} \to \Lambda^{\cdot}_{\widetilde{E}}$ (corresponding to i) *with the following property: the complex*

$$\Lambda^{\cdot}_X = \pi_*\mathcal{E}^{\cdot}_{\widetilde{X}} \oplus j_*\Lambda^{\cdot}_E \oplus (j\circ q)_*\Lambda^{\cdot}_{\widetilde{E}}(-1)$$

whose differential is by definition

$$d : \Lambda^p_X = \pi_*\mathcal{E}^p_{\widetilde{X}} \oplus j_*\Lambda^p_E \oplus (j\circ q)_*\Lambda^{p-1}_{\widetilde{E}} \to \Lambda^{p+1}_X = \pi_*\mathcal{E}^{p+1}_{\widetilde{X}} \oplus j_*\Lambda^{p+1}_E \oplus (j\circ q)_*\Lambda^p_{\widetilde{E}}$$
$$d(\omega,\sigma,\theta) = (d\omega, d\sigma, d\theta + (-1)^p(\psi(\omega) - \phi(\sigma))$$

is a fine resolution of $\mathbb{C}_X$.

NOTATION. Throughout all the paper we will write for simplicity

$$\mathcal{E}^{\cdot}_{\widetilde{X}} \oplus \Lambda^{\cdot}_E \oplus \Lambda^{\cdot}_{\widetilde{E}}(-1)$$

instead of $\pi_*\mathcal{E}^{\cdot}_{\widetilde{X}} \oplus j_*\Lambda^{\cdot}_E \oplus (j\circ q)_*\Lambda^{\cdot}_{\widetilde{E}}(-1)$.

1.2. – Definition of a primary pullback for irreducible spaces

Let $f : X \to Y$ be a morphism of irreducible complex spaces, $\dim X \leq m$, $\dim Y \leq n$, $\Lambda^p_X = \mathcal{E}^p_{\widetilde{X}} \oplus \Lambda^p_E \oplus \Lambda^{p-1}_{\widetilde{E}}$, $\Lambda^p_Y = \mathcal{E}^p_{\widetilde{Y}} \oplus \Lambda^p_F \oplus \Lambda^{p-1}_{\widetilde{F}}$, with $\Lambda^{\cdot}_X \in \mathcal{R}(X)$ and $\Lambda^{\cdot}_Y \in \mathcal{R}(Y)$. Let us consider the corresponding diagrams

$$\begin{array}{ccc} \widetilde{E} & \to & \widetilde{X} \\ \downarrow & & \downarrow \\ E & \xrightarrow{j} & X \end{array} \qquad \begin{array}{ccc} \widetilde{F} & \to & \widetilde{Y} \\ \downarrow & & \downarrow \\ F & \xrightarrow{k} & Y \end{array}$$

In order to define a primary pullback $\phi : \Lambda^{\cdot}_Y \to \Lambda^{\cdot}_X$ we proceed by double induction on (m,n), i.e. $(\mathrm{DP})_{m,n-1}$ and $(\mathrm{DP})_{m-1,n} \Longrightarrow (\mathrm{DPP})_{m,n}$ (for $(\mathrm{DP})_{m,n}$ see the Definition 6 below).

DEFINITION 4 $(\mathrm{DPP})_{m,n}$. We say that ϕ is a primary pullback (corresponding to f) if it satisfies the following properties:

P_0) ϕ is a morphism of complexes, i.e. it commutes with differentials.

P_1) Let $(X_l, h_l)_{l\in L}$, $(Y_s, g_s)_{s\in S}$ the hypercoverings associated to $\Lambda^{\cdot}_X, \Lambda^{\cdot}_Y$, i.e.

$$\Lambda^p_X = \oplus_l \mathcal{E}^{p-q(l)}_{X_l}, \Lambda^p_Y = \oplus_s \mathcal{E}^{p-q(s)}_{Y_s}$$

For every X_l there exist at most one Y_s, having the same rank q as X_l, and a commutative diagram

$$\begin{array}{ccc} X_l & \overset{f_{ls}}{\rightarrow} & Y_s \\ h_l\downarrow & & g_s\downarrow \\ X & \overset{f}{\rightarrow} & Y \end{array}$$

such that the composition

$$\mathcal{E}^{p-q}_{Y_s} \rightarrow \Lambda^p_Y \overset{\phi}{\rightarrow} \Lambda^p_X \rightarrow \mathcal{E}^{p-q}_{X_l}$$

is either identically zero for every p, or coincides with the De Rham pullback f^*_{ls}

P_2) Let $\alpha : \mathcal{E}^{\cdot}_{\widetilde{Y}} \rightarrow \mathcal{E}^{\cdot}_{\widetilde{X}}$ be induced by ϕ. Then $\alpha \equiv 0$ if and only if $f(X) \subset F$; moreover in this case ϕ is the composition $\Lambda^{\cdot}_Y \rightarrow \Lambda^{\cdot}_F \rightarrow \Lambda^{\cdot}_X$ where $\Lambda^{\cdot}_Y \rightarrow \Lambda^{\cdot}_F$ is the projection onto the summand $\Lambda^{\cdot}_F$ and $\Lambda^{\cdot}_F \rightarrow \Lambda^{\cdot}_X$ is a pullback (inductively defined) corresponding to the induced morphism $X \rightarrow F$.

P_3) If $\alpha : \mathcal{E}^{\cdot}_{\widetilde{Y}} \rightarrow \mathcal{E}^{\cdot}_{\widetilde{X}}$ is not identically zero, then, according to P_2, $f(X) \not\subset F$; in that case we assume the following properties:

i) $f^{-1}(F) \subset E$;

ii) the morphism f extends to a morphism $\tilde{f} : \widetilde{X} \rightarrow \widetilde{Y}$ and $\alpha = \tilde{f}^*$ is the ordinary De Rham pullback;

iii) the morphism ϕ is given by

$$\begin{array}{ccccc} \mathcal{E}^{\cdot}_{\widetilde{Y}} & \oplus & \Lambda^{\cdot}_F & \oplus & \Lambda^{\cdot}_{\widetilde{F}}(-1) \\ \alpha\downarrow & \searrow\beta & \gamma\downarrow & \delta\swarrow & \epsilon\downarrow \\ \mathcal{E}^{\cdot}_{\widetilde{X}} & \oplus & \Lambda^{\cdot}_E & \oplus & \Lambda^{\cdot}_{\widetilde{E}}(-1) \end{array}$$

where $(\beta, \gamma, \delta) : \Lambda^{\cdot}_Y \rightarrow \Lambda^{\cdot}_E$ is a pullback corresponding to the composition $f \circ j : E \rightarrow Y$ (inductively defined).

1.3. – Definition of a pullback morphism: the general case

Natural pullback.

Let X be a complex space of dimension m.

Let $X = X_1 \cup \ldots \cup X_r$ be the decomposition of X into its irreducible components. Let $E \subset X$ be a closed subspace such that $X\backslash E$ is smooth and dense in X, and $f : (\widetilde{X}, \widetilde{E}) \rightarrow (X, E)$ be a proper desingularization.

Then: $\widetilde{X} = \widetilde{X}_1 \sqcup \ldots \sqcup \widetilde{X}_r$, $\widetilde{E} = \widetilde{E}_1 \sqcup \ldots \sqcup \widetilde{E}_r$, $\widetilde{E}_i = \widetilde{E} \cap \widetilde{X}$ (here $\sqcup$ denotes disjoint union); moreover, $f_{|\widetilde{X}_i} : (\widetilde{X}_i, \widetilde{E}_i) \to (X_i, E_i)$ is a desingularization. Let $\Lambda^\cdot_X = \mathcal{E}^\cdot_{\widetilde{X}} \oplus \Lambda^\cdot_E \oplus \Lambda^\cdot_{\widetilde{E}}(-1) \in \mathcal{R}(X)$; let us denote by $\phi : \Lambda^\cdot_E \to \Lambda^\cdot_{\widetilde{E}}$, $\psi : \mathcal{E}^\cdot_{\widetilde{X}} \to \Lambda^\cdot_{\widetilde{E}}$ the inner pullback of $\Lambda^\cdot_X$ (see definition (D)$_m$).

Then $\mathcal{E}^\cdot_{\widetilde{X}} = \oplus_i \mathcal{E}^\cdot_{\widetilde{X}_i}$, $\Lambda^\cdot_{\widetilde{E}} = \oplus_i \Lambda^\cdot_{\widetilde{E}_i}$.

Let us denote by $p_i : \mathcal{E}^\cdot_{\widetilde{X}} \to \mathcal{E}^\cdot_{\widetilde{X}_i}$, $q_i : \Lambda^\cdot_{\widetilde{E}} \to \Lambda^\cdot_{\widetilde{E}_i}$ the projections.

Definition 5. A morphism of complexes $\zeta : \Lambda^\cdot_X = \mathcal{E}^\cdot_{\widetilde{X}} \oplus \Lambda^\cdot_E \oplus \Lambda^\cdot_{\widetilde{E}}(-1) \to \Lambda^\cdot_{X_i}$ corresponding to the inclusion $X_i \to X$ is called a **natural pullback** if $\Lambda^\cdot_{X_i} = \mathcal{E}^\cdot_{\widetilde{X}_i} \oplus \Lambda^\cdot_{E_i} \oplus \Lambda^{\cdot,1}_{\widetilde{E}_i}$ with $\Lambda^\cdot_{E_i} \in \mathcal{R}(E_i)$, $\Lambda^{\cdot,1}_{\widetilde{E}_i} \in \mathcal{R}(\widetilde{E}_i)$ and there exists a commutative diagram of pullback

$$\begin{array}{ccccc} & & \Lambda^\cdot_E & \xrightarrow{\eta_i} & \Lambda^\cdot_{E_i} \\ & & \phi \downarrow & & \\ \mathcal{E}^\cdot_{\widetilde{X}} & \xrightarrow{\psi} & \Lambda^\cdot_{\widetilde{E}} & & \phi_i \downarrow \\ p_i \downarrow & & q_i \downarrow & & \\ \mathcal{E}^\cdot_{\widetilde{X}_i} & \xrightarrow{\psi_i} & \Lambda^\cdot_{\widetilde{E}_i} & \xrightarrow{\mu_i} & \Lambda^{\cdot,1}_{\widetilde{E}_i} \end{array} \tag{2}$$

such that

$$\zeta(\omega, \sigma, \theta) = (p_i(\omega), \eta_i(\sigma), \mu_i(q_i(\theta))) \tag{3}$$

Let $f : X \to Y$ be a morphism between (reducible) complex spaces, $X = \bigcup_j X_j$, $Y = \bigcup_k Y_k$ be the respective decompositions into irreducible components. Let $\Lambda^\cdot_X = \mathcal{E}^\cdot_{\widetilde{X}} \oplus \Lambda^\cdot_E \oplus \Lambda^\cdot_{\widetilde{E}}(-1)$, $\Lambda^\cdot_Y = \mathcal{E}^\cdot_{\widetilde{Y}} \oplus \Lambda^\cdot_F \oplus \Lambda^\cdot_{\widetilde{F}}(-1)$.

For a given j two cases can occur:

(a) $f(X_j) \subset F$

(b) $f(X_j) \not\subset F$, and there exists a unique Y_k with $f(X_j) \subset Y_k$ (because F contains, by definition, the singularities of Y).

Definition 6 (DP)$_{m,n}$. A morphism of complexes $\phi : \Lambda^\cdot_Y \to \Lambda^\cdot_X$ is called a pullback corresponding to f if

- it satisfies P_0 and P_1 in definition (DPP)$_{m,n}$
- (P_4) the composition $\Lambda^\cdot_Y \to \Lambda^\cdot_X \to \Lambda^\cdot_E$ (where the second morphism is the projection onto the summand) is a pullback corresponding to the composition $f \circ j : E \to Y$ (inductively defined).
- for every component X_j of X there exists:
  - in case (a) a commutative diagram

$$\begin{array}{ccc} \Lambda^\cdot_Y & \to & \Lambda^\cdot_X \\ \downarrow & & \downarrow \\ \Lambda^\cdot_F & & \Lambda^\cdot_{X_j} \\ & \searrow & \downarrow \\ & & \Lambda^{\cdot,0}_{X_j} \end{array} \tag{4}$$

where $\Lambda^{\cdot}_{F} \to \Lambda^{\cdot,0}_{X_j}$ is a pullback corresponding to $f|_{X_j} : X_j \to F$ (inductively defined), $\Lambda^{\cdot}_{X} \to \Lambda^{\cdot}_{X_j}$ and $\Lambda^{\cdot}_{X_j} \to \Lambda^{\cdot,0}_{X_j}$ are natural pullback,

– in case (b) a commutative diagram

$$
\begin{array}{ccc}
\Lambda^{\cdot}_{Y} & \to & \Lambda^{\cdot}_{X} \\
\downarrow & & \downarrow \\
\Lambda^{\cdot}_{Y_k} & & \Lambda^{\cdot}_{X_j} \\
& \searrow & \downarrow \\
& & \Lambda^{\cdot,0}_{X_j}
\end{array}
\tag{5}
$$

where $\Lambda^{\cdot}_{Y} \to \Lambda^{\cdot}_{Y_k}$, $\Lambda^{\cdot}_{X} \to \Lambda^{\cdot}_{X_j}$ and $\Lambda^{\cdot}_{X_j} \to \Lambda^{\cdot,0}_{X_j}$ are natural pullback, and $\Lambda^{\cdot}_{Y_k} \to \Lambda^{\cdot,0}_{X_j}$ is a primary pullback corresponding to $f|_{X_j} : X_j \to Y_k$.

REMARK 7.

(i) In the above definition we can take $\Lambda^{\cdot}_{X_j} = \Lambda^{\cdot,0}_{X_j}$, because by the Lemma 21 below the composition $\Lambda^{\cdot}_{X} \to \Lambda^{\cdot}_{X_j} \to \Lambda^{\cdot,0}_{X_j}$ is a natural pullback;

(ii) when X and Y are irreducible, we obtain a definition of pullback which is more general that the one in definition $(\text{DPP})_{m,n}$. Although we conjecture that every pullback between irreducible spaces is primary, the reader should keep in mind that a-priori there are pullback between irreducible spaces which are not primary.

1.4. – Existence of primary pullback (the irreducible case)

THEOREM 8. $(\text{EPP})_{m,n}$. *Let $f : X \to Y$ be a morphism between irreducible complex spaces,* $\dim X = m$, $\dim Y = n$ *and fix $\Lambda^{\cdot}_{Y} \in \mathcal{R}(Y)$; there exists a $\Lambda^{\cdot}_{X} \in \mathcal{R}(X)$ and a primary pullback $\Lambda^{\cdot}_{Y} \to \Lambda^{\cdot}_{X}$.*

REMARK 9. In the particular case $X = Y$, $f = id$, $m = n$ it will follow from the proof that in order to obtain $(\text{EPP})_{m,n}$ (i.e. $(\text{EPP})_{m,m}$) we do not need the assumptions $(\text{F})_{m}$, $(\text{EP})_{m,n-1}$, $(\text{U})_{m,n-1}$ (i.e. $(\text{EP})_{m,m-1}$, $(\text{U})_{m,m-1}$).

As a consequence of the proof of the Theorem we will obtain the following more precise statement

THEOREM 10. *Let $f : X \to Y$ be a morphism between irreducible complex spaces,* $\dim X = m$, $\dim Y = n$ *and fix $\Lambda^{\cdot}_{Y} = \mathcal{E}^{\cdot}_{\widetilde{Y}} \oplus \Lambda^{\cdot}_{F} \oplus \Lambda^{\cdot}_{\widetilde{F}}(-1) \in \mathcal{R}(Y)$; let E be a nowhere dense subspace of X, such that*

(i) $f^{-1}(F) \subset E$

(ii) *there are two commutative diagrams*

$$
\begin{array}{ccc}
\widetilde{F} & \to & \widetilde{Y} \\
\downarrow & & p\downarrow \\
F & \to & Y
\end{array}
\qquad
\begin{array}{ccccc}
\widetilde{E} & \to & \widetilde{X} & \xrightarrow{\tilde{f}} & \widetilde{Y} \\
\downarrow & & h\downarrow & & p\downarrow \\
E & \to & X & \xrightarrow{f} & Y
\end{array}
$$

Then there exists a $\Lambda^{\cdot}_X = \mathcal{E}^{\cdot}_{\widetilde{X}} \oplus \Lambda^{\cdot}_E \oplus \Lambda^{\cdot}_{\widetilde{E}}(-1) \in \mathcal{R}(X)$ *and a primary pullback* $\Lambda^{\cdot}_Y \to \Lambda^{\cdot}_X$.

REMARK 11. In particular, if we already know that there is a pullback

$$\Lambda^{\cdot}_Y = \mathcal{E}^{\cdot}_{\widetilde{Y}} \oplus \Lambda^{\cdot}_F \oplus \Lambda^{\cdot}_{\widetilde{F}}(-1) \to \Lambda^{\cdot}_X = \mathcal{E}^{\cdot}_{\widetilde{X}} \oplus \Lambda^{\cdot}_E \oplus \Lambda^{\cdot}_{\widetilde{E}}(-1)$$

then for any $\Lambda^{\cdot,1}_Y$ of the form $\mathcal{E}^{\cdot}_{\widetilde{Y}} \oplus \Lambda^{\cdot,1}_F \oplus \Lambda^{\cdot,1}_{\widetilde{F}}(-1)$ there exist a $\Lambda^{\cdot,1}_X = \mathcal{E}^{\cdot}_{\widetilde{X}} \oplus \Lambda^{\cdot,1}_E \oplus \Lambda^{\cdot,1}_{\widetilde{E}}(-1)$ and a pullback $\Lambda^{\cdot,1}_Y \to \Lambda^{\cdot,1}_X$

1.5. – Uniqueness of the primary pullback (the irreducible case)

THEOREM 12. $(\mathrm{UP})_{m,n}$. *Let* $f : X \to Y$ *be a morphism between irreducible complex spaces,* $\dim X = m$, $\dim Y = n$ *and let* $\phi_j : \Lambda^{\cdot}_Y \to \Lambda^{\cdot}_X$, $j = 1, 2$ *be two primary pullback corresponding to* f*; then* $\phi_1 = \phi_2$.

1.6. – Existence of pullback: the general case

THEOREM 13. $(\mathrm{EP})_{m,n}$. *Let* $f : X \to Y$ *be a morphism between complex spaces,* $\dim X = m$, $\dim Y = n$ *and fix* $\Lambda^{\cdot}_Y \in \mathcal{R}(Y)$*; then there exist a* $\Lambda^{\cdot}_X \in \mathcal{R}(X)$ *and a pullback* $\Lambda^{\cdot}_Y \to \Lambda^{\cdot}_X$.

REMARK 14. In the particular case $X = Y$, $f = id$, $m = n$ it follows from the proof that in order to obtain $(\mathrm{EP})_{m,n}$ (i.e. $(\mathrm{EP})_{m,m}$) we do not need the assumptions $(\mathrm{EP})_{m,n-1}$, $(\mathrm{U})_{m,n-1}$, (i.e. $(\mathrm{EP})_{m,m-1}$, $(\mathrm{U})_{m,m-1}$); moreover we need $(\mathrm{F})_m$ only for the (irreducible) X_j.

1.7. – Uniqueness of the pullback: the general case

THEOREM 15. $(\mathrm{U})_{m,n}$. *Let* $f : X \to Y$ *be a morphism between complex spaces,* $\dim X = m$, $\dim Y = n$ *and let* $\phi_j : \Lambda^{\cdot}_Y \to \Lambda^{\cdot}_X$, $j = 1, 2$ *be two pullback corresponding to* f*; then* $\phi_1 = \phi_2$.

1.8. – Composition of primary pullback (the irreducible case)

THEOREM 16. $(\mathrm{CP})_{k,m,n}$. *Let* Z, X, Y, *be irreducible complex spaces,* $g : Z \to X$, $f : X \to Y$ *two morphisms,* $\psi : \Lambda^{\cdot}_X \to \Lambda^{\cdot}_Z$, $\phi : \Lambda^{\cdot}_Y \to \Lambda^{\cdot}_X$ *be two primary pullback corresponding to* g *and* f *respectively. Then the composition* $\psi \circ \phi : \Lambda^{\cdot}_Y \to \Lambda^{\cdot}_Z$ *is a primary pullback corresponding to* $f \circ g$.

1.9. – Composition of pullback: the general case

THEOREM 17. $(\mathrm{C})_{k,m,n}$. *Let* Z, X, Y, *be complex spaces,* $g : Z \to X$, $f : X \to Y$ *two morphisms,* $\psi : \Lambda^{\cdot}_X \to \Lambda^{\cdot}_Z$, $\phi : \Lambda^{\cdot}_Y \to \Lambda^{\cdot}_X$ *be two pullback corresponding to* g *and* f *respectively. Then the composition* $\psi \circ \phi : \Lambda^{\cdot}_Y \to \Lambda^{\cdot}_Z$ *is a pullback corresponding to* $f \circ g$.

1.10. – The filtration property

THEOREM 18. $(\mathrm{F})_m$. *Let* X *be a complex space,* $\dim X \le m$. *If* $\Lambda^{\cdot,1}_X$, $\Lambda^{\cdot,2}_X \in \mathcal{R}(X)$, *there exists a third* $\Lambda^{\cdot}_X \in \mathcal{R}(X)$ *and two pullback* $\Lambda^{\cdot,1}_X \to \Lambda^{\cdot}_X$, $\Lambda^{\cdot,2}_X \to \Lambda^{\cdot}_X$.

REMARK 19. If $\Lambda_X^{\cdot,1} = \mathcal{E}^{\cdot}_{\widetilde{X}_1} \oplus \Lambda_E^{\cdot,1} \oplus \Lambda_{\widetilde{E}_1}^{\cdot,1}(-1)$, $\Lambda_X^{\cdot,2} = \mathcal{E}^{\cdot}_{\widetilde{X}_2} \oplus \Lambda_E^{\cdot,2} \oplus \Lambda_{\widetilde{E}_2}^{\cdot,2}(-1)$, then it is possible to choose $\Lambda_X^{\cdot,3} = \mathcal{E}^{\cdot}_{\widetilde{X}_3} \oplus \Lambda_E^{\cdot,3} \oplus \Lambda_{\widetilde{E}_3}^{\cdot,3}(-1)$

2. – The induction procedure

Let $s, t \in \mathbb{N} \times \mathbb{N}$, $s = (m, n), t = (p, q)$. We define the following order on $\mathbb{N} \times \mathbb{N}$: $s > t$ if $\sup(m, n) > \sup(p, q)$ or $\sup(m, n) = \sup(p, q)$ and $(m, n) > (p, q)$ in the lexicographic order.

We write $(\mathrm{EP})_s$, $(\mathrm{U})_s \ldots$ instead of $(\mathrm{EP})_{m,n}$, $(\mathrm{U})_{m,n}$.

Then we prove the following implications:

$(\mathrm{E})_p + (\mathrm{F})_q + (\mathrm{EP})_t + (\mathrm{U})_t$ for $p < n$, $q < n$, $t < (0, n) \Longrightarrow (\mathrm{E})_n$
$(\mathrm{E})_p + (\mathrm{F})_q + (\mathrm{EP})_t + (\mathrm{U})_t$ for $p < n$, $q < n$, $t < (n, 0) \Longrightarrow (\mathrm{F})_n$
$(\mathrm{E})_p + (\mathrm{F})_q + (\mathrm{EP})_t + (\mathrm{U})_t$ for $(0, p) \le s$, $(q, 0) \le s$, $t < s \Longrightarrow (\mathrm{EPP})_s$ and $(\mathrm{EP})_s$
$(\mathrm{E})_p + (\mathrm{F})_q + (\mathrm{EP})_t + (\mathrm{U})_t$ for $(0, p) \le s$, $(q, 0) \le s$, $t < s \Longrightarrow (\mathrm{UP})_s$ and $(\mathrm{U})_s$

Also the definitions $(\mathrm{D})_t$, $(\mathrm{DPP})_t$, $(\mathrm{DP})_t$, are given by induction. Their order is as follows: $(\mathrm{U})_r$ for $r < t$, $(\mathrm{D})_t$, $(\mathrm{DPP})_t$, $(\mathrm{DP})_t$, $(\mathrm{E})_t$.

Finally, $(\mathrm{CP})_{m,n,k}$ preceeds $(\mathrm{C})_{m,n,k}$, and $(\mathrm{CP})_{m,n,k}$ (resp. $(\mathrm{C})_{m,n,k}$) must be proved after $(\mathrm{DPP})_{(m,n)}$, $(\mathrm{DPP})_{(n,k)}$, $(\mathrm{DPP})_{(m,k)}$ (resp. $(\mathrm{DP})_{(m,n)}$, $(\mathrm{DP})_{(n,k)}$, $(\mathrm{DP})_{(m,k)}$. The induction scheme is

$(\mathrm{C})_{k-1,m,n} + (\mathrm{C})_{k,m-1,n} + (\mathrm{C})_{k,m,n-1} \Longrightarrow (\mathrm{CP})_{k,m,n}$ and $(\mathrm{C})_{k,m,n}$.

In the course of each proof the corresponding induction assumptions will be made more explicit.

3. – The proofs

PROPOSITION 20. *We suppose that* $(\mathrm{EP})_{m-1,m-1}$, $(\mathrm{F})_{m-1}$, $(\mathrm{U})_{m-1,m-1}$, $(\mathrm{E})_m$ *are already proved. Let X be a complex space of dimension m, X_i an irreducible component of X.*

i) *Given* $\Lambda_X^{\cdot} = \mathcal{E}^{\cdot}_{\widetilde{X}} \oplus \Lambda_E^{\cdot} \oplus \Lambda_{\widetilde{E}}^{\cdot}(-1)$ *there exists a natural pullback* $\zeta : \Lambda_X^{\cdot} \to \Lambda_{X_i}^{\cdot}$.

ii) *Let* $\zeta_1 : \Lambda_X^{\cdot} \to \Lambda_{X_i}^{\cdot,1}$, $\zeta_2 : \Lambda_X^{\cdot} \to \Lambda_{X_i}^{\cdot,2}$ *two natural pullback; then there exist* $\Lambda_{X_i}^{\cdot,3}$ *and pullback* $\beta_1 : \Lambda_{X_i}^{\cdot,1} \to \Lambda_{X_i}^{\cdot,3}$, $\beta_2 : \Lambda_{X_i}^{\cdot,2} \to \Lambda_{X_i}^{\cdot,3}$ *such that* $\beta_1 \circ \zeta_1 = \beta_2 \circ \zeta_2$ *and the composition* $\Lambda_X^{\cdot} \to \Lambda_{X_i}^{\cdot,3}$ *is a natural pullback.*

PROOF.

i) By $(\mathrm{EP})_{m-1,m-1}$ there exist pullback $\eta_i : \Lambda_E^{\cdot} \to \Lambda_{E_i}^{\cdot}$, $a : \Lambda_{E_i}^{\cdot} \to \Lambda_{E_i}^{\cdot,2}$ and $b : \Lambda_{\widetilde{E}_i}^{\cdot} \to \Lambda_{\widetilde{E}_i}^{\cdot,3}$; by $(\mathrm{F})_{m-1}$ there are pullback $c : \Lambda_{\widetilde{E}_i}^{\cdot,2} \to \Lambda_{\widetilde{E}_i}^{\cdot,1}$ and

$e : \Lambda^{\cdot,3}_{\widetilde{E}_i} \to \Lambda^{\cdot,1}_{\widetilde{E}_i}$; if we apply $(\mathrm{U})_{m-1,m-1}$ to the pullback from $\Lambda^{\cdot}_E$ to $\Lambda^{\cdot,1}_{\widetilde{E}_i}$ we obtain $c \circ a \circ \eta_i = e \circ b \circ q_i \circ \phi$. Putting $\phi_i = c \circ a$ and $\theta_i = e \circ b$ we obtain a commutative diagram like (2) and we define ζ by formula (3).

ii) Let $\Lambda^{\cdot,j}_{X_i} = \mathcal{E}^{\cdot,j}_{\widetilde{X}_i} \oplus \Lambda^{\cdot,j}_{E_i} \oplus \Lambda^{\cdot,j}_{\widetilde{E}_i}(-1)$, $j = 1, 2$; the two pullback ζ_1 and ζ_2 differ only by the morphisms $\Lambda^{\cdot}_E \to \Lambda^{\cdot,j}_{E_i}$ and $\Lambda^{\cdot}_{\widetilde{E}}(-1) \to \Lambda^{\cdot,j}_{\widetilde{E}_i}(-1)$. Arguing as in i), we find $\Lambda^{\cdot,3}_{E_i}$ and $\Lambda^{\cdot,3}_{\widetilde{E}_i}$ and commutative diagrams

$$\begin{array}{ccc} \Lambda^{\cdot,j}_{E_i} & \to & \Lambda^{\cdot,3}_{E_i} \\ \downarrow & & \downarrow \\ \Lambda^{\cdot,j}_{\widetilde{E}_i} & \to & \Lambda^{\cdot,3}_{\widetilde{E}_i} \end{array}$$

$(j = 1, 2)$ from which we easily construct the morphisms β_1 and β_2.

LEMMA 21. *Let $\Lambda^{\cdot}_Y \to \Lambda^{\cdot}_X$ be a pullback corresponding to $f : X \to Y$,* $\dim X = m, \dim Y = n$. *We suppose that* $(\mathrm{EP})_{m-1,n-1}$, $(\mathrm{U})_{m-1,n-1}$, $(\mathrm{F})_q$, $(\mathrm{E})_q$ $q = \sup(m, n)$ *are already proved, as well as* $(\mathrm{EPP})_{m,n}$ *and* $(\mathrm{UP})_{m,n}$ *(for morphisms between irreducible spaces). Then*

a) *If $f(X_j) \subset F$, for any natural pullback $\Lambda^{\cdot}_X \to \Lambda^{\cdot}_{X_j}$ there exists a commutative diagram (4).*

b) *If $f(X_j) \not\subset F$ and $f(X_j) \subset Y_k$, for every natural pullback $\Lambda^{\cdot}_X \to \Lambda^{\cdot}_{X_j}$ and every natural pullback $\Lambda^{\cdot}_Y \to \Lambda^{\cdot}_{Y_k}$ there exists a commutative diagram* (5).

PROOF. We check b), leaving to the reader the proof of a), which is similar. By definition there exists a commutative diagram

$$\begin{array}{ccc} \Lambda^{\cdot}_Y & \to & \Lambda^{\cdot}_X \\ \downarrow & & \downarrow \\ \Lambda^{\cdot,1}_{Y_k} & & \Lambda^{\cdot,1}_{X_j} \\ & \searrow & \downarrow \\ & & \Lambda^{\cdot,2}_{X_j} \end{array} \tag{6}$$

where $\Lambda^{\cdot}_Y \to \Lambda^{\cdot,1}_{Y_k}$, $\Lambda^{\cdot}_X \to \Lambda^{\cdot,1}_{X_j}$ and $\Lambda^{\cdot,1}_{X_j} \to \Lambda^{\cdot,2}_{X_j}$ are natural pullback, and $\Lambda^{\cdot,1}_{Y_k} \to \Lambda^{\cdot,2}_{X_j}$ is a pullback corresponding to $f|_{X_j} : X_j \to Y_k$. By Proposition 20, (ii), we find pullback $\Lambda^{\cdot,1}_{Y_k} \to \Lambda^{\cdot,3}_{Y_k}$ and $\Lambda^{\cdot}_{Y_k} \to \Lambda^{\cdot,3}_{Y_k}$ such that their compositions $\Lambda^{\cdot}_Y \to \Lambda^{\cdot,3}_{Y_k}$ are identical, and are natural pullback. It is clear that for our purposes we can replace $\Lambda^{\cdot}_{Y_k}$ by $\Lambda^{\cdot,3}_{Y_k}$ hence we can suppose from the beginning that $\Lambda^{\cdot}_Y \to \Lambda^{\cdot}_{Y_k}$ decomposes through $\Lambda^{\cdot}_Y \to \Lambda^{\cdot,1}_{Y_k} \to \Lambda^{\cdot}_{Y_k}$; the same argument shows that we can suppose that the morphism $\Lambda^{\cdot}_X \to \Lambda^{\cdot}_{X_j}$ decomposes through $\Lambda^{\cdot}_Y \to \Lambda^{\cdot,1}_{X_j} \to \Lambda^{\cdot}_{X_j}$. We apply $(\mathrm{EPP})_{m,n}$ to $f|_{X_j} : X_j \to Y_k$ and we get a pullback $\Lambda^{\cdot}_{Y_k} \to \Lambda^{\cdot,4}_{X_j}$; moreover by Remark 11 we can suppose $\Lambda^{\cdot,4}_{X_j} =$

$\mathcal{E}^{\cdot}_{\widetilde{X}_i} \oplus \Lambda^{\cdot,4}_{E_i} \oplus \Lambda^{\cdot,4}_{\widetilde{E}_i}(-1)$; by (F)$_m$ and Remark 19 (for X_j) there are natural pullback $\Lambda^{\cdot,2}_{X_j} \to \Lambda^{\cdot,0}_{X_j}$ and $\Lambda^{\cdot,4}_{X_j} \to \Lambda^{\cdot,0}_{X_j}$; finally we apply (UP)$_{m,n}$ to $f|_{X_j} : X_j \to Y_k$: the two compositions $\Lambda^{\cdot,1}_{Y_k} \to \Lambda^{\cdot,2}_{X_j} \to \Lambda^{\cdot,0}_{X_j}$ and $\Lambda^{\cdot,1}_{Y_k} \to \Lambda^{\cdot}_{Y_k} \to \Lambda^{\cdot,4}_{X_j} \to \Lambda^{\cdot,0}_{X_j}$ agree. This completes the proof.

COROLLARY 22.

i) *If $\Lambda^{\cdot}_Y \to \Lambda^{\cdot}_X$ is a pullback, and $\Lambda^{\cdot}_X \to \Lambda^{\cdot}_{X_j}$ is a natural pullback, the composition $\Lambda^{\cdot}_Y \to \Lambda^{\cdot}_{X_j}$ is a pullback.*

(ii) *Conversely let $\Lambda^{\cdot}_Y \to \Lambda^{\cdot}_X$ be a morphism of complexes satisfying* (P_0), (P_1), (P_4) *in Definition* 6, *and suppose that for every irreducible component X_j of X there exists a natural pullback $\Lambda^{\cdot}_X \to \Lambda^{\cdot}_{X_j}$ such that the composition $\Lambda^{\cdot}_Y \to \Lambda^{\cdot}_{X_j}$ is a pullback; then $\Lambda^{\cdot}_Y \to \Lambda^{\cdot}_X$ is a pullback.*

3.1. – Proof of Theorem 16: composition of primary pullback (the irreducible case)

The proof is by triple induction on (k, m, n) where $\dim Z \le k, \dim X \le m, \dim Y \le n$. More precisely we prove

(C)$_{k-1,m,n}$ and (C)$_{k,m-1,n}$ and (C)$_{k,m,n-1} \Longrightarrow$ (CP)$_{k,m,n}$ for primary pullback.

It is obvious that $\psi \circ \phi$ commutes to differentials and satisfies the property (P_1) in the Definition 4. Let $\Lambda^{\cdot}_Z = \mathcal{E}^{\cdot}_{\widetilde{Z}} \oplus \Lambda^{\cdot}_G \oplus \Lambda^{\cdot}_{\widetilde{G}}(-1)$, $\Lambda^{\cdot}_X = \mathcal{E}^{\cdot}_{\widetilde{X}} \oplus \Lambda^{\cdot}_E \oplus \Lambda^{\cdot}_{\widetilde{E}}(-1)$, $\Lambda^{\cdot}_Y = \mathcal{E}^{\cdot}_{\widetilde{Y}} \oplus \Lambda^{\cdot}_F \oplus \Lambda^{\cdot}_{\widetilde{F}}(-1)$, $\alpha : \mathcal{E}^{\cdot}_{\widetilde{Y}} \to \mathcal{E}^{\cdot}_{\widetilde{X}}$, $\alpha' : \mathcal{E}^{\cdot}_{\widetilde{X}} \to \mathcal{E}^{\cdot}_{\widetilde{Z}}$, be induced by ϕ and ψ respectively.

1) Case $\alpha \neq 0, \alpha' \neq 0$. Then we find a commutative diagram

$$\begin{array}{ccccc}
\mathcal{E}^{\cdot}_{\widetilde{Y}} & \oplus & \Lambda^{\cdot}_F & \oplus & \Lambda^{\cdot}_{\widetilde{F}}(-1) \\
\alpha \downarrow & \searrow \beta & \gamma \downarrow & \delta \swarrow & \epsilon \downarrow \\
\mathcal{E}^{\cdot}_{\widetilde{X}} & \oplus & \Lambda^{\cdot}_E & \oplus & \Lambda^{\cdot}_{\widetilde{E}}(-1) \\
\alpha' \downarrow & \searrow \beta' & \gamma' \downarrow & \delta' \swarrow & \epsilon' \downarrow \\
\mathcal{E}^{\cdot}_{\widetilde{Z}} & \oplus & \Lambda^{\cdot}_G & \oplus & \Lambda^{\cdot}_{\widetilde{G}}(-1)
\end{array}$$

Since $\alpha \neq 0, \alpha' \neq 0$, f and g extend respectively to $\tilde{f} : \widetilde{X} \to \widetilde{Y}$ and $\tilde{g} : \widetilde{Z} \to \widetilde{X}$ so that $f \circ g$ extends to $\tilde{f} \circ \tilde{g} : \widetilde{Z} \to \widetilde{Y}$, and $\alpha' \circ \alpha$ is the ordinary De Rham pullback $(\tilde{f} \circ \tilde{g})^*$. Hence it remains to check that $\psi \circ \phi$ satisfies (P_3) (i) (iii) in Definition 4.

We check (P_3) (i). Since $\alpha \neq 0$, $\alpha' \neq 0$, we see that ϕ and ψ satisfy (P_3) (i); hence $f^{-1}(F) \subset E$, $g^{-1}(E) \subset G$ and finally $(f \circ g)^{-1}(E) \subset G$

The morphism $\Lambda^{\cdot}_Y \to \Lambda^{\cdot}_G$ is the composition $\Lambda^{\cdot}_Y \to \Lambda^{\cdot}_X \to \Lambda^{\cdot}_G$; by ($P_3$) (iii) applied to ψ, $\Lambda^{\cdot}_X \to \Lambda^{\cdot}_G$ is a pullback; since $\dim G \le k-1$, it follows by (C)$_{k-1,m,n}$ that $\Lambda^{\cdot}_Y \to \Lambda^{\cdot}_G$ is a pullback, which is (P_3) (iii).

2) Case $\alpha = 0$. In this case $\alpha' \circ \alpha = 0$ so we must check P_2 for $\psi \circ \phi$. $\alpha = 0$ implies $f(X) \subset F$, hence $(f \circ g)(Z) \subset F$; the morphism $\Lambda^{\cdot}_Y \to \Lambda^{\cdot}_X$ decomposes through $\Lambda^{\cdot}_Y \to \Lambda^{\cdot}_F \to \Lambda^{\cdot}_X$. The composition $\Lambda^{\cdot}_F \to \Lambda^{\cdot}_X \to \Lambda^{\cdot}_Z$ is a pullback

by $(C)_{k,m,n-1}$, so that $\Lambda^{\cdot}_Y \to \Lambda^{\cdot}_Z$ decomposes through $\Lambda^{\cdot}_Y \to \Lambda^{\cdot}_F \to \Lambda^{\cdot}_Z$, which is exactly (P_2).

3) Case $\alpha' = 0, \alpha \neq 0$. In this case $g(Z) \subset E$ and $\Lambda^{\cdot}_X \to \Lambda^{\cdot}_Z$ decomposes through $\Lambda^{\cdot}_X \to \Lambda^{\cdot}_E \to \Lambda^{\cdot}_Z$, where $\Lambda^{\cdot}_E \to \Lambda^{\cdot}_Z$ is a pullback. The composite mapping $\Lambda^{\cdot}_Y \to \Lambda^{\cdot}_X \to \Lambda^{\cdot}_E$ is a pullback by (P_2) ; finally the composition $\Lambda^{\cdot}_Y \to \Lambda^{\cdot}_E \to \Lambda^{\cdot}_Z$ is a pullback by $(C)_{k,m-1,n}$.

3.2. – Proof of Theorem 17: composition of pullback (the general case)

The proof is by triple induction on (k, m, n) where $\dim Z \leq k, \dim X \leq m, \dim Y \leq n$. More precisely we prove

$$(C)_{k-1,m,n} \text{ and } (C)_{k,m-1,n} \text{ and } (C)_{k,m,n-1} \Longrightarrow (C)_{k,m,n}.$$

We already know $(CP)_{k,m,n}$ true for primary pullback between irreducible spaces. It is obvious that $\psi \circ \phi$ commutes to differentials and satisfies the property (P_1) in the Definition 6. Let $\Lambda^{\cdot}_Z = \mathcal{E}^{\cdot}_{\widetilde{Z}} \oplus \Lambda^{\cdot}_G \oplus \Lambda^{\cdot}_{\widetilde{G}}(-1)$, $\Lambda^{\cdot}_X = \mathcal{E}^{\cdot}_{\widetilde{X}} \oplus \Lambda^{\cdot}_E \oplus \Lambda^{\cdot}_{\widetilde{E}}(-1)$, $\Lambda^{\cdot}_Y = \mathcal{E}^{\cdot}_{\widetilde{Y}} \oplus \Lambda^{\cdot}_F \oplus \Lambda^{\cdot}_{\widetilde{F}}(-1)$; the composition $\Lambda^{\cdot}_X \to \Lambda^{\cdot}_Z \to \Lambda^{\cdot}_G$ is a pullback, hence the composite mapping $\Lambda^{\cdot}_Y \to \Lambda^{\cdot}_G$ is a pullback by $(C)_{k-1,m,n}$.

– Let Z_l be an irreducible component of Z; if $g(Z_l) \subset E$ there exists a commutative diagram

$$\begin{array}{ccc} \Lambda^{\cdot}_X & \to & \Lambda^{\cdot}_Z \\ \downarrow & & \downarrow \\ \Lambda^{\cdot}_E & & \Lambda^{\cdot}_{Z_l} \\ & \searrow & \downarrow \\ & & \Lambda^{\cdot,0}_{Z_l} \end{array}$$

The composition $\Lambda^{\cdot}_Y \to \Lambda^{\cdot}_X \to \Lambda^{\cdot}_E$ is a pullback because $\Lambda^{\cdot}_Y \to \Lambda^{\cdot}_X$ is, hence the composition $\Lambda^{\cdot}_Y \to \Lambda^{\cdot}_E \to \Lambda^{\cdot,0}_{Z_l}$ is a pullback by $(C)_{k,m-1,n}$. If $g(f(Z_l) \not\subset F$, by the Lemma 21, b) (applied to the spaces Z_l and Y) there exists a commutative diagram

$$\begin{array}{ccc} \Lambda^{\cdot}_Y & \to & \Lambda^{\cdot,0}_{Z_l} \\ \downarrow & & \downarrow \\ \Lambda^{\cdot}_{Y_k} & & \Lambda^{\cdot,1}_{Z_l} \\ & \searrow & \downarrow \\ & & \Lambda^{\cdot,2}_{Z_l} \end{array}$$

where $\Lambda^{\cdot}_{Y_k} \to \Lambda^{\cdot,2}_{Z_l}$ is a primary pullback. So we obtain a commutative diagram

$$\begin{array}{ccc} \Lambda^{\cdot}_Y & \to & \Lambda^{\cdot}_Z \\ \downarrow & & \downarrow \\ \Lambda^{\cdot}_{Y_k} & \to & \Lambda^{\cdot,2}_{Z_l} \end{array}$$

– if $g(f(Z_l) \subset F$ arguing in the same way we find a commutative diagram

$$\begin{array}{ccc} \Lambda^{\cdot}_Y & \to & \Lambda^{\cdot}_Z \\ \downarrow & & \downarrow \\ \Lambda^{\cdot}_F & \to & \Lambda^{\cdot,3}_{Z_l} \end{array}$$

These last two diagrams imply that $\Lambda^{\cdot}_Y \to \Lambda^{\cdot}_Z$ is a pullback.

– if $g(Z_l) \not\subset E$ there exists a unique irreducible component X_j of X with $g(Z_l) \subset X_j$; for simplicity we suppose that there is a unique irreducible component Y_k of Y with $f(X_j) \subset Y_k$ (we leave to the reader the case $f(X_j) \subset F$). According to the Definition 6 there is a commutative diagram

$$\begin{array}{ccc} \Lambda^{\cdot}_Y & \to & \Lambda^{\cdot}_X \\ \downarrow & & \downarrow \\ \Lambda^{\cdot}_{Y_k} & & \Lambda^{\cdot}_{X_j} \\ & \searrow & \downarrow \\ & & \Lambda^{\cdot,0}_{X_j} \end{array}$$

and according to the Lemma 21 another commutative diagram

$$\begin{array}{ccc} \Lambda^{\cdot}_X & \to & \Lambda^{\cdot}_Z \\ \downarrow & & \downarrow \\ \Lambda^{\cdot,0}_{X_j} & & \Lambda^{\cdot}_{Z_l} \\ & \searrow & \downarrow \\ & & \Lambda^{\cdot,0}_{Z_l} \end{array}$$

The composite morphism $\Lambda^{\cdot}_{Y_k} \to \Lambda^{\cdot,0}_{X_j} \to \Lambda^{\cdot,0}_{Z_l}$ is the composition of two primary pullback between irreducible spaces, hence it is a pullback by $(\mathrm{CP})_{m,n}$. If $f(g(Z_l) \not\subset F$ the proof is finished. If $f(g(Z_l) \subset F$, then by Definition 4 (primary pullback for irreducible spaces) the morphism $\Lambda^{\cdot}_{Y_k} \to \Lambda^{\cdot}_{Z_l}$ decomposes as $\Lambda^{\cdot}_{Y_k} \to \Lambda^{\cdot}_{F_k} \to \Lambda^{\cdot,0}_{Z_l}$ $(F_k = F \cap Y_k)$. Taking into account the commutative diagram

$$\begin{array}{ccc} \Lambda^{\cdot}_Y & \to & \Lambda^{\cdot}_{Y_k} \\ \downarrow & & \downarrow \\ \Lambda^{\cdot}_F & \to & \Lambda^{\cdot}_{F_k} \end{array}$$

we finally get by composition the diagram

$$\begin{array}{ccc} \Lambda^{\cdot}_Y & \to & \Lambda^{\cdot}_Z \\ \downarrow & & \downarrow \\ \Lambda^{\cdot}_F & \to & \Lambda^{\cdot,0}_{Z_l} \end{array}$$

which concludes the proof.

3.3. – Proof of Theorem 3: construction-existence theorem

We suppose that $(\mathrm{EP})_{n-1,n-1}$, $(\mathrm{U})_{n-1,n-1}$, $(\mathrm{F})_{n-1}$, $(\mathrm{E})_{n-1}$ are already proved. Let us establish first the following

LEMMA. *Let $f : X \to Y$ be a morphism of complex spaces, with Y smooth,* $\dim(X) = r \leq n-1$, $\dim(Y) \leq n$. *We suppose that* $(\mathrm{C})_{k,m,l}$, $(\mathrm{EP})_{m,l}$, $(\mathrm{F})_m$, $(\mathrm{U})_{m,l}$, $(\mathrm{E})_m$ *are already proved for $k < n$, $m < n$, $l < n$. Then*

(i) *There exists a $\Lambda^{\cdot}_X \in \mathcal{R}(X)$ and a pullback $\psi : \mathcal{E}^{\cdot}_Y \to \Lambda^{\cdot}_X$ (corresponding to f).*

(ii) *Moreover if $\psi_1 : \mathcal{E}^{\cdot}_Y \to \Lambda^{\cdot}_X$ is another pullback corresponding to f then $\psi = \psi_1$.*

The proof of the lemma is by induction on r, the case $r = 0$ being trivial. First we check (i). Let $E = \mathrm{Sing}(X)$. Since $\dim(E) < r$ there is a $\Lambda^{\cdot}_E \in \mathcal{R}(E)$ and a pullback $\rho : \mathcal{E}^{\cdot}_Y \to \Lambda^{\cdot}_E$. Let us take a desigularisation $\pi : \widetilde{X} \to X$ with exceptional space $\widetilde{E} = \pi^{-1}(E)$.

By $(\mathrm{E})_{n-1}$ we construct $\Lambda^{\cdot}_X = \mathcal{E}^{\cdot}_{\widetilde{X}} \oplus \Lambda^{\cdot}_E \oplus \Lambda^{\cdot}_{\widetilde{E}}(-1)$. Let us consider the commutative diagram of pullback

$$\begin{array}{ccc} \mathcal{E}^{\cdot}_Y & \stackrel{\rho}{\to} & \Lambda^{\cdot}_E \\ \alpha\downarrow & & \phi\downarrow \\ \mathcal{E}^{\cdot}_{\widetilde{X}} & \stackrel{\theta}{\to} & \Lambda^{\cdot}_{\widetilde{E}} \end{array}$$

where $\alpha = (f \circ \pi)^*$ is the ordinary De Rham pullback, and ϕ, θ are the inner pullback of $\Lambda^{\cdot}_X$; using (ii) in smaller dimension (replace X by E) we see that $\phi \circ \rho = \theta \circ \alpha$.

Then we define $\psi : \mathcal{E}^{\cdot}_Y \to \Lambda^{\cdot}_X$ by $\psi(\omega) = ((f \circ \pi)^*(\omega), \rho(\omega), 0)$, which is a pullback (left to the reader).

We check the uniqueness (ii); we remark that in our situation any pullback $\psi_1 : \mathcal{E}^{\cdot}_Y \to \Lambda^{\cdot}_X$ is given by

$$\mathcal{E}^{\cdot}_Y \xrightarrow{(\alpha_1,\beta_1,0)} \mathcal{E}^{\cdot}_{\widetilde{X}} \oplus \Lambda^{\cdot}_E \oplus \Lambda^{\cdot}_{\widetilde{E}}(-1)$$

where $\alpha_1 = (f \circ \pi)*$ and $\beta_1 : \Lambda^{\cdot}_Y \to \Lambda^{\cdot}_E$ is a pullback corresponding to the composition $f \circ j : E \to Y$ (inductively defined). Hence we immediately get $\alpha_1 = \alpha$, and $\beta_1 = \rho$ follows from $(\mathrm{U})_{n-1,n-1}$.

Let us go back to the construction-existence theorem. By the above lemma there is a pullback $\rho : \mathcal{E}^{\cdot}_{\widetilde{X}} \to \Lambda^{\cdot,1}_{\widetilde{E}}$; by $(\mathrm{EP})_{n-1,n-1}$ there is a pullback $\Lambda^{\cdot}_E \to \Lambda^{\cdot,2}_{\widetilde{E}}$; by $(\mathrm{F})_{n-1}$ and $(\mathrm{EP})_{n-1,n-1}$ there exists $\Lambda^{\cdot}_{\widetilde{E}}$ and morphisms $\Lambda^{\cdot,1}_{\widetilde{E}} \to \Lambda^{\cdot}_{\widetilde{E}}$ and $\Lambda^{\cdot,2}_{\widetilde{E}} \to \Lambda^{\cdot}_{\widetilde{E}}$; by composition we obtain $\phi : \Lambda^{\cdot}_E \to \Lambda^{\cdot}_{\widetilde{E}}$ (corresponding to q), and $\psi : \mathcal{E}^{\cdot}_{\widetilde{X}} \to \Lambda^{\cdot,1}_{\widetilde{E}}$ (corresponding to i). Let us prove that the complex $\Lambda^{\cdot}_X = \mathcal{E}^{\cdot}_{\widetilde{X}} \oplus \Lambda^{\cdot}_E \oplus \Lambda^{\cdot}_{\widetilde{E}}(-1)$ is a resolution of $\mathbb{C}_X$. Let $x \in X$, U a neighbhorhood of x in X, and $(\omega, \sigma, \theta) \in \Lambda^p_X(U)$, i.e. $\omega \in \Lambda^p_X(\pi^{-1}(U)$, $\sigma \in \Lambda^p_E(U \cap E)$, $\theta \in \Lambda^{p-1}_{\widetilde{E}}(q^{-1}(U \cap E))$, with $(d\omega, d\sigma, d\theta + (-1)^p(\psi(\omega) - \phi(\sigma)) = (0, 0, 0)$. Then $d\sigma = 0$ implies by induction on the dimension of X that $\sigma = d\sigma'$ (after possibly shrinking U). Then on $q^{-1}(U \cap E)$ we have

$$d\omega = 0, \qquad d(\theta - (-1)^p\phi(\sigma')) = -(-1)^p\psi(\omega)$$

which implies that ω gives a cohomology class in $H^p(p^{-1}(U), \mathbb{C})$ whose restriction to $H^p(p^{-1}(U \cap E), \mathbb{C})$ is zero; by Proposition 1 we can write $\omega = d\omega'$ where $\omega' \in \mathcal{E}^{p-1}_{\widetilde{X}}(\pi^{-1}(U))$ (again after possibly shrinking U); it follows

$$d[\theta + (-1)^p\psi(\omega') - (-1)^p\phi(\sigma')] = 0$$

thus $\theta+(-1)^p\psi(\omega')-(-1)^p\phi(\sigma')$ gives a cohomology class in $H^{p-1}(q^{-1}(U\cap E),\mathbb{C})$. Again by Proposition 1 we can write

$$\theta+(-1)^p\psi(\omega')-(-1)^p\phi(\sigma')=(-1)^{p-1}\psi(\omega'')+d\theta'$$

where $\omega''\in\mathcal{E}^{p-1}_{\widetilde{X}}(\pi^{-1}(U))$, $d\omega''=0$, and $\theta'\in\Lambda^{p-2}_{\widetilde{E}}(q^{-1}(U\cap E))$ (here we suppose of course $p\geq 2$: the case $p\leq 1$ needs minor modifications). As a consequence

$$(\omega,\sigma,\theta)=d(\omega'+\omega'',\sigma',\theta')$$

Finally, for every p, $\Lambda^p_X=\mathcal{E}^p_{\widetilde{X}}\oplus\Lambda^p_E\oplus\Lambda^{p-1}_{\widetilde{E}}$ is a fine sheaf; in fact it is a direct sum of direct images of fine sheaves (we use again induction on the dimension).

3.4. – Proof of Theorem 8: existence of primary pullback (the irreducible case)

We suppose that $(\mathrm{EP})_{m-1,n}$, $(\mathrm{EP})_{m,n-1}$, $(\mathrm{EP})_{m-1,m}$, $(\mathrm{U})_{m-1,n}$, $(\mathrm{F})_m$ $(\mathrm{E})_q$, $q=\sup(m,n)$ are already proved.

Let $\Lambda^p_Y=\mathcal{E}^p_{\widetilde{Y}}\oplus\Lambda^p_F\oplus\Lambda^{p-1}_{\widetilde{F}}$ be defined by a proper modification

$$\begin{array}{ccc}\widetilde{F} & \to & \widetilde{Y}\\ \downarrow & & p\downarrow\\ F & \to & Y\end{array}$$

with inner pullback given by $u_1:\mathcal{E}^{\cdot}_{\widetilde{Y}}\to\Lambda^{\cdot}_{\widetilde{F}}$ and $p_1:\Lambda^{\cdot}_F\to\Lambda^{\cdot}_{\widetilde{F}}$. If $f(X)\subset F$, by $(\mathrm{EP})_{m,n-1}$ there exists a pullback $\Lambda^{\cdot}_F\to\Lambda^{\cdot}_X$; hence we obtain the required pullback as composition $\Lambda^{\cdot}_Y\to\Lambda^{\cdot}_F\to\Lambda^{\cdot}_X$, where $\Lambda^{\cdot}_Y\to\Lambda^{\cdot}_F$ is the projection onto the summand.

From now on we suppose $f(X)\not\subset F$. Since X is irreducible, the subspace $E:=f^{-1}(F)\cup\mathrm{Sing}(X)$ is nowhere dense in X.

LEMMA. *We can construct two commutative diagrams*

$$\begin{array}{ccccc}\widetilde{E} & \to & \widetilde{X} & \xrightarrow{\tilde{f}} & \widetilde{Y}\\ \downarrow & & h\downarrow & & p\downarrow\\ E & \to & X & \xrightarrow{f} & Y\end{array}$$

where h is a proper desingularization of X.

PROOF OF THE LEMMA. The modification $f:\widetilde{Y}\to Y$ is dominated by a blowing-up $Y_1\to Y$ centered at an ideal $\mathcal{I}$ of $\mathcal{O}_Y$ supported on F; we obtain a commutative diagram

$$\begin{array}{ccc}X_1 & \xrightarrow{f_1} & \widetilde{Y}\\ u\downarrow & & p\downarrow\\ X & \xrightarrow{f} & Y\end{array}$$

where u is the blowing-up of X centered at the ideal $\mathcal{I}\mathcal{O}_X \subset \mathcal{O}_X$. We consider the proper transform Q of $\mathrm{Sing}(X)$ in X_1 and we take as $\widetilde{X}$ a desingularization of the blow-up of X_1 centered at Q. This proves the lemma.

Now, $\dim \widetilde{F} < n, \dim \tilde{f}^{-1}(\widetilde{F}) < m$, so that by $(\mathrm{EP})_{m-1,m}$ and $(\mathrm{EP})_{m-1,n-1}$ there exist two pullback

$u_2 : \mathcal{E}^{\cdot}_{\widetilde{X}} \to \Lambda^{\cdot}_{\tilde{f}^{-1}(\widetilde{F})}$ and

$p_2 : \Lambda^{\cdot}_{\widetilde{F}} \to \Lambda^{\cdot}_{\tilde{f}^{-1}(\widetilde{F})}$

(here we need also $(\mathrm{F})_{m-1}$ in order to insure the same target $\Lambda^{\cdot}_{\tilde{f}^{-1}(\widetilde{F})}$).

By $(\mathrm{U})_{m-1,n}$

$$p_2 \circ u_1 = u_2 \circ \tilde{f}^*$$

From the trivial proper modification $id : (\widetilde{X}, \tilde{f}^{-1}(\widetilde{F})) \to (\widetilde{X}, \tilde{f}^{-1}(\widetilde{F}))$ we build the complex $\Lambda^{\cdot}_{\widetilde{F}} \in \mathcal{R}(\widetilde{X})$:

$$\Lambda^p_{\widetilde{X}} = \mathcal{E}^p_{\widetilde{X}} \oplus \Lambda^p_{\tilde{f}^{-1}(\widetilde{F})} \oplus \Lambda^{p-1}_{\tilde{f}^{-1}(\widetilde{F})}$$

with inner pullback given by $u_2 : \mathcal{E}^p_{\widetilde{X}} \to \Lambda^p_{\tilde{f}^{-1}(\widetilde{F})}$ and $id : \Lambda^p_{\tilde{f}^{-1}(\widetilde{F})} \to \Lambda^p_{\tilde{f}^{-1}(\widetilde{F})}$. Let us define the morphism

$$\psi : \Lambda^p_Y = \mathcal{E}^p_{\widetilde{Y}} \oplus \Lambda^p_F \oplus \Lambda^{p-1}_{\widetilde{F}} \to \Lambda^p_{\widetilde{X}} = \mathcal{E}^p_{\widetilde{X}} \oplus \Lambda^p_{\tilde{f}^{-1}(\widetilde{F})} \oplus \Lambda^{p-1}_{\tilde{f}^{-1}(\widetilde{F})}$$
$$\psi(\omega, \sigma, \theta) = (\tilde{f}^*\omega, (p_2 \circ p_1)(\sigma), p_2(\theta))$$

We check that ψ is a primary pullback corresponding to the morphism $f \circ h : \widetilde{X} \to Y$; in fact it commutes with the differential because of $p_2 \circ u_1 = u_2 \circ \tilde{f}^*$; the condition P_1 on the hypercoverings of $\widetilde{X}$ and Y is clearly satisfied; the condition $(f \circ h)^{-1}(F) \subset \tilde{f}^{-1}(\widetilde{F})$ is true because of $f \circ h = p \circ \tilde{f}$; $f \circ h$ extends trivially to $\tilde{f} : \widetilde{X} \to Y$; finally the composition $\Lambda^p_Y \to \Lambda^p_{\widetilde{X}} \to \Lambda^p_{\tilde{f}^{-1}(\widetilde{F})}$ coincides with $p_2 \circ p_1$, therefore is a pullback.

Next we define

$$t : \mathcal{E}^p_{\widetilde{X}} \to \Lambda^p_{\widetilde{X}} = \mathcal{E}^p_{\widetilde{X}} \oplus \Lambda^p_{\tilde{f}^{-1}(\widetilde{F})} \oplus \Lambda^{p-1}_{\tilde{f}^{-1}(\widetilde{F})}$$
$$t(\rho) = (\rho, u_2(\rho), 0)$$

which is also a pullback corresponding to the identity $id : \widetilde{X} \to \widetilde{X}$ (left to the reader). Now we use induction: by $(\mathrm{EP})_{m-1,n}$ there is a pullback $w : \Lambda^{\cdot}_Y \to \Lambda^{\cdot}_E$ corresponding to the composite morphism $E \to X \to Y$: again by $(\mathrm{EP})_{m-1,m}$ there is a pullback $z : \Lambda^{\cdot}_{\widetilde{X}} \to \Lambda^{\cdot}_{\widetilde{E}}$ corresponding to the embedding $\widetilde{E} \to \widetilde{X}$; by $(\mathrm{EP})_{m-1,m-1}$ there is a a pullback $q_1 : \Lambda^{\cdot}_E \to \Lambda^{\cdot,1}_{\widetilde{E}}$ corresponding to the morphism $\widetilde{E} \to E$. Using $(\mathrm{F})_{m-1}$ we can suppose $\Lambda^{\cdot,1}_{\widetilde{E}} = \Lambda^{\cdot}_{\widetilde{E}}$.

Because of $(\mathrm{U})_{m-1,n}$ the two pullback $z \circ \psi, q_1 \circ w : \Lambda^{\cdot}_{Y} \to \Lambda^{\cdot}_{\widetilde{E}}$ coincide:

$$z \circ \psi = q_1 \circ w$$

We define

$$\Lambda^{p}_{X} = \mathcal{E}^{p}_{\widetilde{X}} \oplus \Lambda^{p}_{E} \oplus \Lambda^{p-1}_{\widetilde{E}}$$

where the inner pullback are defined by $z \circ t : \mathcal{E}^{\cdot}_{\widetilde{X}} \to \Lambda^{\cdot}_{\widetilde{E}}$ and $q_1 : \Lambda^{\cdot}_{E} \to \Lambda^{\cdot}_{\widetilde{E}}$.

Finally we define

$$\begin{aligned} &\phi : \Lambda^{p}_{Y} = \mathcal{E}^{p}_{\widetilde{Y}} \oplus \Lambda^{p}_{F} \oplus \Lambda^{p-1}_{\widetilde{F}} \to \Lambda^{p}_{X} = \mathcal{E}^{p}_{\widetilde{X}} \oplus \Lambda^{p}_{E} \oplus \Lambda^{p-1}_{\widetilde{E}} \\ &\phi(\omega, \sigma, \theta) = (\tilde{f}^{*}\omega, w(\omega, \sigma, \theta), z(0, p_2(\theta), 0)) \end{aligned}$$

Here we notice that $p_2(\theta) \in \Lambda^{p-1}_{\tilde{f}^{-1}(\widetilde{F})}$, $(0, p_2(\theta), 0) \in \Lambda^{p-1}_{X}$ so that $z(0, p_2(\theta), 0) \in \Lambda^{p-1}_{\widetilde{E}}$. In order to prove that ϕ is a primary pullback, the only non trivial property is that it commutes with differentials. Let us check it.

$$\begin{aligned} d(\omega, \sigma, \theta) &= (d\omega, d\sigma, d\theta + (-1)^{p}(u_1(\omega) - p_1(\sigma)) \\ \phi(d(\omega, \sigma, \theta)) &= (\tilde{f}^{*}d\omega, w(d(\omega, \sigma, \theta)), z(0, p_2(d\theta + (-1)^{p}(u_1(\omega) - p_1(\sigma)), 0) \end{aligned}$$

On the other hand

$$\begin{aligned} d(\phi(\omega, \sigma, \theta)) &= d(\tilde{f}^{*}\omega, w(\omega, \sigma, \theta), z(0, p_2(\theta), 0)) \\ &= (d\tilde{f}^{*}\omega, d(w(\omega, \sigma, \theta)), d(z(0, p_2(d\theta), 0)) \\ &\quad + (-1)^{p}[(z \circ t)(\tilde{f}^{*}\omega) - q_1(w(\omega, \sigma, \theta))]) \end{aligned}$$

Since $\tilde{f}^{*}d\omega = d\tilde{f}^{*}\omega$, $w(d(\omega, \sigma, \theta)) = d(w(\omega, \sigma, \theta)$, it remains to check the equality of the third components. We must be careful about signs: $d(z(0, p_2(d\theta), 0)$ in the above formula(s) is a differential of a $(p-1)$-form.

Recalling that

$$\psi(\omega, \sigma, \theta) = (\tilde{f}^{*}\omega, (p_2 \circ p_1)(\sigma), p_2(\theta))$$

and

$$t(\tilde{f}^{*}\omega) = (\tilde{f}^{*}\omega, u_2(\tilde{f}^{*}\omega), 0) = (\tilde{f}^{*}\omega, (p_2 \circ u_1)(\omega), 0)$$

we obtain

$$\begin{aligned} &d(z(0, p_2(d\theta), 0)) + (-1)^{p}[(z \circ t)(\tilde{f}^{*}\omega) - q_1(w(\omega, \sigma, \theta))] \\ &= z(0, dp_2(\theta), (-1)^{p} p_2(\theta)) + (-1)^{p}[(z \circ t)(\tilde{f}^{*}\omega) - (z \circ \psi)(\omega, \sigma, \theta))] \\ &= z\{(0, dp_2(\theta), (-1)^{p} p_2(\theta)) + (-1)^{p}[\tilde{f}^{*}\omega - \psi(\omega, \sigma, \theta)]\} \\ &= z(0, dp_2(\theta) + (-1)^{p}[(p_2 \circ u_1)(\omega) - (p_2 \circ p_1)(\sigma)], 0) \end{aligned}$$

which gives the result.

It is clear that the above proof implies the more precise statement Theorem 10 and Remark 9 and 11.

3.5. – Proof of Theorem 12: uniqueness of the primary pullback (the irreducible case)

We suppose that $(\mathrm{EP})_{m-1,n}$, $(\mathrm{EP})_{m,n-1}$, $(\mathrm{EP})_{n-1,n-1}$, $(\mathrm{U})_{m-1,n}$, $(\mathrm{U})_{m,n-1}$, $(\mathrm{U})_{n-1,n-1}$, $(\mathrm{F})_m$, $(\mathrm{E})_q$, $q = \sup(m,n)$ are already proved.

Let $\Lambda^{\cdot}_Y = \mathcal{E}^{\cdot}_{\widetilde{Y}} \oplus \Lambda^{\cdot}_F \oplus \Lambda^{\cdot}_{\widetilde{F}}(-1)$, $\Lambda^{\cdot}_X = \mathcal{E}^{\cdot}_{\widetilde{X}} \oplus \Lambda^{\cdot}_E \oplus \Lambda^{\cdot}_{\widetilde{E}}(-1)$.

We suppose first that $f(X) \subset F$, so that both ϕ_1 and ϕ_2 decompose through the projection $\Lambda^{\cdot}_Y \to \Lambda^{\cdot}_F$; the result follows if we apply $(\mathrm{U})_{m,n-1}$ to the induced morphism $X \to F$.

So we assume $f(X) \not\subset F$.

According to (P_3) in Definition 4

(i) $f^{-1}(F) \subset E$;
(ii) the morphism f extends to a morphism $\tilde{f} : \widetilde{X} \to \widetilde{Y}$;
(iii) the morphism ϕ_j is given by

$$
\begin{array}{ccccc}
\mathcal{E}^{\cdot}_{\widetilde{Y}} & \oplus & \Lambda^{\cdot}_F & \oplus & \Lambda^{\cdot}_{\widetilde{F}}(-1) \\
\alpha_j \downarrow & \searrow \beta_j & \gamma_j \downarrow & \delta_j \swarrow & \epsilon_j \downarrow \\
\mathcal{E}^{\cdot}_{\widetilde{X}} & \oplus & \Lambda^{\cdot}_E & \oplus & \Lambda^{\cdot}_{\widetilde{E}}(-1)
\end{array} \tag{7}
$$

where $\mu_j = (\beta_j, \gamma_j, \delta_j) : \Lambda^{\cdot}_Y \to \Lambda^{\cdot}_E$ is a pullback corresponding to the composition $E \to X \to Y$, and $\alpha_j = \tilde{f}^*$ is the ordinary De Rham pullback. It follows $\alpha_1 = \tilde{f}^* = \alpha_2$; by $(\mathrm{U})_{m-1,n}$ $\mu_1 = \mu_2$;
The conclusion will follow from the next lemma.

LEMMA 23. *Let $\phi_j : \Lambda^{\cdot}_Y \to \Lambda^{\cdot}_X$, $j = 1, 2$, two primary pullback given by* (7), *and suppose $\alpha_1 = \alpha_2$, $\beta_1 = \beta_2$, $\gamma_1 = \gamma_2$, $\delta_1 = \delta_2$. Then also $\epsilon_1 = \epsilon_2$.*

In order to prove the lemma we need the following two propositions

PROPOSITION 24. *Let $\phi : \mathcal{E}^{\cdot}_X \to \Lambda^{\cdot}_Z$ be a pullback corresponding to a morphism $f : Z \to X$, X being smooth, and let $(Z_s, g_s)_{s\in S}$ the hypercovering associated to $\Lambda^{\cdot}_Z$. For every Z_s of rank 0 in Z the pullback ϕ induces the De Rham pullback $f_s^* : \mathcal{E}^{\cdot}_X \to \mathcal{E}^{\cdot}_{Z_s}$.*

The proof of Proposition 24 is by induction on the dimension of Z. Let $\Lambda^{\cdot\cdot}_Z = \mathcal{E}^{\cdot}_{\widetilde{Z}} \oplus \Lambda^{\cdot\cdot}_G \oplus \Lambda^{\cdot\cdot}_{\widetilde{G}}(-1)$ If Z_s appears in the hypercovering of G, we conclude by induction applied to the pullback $\mathcal{E}^{\cdot}_X \to \Lambda^{\cdot}_G$. Otherwise, because of its rank, Z_s cannot appear in the hypercovering of $\widetilde{G}$ so it must be a connected component of the desingularisation $\widetilde{Z}$, so the result is an immediate consequence of the definition of pullback.

PROPOSITION 25. *Let $\Lambda^{\cdot}_X \in \mathcal{R}(X)$ and $(X_l, h_l)_{l\in L}$ the associated hypercovering. For every X_l of positive rank $r > 0$ in X there exists X_m of rank $r - 1$ and an inner differential inducing a De Rham pullback $f^*_{lm} : \mathcal{E}^p_{X_m} \to \mathcal{E}^p_{X_l}$.*

PROOF OF PROPOSITION 25. Here, an inner differential is one of the pullback appearing in the construction of the differential $d : \Lambda^{\cdot}_X \to \Lambda^{\cdot}_X$; more precisely it is one of the following:

- the pullback $\mathcal{E}^{\cdot}_{\widetilde{X}} \to \Lambda^{\cdot}_{\widetilde{E}}$
- the pullback $\Lambda^{\cdot}_E \to \Lambda^{\cdot}_{\widetilde{E}}$
- an inner differential in $\Lambda^{\cdot}_E$ (inductively defined)
- an inner differential in $\Lambda^{\cdot}_{\widetilde{E}}$ (inductively defined)

The proof is by induction on $m = \dim X$. If X_l appears in the hypercovering of E, we conclude by induction. If X_l appears in the hypercovering of $\widetilde{E}$, then the rank s of X_l in $\widetilde{E}$ is $r-1$; if $r > 1$, then $s > 0$ and we conclude by induction ($\widetilde{E}$ in the place of X); if $r = 1$, then $s = 0$, and we apply the Proposition 24 to the inner pullback $\mathcal{E}^{\cdot}_{\widetilde{X}} \to \Lambda^{\cdot}_{\widetilde{E}}$.

PROOF OF THE LEMMA 23. By the hypothesis we define $\alpha = \alpha_1 = \alpha_2$, $\beta = \beta_1 = \beta_2$, $\gamma = \gamma_1 = \gamma_2$, $\delta = \delta_1 = \delta_2$.

Let (u, p) be the inner pullback in $\Lambda^{\cdot}_X$, and (v, q) those in $\Lambda^{\cdot}_Y$.

Then $\phi \circ d = d \circ \phi$ implies for $\epsilon = \epsilon_1$ or $\epsilon = \epsilon_2$:

$$\epsilon(v(\omega)) = u(f^*(\omega)) - (p \circ \phi)(\omega, 0, 0) \tag{8}$$

$$(\epsilon \circ d - d \circ \epsilon)(\gamma) = (-1)^p (p \circ \phi)(0, 0, \gamma) \tag{9}$$

Let (X_l) and (Y_s) the hypercoverings corresponding to $\Lambda^{\cdot}_X$ and $\Lambda^{\cdot}_Y$; for every X_l appearing in $\widetilde{E}$ there is at most a Y_s appearing in $\widetilde{F}$ such that the morphism $\epsilon_{1,ls} : \mathcal{E}^p_{Y_s} \to \mathcal{E}^p_{X_l}$ induced a by ϵ_1 is a De Rham pullback; otherwise $\epsilon_{1,ls} = 0$; the same for $\epsilon_{2,ls}$. Since $\epsilon_{1,ls}$ and $\epsilon_{2,ls}$ completely determine ϵ_1 and ϵ_2, it will be enough to prove that $\epsilon_{1,ls} = 0$ if and only if $\epsilon_{2,ls} = 0$; more precisely we check that (8) and (9) completely determine whether ϵ_{ls} is zero or nonzero.

Let $s \geq 0$ be the rank of X_l in $\widetilde{E}$, so that the rank of X_l in X is $r = s + 1 > 0$. We proceed by induction on s.

Let first s be 0. We apply Proposition 24 to the inner pullback $v : \mathcal{E}^{\cdot}_{\widetilde{Y}} \to \Lambda^{\cdot}_{\widetilde{F}}$; hence for every s such that Y_s appears in $\widetilde{F}$ with rank 0 there is $\omega \in \mathcal{E}^0_{\widetilde{Y}}$ such that $v(\omega) \neq 0 \in \mathcal{E}^0_{Y_s}$. it follows that $\epsilon_{ls} = 0$ or $\neq 0$ according to the second member of (8), which is the same for ϵ_1 and ϵ_2.

We consider now the case $s > 0$. Let us fix l and s; by Proposition 25 there exists Y_k of rank $s - 1 \geq 0$ in $\widetilde{F}$ and an inner pullback inducing a real De Rham pullback $a_{sk} : \mathcal{E}^{\cdot}_{Y_k} \to \mathcal{E}^{\cdot}_{Y_s}$.

Let $\theta \in \Lambda^{\cdot}_{\widetilde{F}}$ be defined as $\theta = (0, \dots, \theta_k, \dots, 0)$; then $(d\theta)_l = \pm a_{sk}(\theta_k)$. Moreover $\epsilon(\theta) = (\epsilon_{tk}(\theta_k))_t$ where $\epsilon_{tk} : \mathcal{E}^p_{Y_k} \to \mathcal{E}^p_{X_t}$ involves only components Y_k and X_t of rank $< s$. hence by induction $\epsilon_{1,tk} = \epsilon_{2,tk}$ and $d\epsilon_1(\theta) = d\epsilon_2(\theta)$. from (9) it follows that $\epsilon_{sl}(a_{sk}(\theta_k)$ is the same for ϵ_1 and ϵ_2. Taking any $\theta_k \neq 0 \in \mathcal{E}^0_{Y_k}$ we obtain $a_{sk}(\theta_k) \neq 0 \in \mathcal{E}^0_{Y_s}$ so that $\epsilon_{1,ls} = 0$ if and only if $\epsilon_{2,ls} = 0$.

3.6. – Proof of Theorem 13: existence of pullback (the general case)

We suppose that $(\mathrm{EP})_{m-1,n}$,$(\mathrm{EP})_{m-1,m-1}$, $(\mathrm{U})_{m-1,n}$, $(\mathrm{F})_m$,$(\mathrm{E})_q$, $q=\sup(m,n)$ are already proved. Moreover we suppose $(\mathrm{EPP})_{m,n}$ true for morphisms between irreducible spaces.

First we deal with the case X irreducible. There exists an irreducible component Y_k of Y such that $f(X) \subset Y_k$; let $\Lambda^{\cdot}_Y \to \Lambda^{\cdot}_{Y_k}$ be a natural pullback; by $(\mathrm{EPP})_{m,n}$ there exists a pullback $\Lambda^{\cdot}_{Y_k} \to \Lambda^{\cdot}_X$. By composition we get the required pullback.

Let us consider the case where X is reducible; let $X = X_1 \cup \ldots \cup X_r$ be the decomposition of X into its irreducible components. By the above case for every j there exists a pullback $\phi_j : \Lambda^{\cdot}_Y \to \Lambda^{\cdot}_{X_j}$. Let $\Lambda^{\cdot}_{X_j} = \mathcal{E}^{\cdot}_{\widetilde{X}_j} \oplus \Lambda^{\cdot}_{E_j} \oplus \Lambda^{\cdot}_{\widetilde{E}_j}(-1)$ and denote by $\rho_j : \Lambda^{\cdot}_{E_j} \to \Lambda^{\cdot}_{\widetilde{E}_j}$ and $\eta_j : \mathcal{E}^{\cdot}_{\widetilde{X}_j} \to \Lambda^{\cdot}_{\widetilde{E}_j}$ the inner pullback. Then $\mathrm{Sing}(X_j) \subset E_j$; using $(\mathrm{F})_m$ and $(\mathrm{E})_m$ we can enlarge E_j in order to obtain $\mathrm{Sing}(X) \cap X_j \subset E_j$, so that if we define $E = E_1 \cup \ldots \cup E_r$ we obtain $\mathrm{Sing}(X) \subset E$. Let moreover $\widetilde{X} = \widetilde{X_1} \sqcup \ldots \sqcup \widetilde{X_r}$, $\widetilde{E} = \widetilde{E_1} \sqcup \ldots \sqcup \widetilde{E_r}$.

By $(\mathrm{EP})_{m-1,n}$ and $(\mathrm{EP})_{m-1,m-1}$ there are pullback $\beta : \Lambda^{\cdot}_Y \to \Lambda^{\cdot}_E$ and $\mu_j : \Lambda^{\cdot}_E \to \Lambda^{\cdot,1}_{E_j}$; if we apply $(\mathrm{F})_{m-1}$ to E_j we find two pullback $\Lambda^{\cdot}_{E_j} \to \Lambda^{\cdot,0}_{E_j}, \Lambda^{\cdot,1}_{E_j} \to \Lambda^{\cdot,0}_{E_j}$; by $(\mathrm{E})_m$ there exists $\Lambda^{\cdot,0}_{X_j} = \mathcal{E}^{\cdot}_{\widetilde{X}_j} \oplus \Lambda^{\cdot,0}_{E_j} \oplus \Lambda^{\cdot,0}_{\widetilde{E}_j}(-1)$ and a pullback $\theta_j : \Lambda^{\cdot}_{X_j} \to \Lambda^{\cdot,0}_{X_j}$. Finally, after replacing $\Lambda^{\cdot}_{E_j}$ by $\Lambda^{\cdot,0}_{E_j}$, $\Lambda^{\cdot}_{X_j}$ by $\Lambda^{\cdot,0}_{X_j}$ and ϕ_j by $\theta_j \circ \phi_j$ we can assume that $\mu_j : \Lambda^{\cdot}_E \to \Lambda^{\cdot}_{E_j}$. Let ϕ_j be given by

$$\Lambda^p_Y \to \Lambda^p_{X_j} = \mathcal{E}^p_{\widetilde{X}_j} \oplus \Lambda^p_{E_j} \oplus \Lambda^{p-1}_{\widetilde{E}_j}$$
$$\omega \to (\alpha_j(\omega), \beta_j(\omega), \theta_j(\omega))$$

Recall that by the definition of pullback, $\beta_j : \Lambda^{\cdot}_Y \to \Lambda^{\cdot}_{E_j}$ is a pullback. By $(\mathrm{U})_{m-1,n}$

$$\beta_j = \mu_j \circ \beta$$

We can now define

$$\Lambda^{\cdot}_X = \mathcal{E}^{\cdot}_{\widetilde{X}} \oplus \Lambda^{\cdot}_E \oplus \Lambda^{\cdot}_{\widetilde{E}}(-1)$$

with inner pullback given by

$$\xi : \Lambda^{\cdot}_E \to \Lambda^{\cdot}_{\widetilde{E}} = \oplus_j \Lambda^{\cdot}_{\widetilde{E}_j}$$
$$\sigma \to \oplus_j \rho_j(\mu_j(\sigma))$$

and

$$\lambda : \mathcal{E}^{\cdot}_{\widetilde{X}} = \oplus_j \mathcal{E}^{\cdot}_{\widetilde{X}_j} \to \Lambda^{\cdot}_{\widetilde{E}} = \oplus_j \Lambda^{\cdot}_{\widetilde{E}_j}$$
$$(\tau_j) \to \oplus_j \eta_j(\tau_j)$$

Finally we define the morphism

$$\begin{aligned}\phi : \Lambda^p_Y \to \Lambda^p_X = \oplus_j \mathcal{E}^p_{\widetilde{X}_j} \oplus \Lambda^p_E \oplus \oplus_j \Lambda^{p-1}_{\widetilde{E}_j} \\ \omega \to (\oplus_j \alpha_j(\omega), \beta(\omega), \oplus_j \theta_j(\omega))\end{aligned}$$

We check that ϕ commutes with differentials. Since ϕ_j commutes with differentials, we get for $\omega \in \Lambda^p_Y$:

$$\begin{aligned}d\alpha_j(\omega) &= \alpha_j(d\omega) \\ d\beta_j(\omega) &= \beta_j(d\omega) \\ d\theta_j(\omega) &= \theta_j(d\omega) - (-1)^p[\eta_j(\alpha_j(\omega)) - \rho_j(\beta_j(\omega))]\end{aligned}$$

Using the equality $\rho_j \circ \beta_j = \rho_j \circ \mu_j \circ \beta$ a simple computation shows that ϕ commutes to d. It is clear that by construction ϕ satisfies all the properties in the Definition 6 so it is the required pullback.

As to Remark 14, it is clear from the proof that if $X = Y$, $f = id$, $m = n$ we do not need the assumptions (EP)$_{m,m-1}$, (U)$_{m,m-1}$, because for every j $f(X_j) \not\subset E$; moreover we need (F)$_m$ only for the (irreducible) X_j.

3.7. – Proof of Theorem 15: uniqueness of the pullback (the general case)

We suppose that (U)$_{m-1,n}$, (F)$_m$, (E)$_q$, $q = \sup(m, n)$ are already proved. Moreover we suppose (UP)$_{m,n}$ true for morphisms between irreducible spaces.

Let $f : X \to Y$ be a morphism between (reducible) complex spaces, $X = \bigcup_j X_j$, $Y = \bigcup_k Y_k$ be the respective decompositions into irreducible components. Let $\Lambda^{\cdot}_X = \mathcal{E}^{\cdot}_{\widetilde{X}} \oplus \Lambda^{\cdot}_E \oplus \Lambda^{\cdot}_{\widetilde{E}}(-1)$, $\Lambda^{\cdot}_Y = \mathcal{E}^{\cdot}_{\widetilde{Y}} \oplus \Lambda^{\cdot}_F \oplus \Lambda^{\cdot}_{\widetilde{F}}(-1)$. Let $\phi_i : \Lambda^{\cdot}_Y \to \Lambda^{\cdot}_X$, $i = 1, 2$ two pullback. Let (u, p) be the inner pullback in $\Lambda^{\cdot}_X$, and (v, q) those in $\Lambda^{\cdot}_Y$.

For $(\omega, \sigma, \theta) \in \Lambda^p_Y$, we have

$$\begin{aligned}\phi_i(\omega, 0, 0) &= (\alpha_i(\omega), \beta_i(\omega), 0) \\ \phi_i(0, \sigma, 0) &= (\rho_i(\sigma), \gamma_i(\sigma), \eta_i(\sigma)) \\ \phi_i(0, 0, \theta) &= (0, \delta_i(\theta), \epsilon_i(\theta))\end{aligned}$$

In fact the morphism $\mathcal{E}^p_{\widetilde{Y}} \to \Lambda^{p-1}_{\widetilde{E}}$ induced by ϕ_i is identically zero because the components of $\widetilde{Y}$ have rank 0 in Y, while the components of the hypercovering of $\Lambda^{\cdot}_X$ belonging to $\Lambda^{\cdot}_{\widetilde{E}}$ have strictly positive rank in X; for an analogous reason the morphism $\Lambda^{p-1}_{\widetilde{F}} \to \mathcal{E}^p_{\widetilde{X}}$ is zero.

By (P$_4$) in the Definition 6, and (U)$_{m-1,n}$ we obtain $\beta_1 = \beta_2 = \beta, \gamma_1 = \gamma_2 = \gamma, \delta_1 = \delta_2 = \delta$.

Moreover it is clear that $\alpha_i = 0$ on $\widetilde{X}_j$ in case $f(X_j) \subset F$, while α_i is an ordinary De Rham pullback on $\widetilde{X}_j$ if $f(X_j) \not\subset F$; it follows $\alpha_1 = \alpha_2 = \alpha$.

For any irreducible component X_j of X we consider a commutative diagram like (5) (or (4): left to the reader) for ϕ_1. Becuase of Lemma 21 we have for ϕ_2 an analogous diagram, with the same $\Lambda^{\cdot,0}_{X_j}$ and $\Lambda^{\cdot}_{Y_k}$, and same natural pullback $\Lambda^{\cdot}_X \to \Lambda^{\cdot,0}_{X_j}$ and $\Lambda^{\cdot}_Y \to \Lambda^{\cdot}_{Y_k}$.

By $(\mathrm{UP})_{m,n}$ applied to the morphism of irreducible spaces $X_j \to Y_k$ we can suppose that the diagrams (5) for ϕ_1 and ϕ_2 coincide, so that the same is true for the composite morphisms $\pi_i : \Lambda^{\cdot}_Y \to \Lambda^{\cdot}_{Y_k} \to \Lambda^{\cdot,0}_{X_j}$, i.e. $\pi_1 = \pi_2$. Since by construction π_i and ϕ_i induce the same morphisms $\Lambda^{p}_{\widetilde{F}} \to \mathcal{E}^{p}_{\widetilde{X}_j}$ for every j, we conclude that also $\rho_1 = \rho_2$.

Since ϕ_i commutes to the differentials we can deduce

$$\epsilon_i(v(\omega)) = u(\alpha(\omega)) - p(\beta(\omega)) \tag{10}$$

$$(\epsilon_i \circ d - d \circ \epsilon_i)(\theta) = (-1)^p (p(\delta(\theta)) \tag{11}$$

$$(\eta_i \circ d - d \circ \eta_i)(\sigma) = (-1)^p \epsilon_i(q(\sigma)) + (-1)^p [u(d\rho(\sigma)) - q(d\gamma(\sigma)] \tag{12}$$

Arguing as in the proof of Lemma 23, by (10) and (11) we obtain $\epsilon_1 = \epsilon_2$; then using (12) (instead of (9)) again an argument similar to that in the proof of the Lemma 23 shows that $\eta_1 = \eta_2$ (the reader should note that η_1 and η_2 do not involve $\widetilde{Y}$).

3.8. – Proof of Theorem 18: filtering

We suppose that $(\mathrm{EP})_{m-1,m-1}$, $(\mathrm{U})_{m-1,m-1}$, $(\mathrm{F})_{m-1}$, $(\mathrm{E})_m$, are already proved. Let $\Lambda^{\cdot,1}_X = \mathcal{E}^{\cdot}_{\widetilde{X_1}} \oplus \Lambda^{\cdot,1}_E \oplus \Lambda^{\cdot,1}_{\widetilde{E}}(-1)$, $\Lambda^{\cdot,2}_X = \mathcal{E}^{\cdot}_{\widetilde{X_2}} \oplus \Lambda^{\cdot,2}_F \oplus \Lambda^{\cdot,2}_{\widetilde{F}}(-1)$.

1) First we consider the case $\widetilde{X_1} = \widetilde{X_2} = \widetilde{X}$, $E = F$, $\widetilde{E} = \widetilde{F}$. By $(\mathrm{F})_{m-1}$ there are pullback $\Lambda^{\cdot,1}_E \to \Lambda^{\cdot}_E$ $\Lambda^{\cdot,2}_E \to \Lambda^{\cdot}_E$, $\Lambda^{\cdot,1}_{\widetilde{E}} \to \Lambda^{\cdot}_{\widetilde{E}}$, $\Lambda^{\cdot,2}_{\widetilde{E}} \to \Lambda^{\cdot}_{\widetilde{E}}$. Using $(\mathrm{EP})_{m-1,m-1}$, $(\mathrm{U})_{m-1,m-1}$ and again $(\mathrm{F})_{m-1}$ we find a pullback $\Lambda^{\cdot}_E \to \Lambda^{\cdot}_{\widetilde{E}}$ such that the two following diagrams for $j = 1, 2$ are commutative

$$\begin{array}{ccc} \Lambda^{\cdot,j}_E & \to & \Lambda^{\cdot}_E \\ \downarrow & & \downarrow \\ \Lambda^{\cdot,j}_{\widetilde{E}} & \to & \Lambda^{\cdot}_{\widetilde{E}} \end{array}$$

where the left vertical arrows come from the inner differentials in $\Lambda^{\cdot,j}_X$. Finally by $(\mathrm{E})_m$ we find $\Lambda^{\cdot}_X = \mathcal{E}^{\cdot}_{\widetilde{X}} \oplus \Lambda^{\cdot}_E \oplus \Lambda^{\cdot}_{\widetilde{E}}(-1)$ and obvious pullback $\Lambda^{\cdot,j}_X \to \Lambda^{\cdot}_X$.

LEMMA. *Under the assumptions of the theorem, let* $\Lambda^{\cdot}_X = \mathcal{E}^{\cdot}_{\widetilde{X}} \oplus \Lambda^{\cdot}_E \oplus \Lambda^{\cdot}_{\widetilde{E}}(-1)$, *and let* G *be a nowhere dense closed subspace of* X *with* $E \subset G$. *Then there exists* $\Lambda^{\cdot,0}_X = \mathcal{E}^{\cdot}_{\widetilde{X}} \oplus \Lambda^{\cdot}_G \oplus \Lambda^{\cdot}_{\widetilde{G}}(-1) \in \mathcal{R}(X)$ *and a pullback* $\Lambda^{\cdot}_X \to \Lambda^{\cdot,0}_X$ *corresponding to the identity.*

It is now clear how to prove (F)$_m$: we take $G = E \cup F$. By the lemma we find $\Lambda_X^{\cdot,0} = \mathcal{E}_{\widetilde{X}}^{\cdot} \oplus \Lambda_G^{\cdot} \oplus \Lambda_{\widetilde{G}}^{\cdot}(-1)$ and two pullback $\Lambda_X^{\cdot,j} \to \Lambda_X^{\cdot,0}$, $j = 1, 2$.

Sketch of the proof of the lemma. In the case where X is irreducible, the lemma is a consequence of the Remark 9. In the general case it follows from the Remark 14, because meanwhile (F)$_m$ has been already proved for each irreducible component of X.

4. – The Hodge-Deligne mixed structure

The complexes of differential forms defined above give a natural approach to the Deligne theory of mixed Hodge structures. We refer to [D], [A], [E], for different approaches to this theory. In [AG7] there is a self contained, quite elementary treatment, which avoids the use of cohomological descent theory.

For the construction of the Deligne mixed Hodge structure it is necessary to put two filtrations on the cohomology of X. On the level of the forms on X we introduce them in the following way.

For a fixed $\Lambda_X^{\cdot} \in \mathcal{R}(X)$ we define:

i) The (increasing) weight filtration

$$W_m(\Lambda_X^p) := W_m(\mathcal{E}_{\widetilde{X}}^p) \oplus W_m(\Lambda_E^p) \oplus W_{m+1}(\Lambda_{\widetilde{E}}^{p-1})$$

by induction on $\dim(X)$: $W_m(\Lambda_E^p)$ and $W_m(\Lambda_{\widetilde{E}}^{p-1})$ are already defined because $\dim E < \dim X$, $\dim \widetilde{E} < \dim X$, while $W_m(\mathcal{E}_{\widetilde{X}}^p)$ is equal to $\mathcal{E}_{\widetilde{X}}^p$ for $m \geq 0$ and zero otherwise.

More constructively, if $(X_l, h_l)_{l \in L}$ is the hypercovering associated to Λ_X^p,

$$W_m(\Lambda_X^p) = \oplus_{l: -q(l) \leq m} \mathcal{E}_{X_l}^{p-q(l)}$$

ii) The (decreasing) Hodge filtration

$$F^r(\Lambda_X^p) := F^r(\mathcal{E}_{\widetilde{X}}^p) \oplus F^r(\Lambda_E^p) \oplus F^r(\Lambda_{\widetilde{E}}^{p-1})$$

again by induction (here $F^r(\mathcal{E}_{\widetilde{X}}^p)$ is the usual Hodge filtration in the smooth case). Alternatively

$$F^r(\Lambda_X^p) = \oplus_l F^r \mathcal{E}_{X_l}^{p-q(l)}\,.$$

It is not hard to see that $d(W_m(\Lambda_X^p)) \subset W_m(\Lambda_X^{p+1})$, $d(F^r(\Lambda_X^p)) \subset F^r(\Lambda_X^p)$; moreover an admissible pullback preserves both the filtrations. As a consequence we get filtrations on the spaces $H^0(X, \Lambda_X^p)$, $Z^p(X) = \{\omega \in H^0(X, \Lambda_X^p) : d\omega = 0\}$, $B^p(X) = dH^0(X, \Lambda_X^{p-1}) \subset H^0(X, \Lambda_X^p)$, and finally on $Z^p(X)/B^p(X) = H^p(X, \mathbb{C})$.

THEOREM 26. *If X is a compact projective variety, for any $\Lambda^{\cdot}_X \in \mathcal{R}(X)$ the weight and the Hodge filtrations induce the Hodge-Deligne mixed structure on the cohomology of X.*

SKETCH OF THE PROOF. By induction on the dimension of X we already know that the conclusion is true for the cohomology of E and $\widetilde{E}$; moreover it is true for the cohomology of $\widetilde{X}$, which is smooth. Then $\Lambda^{\cdot}_X$ coincides (up to shift) with the cone of the morphism of bifiltered complexes

$$(j \circ q)_* \Lambda^p_{\widetilde{E}} \overset{(-1)^p(\psi-\phi)}{\longrightarrow} \pi_* \mathcal{E}^p_{\widetilde{X}} \oplus j_* \Lambda^p_E$$

It follows by [D] that the weight and Hodge filtrations induce a mixed Hodge structure on the cohomology groups $H^p(X, \mathbb{C})$. In other words, once we know that the above filtrations induce the classical pure Hodge structure on the cohomology of smooth projective varieties, and mixed structures on the cohomology of projective varieties of dimension less than $\dim X$, we use the Mayer-Vietoris sequence

$$\cdots \ H^{p-1}(\widetilde{E}, \mathbb{C}) \to H^p(X, \mathbb{C}) \to H^p(\widetilde{X}, \mathbb{C}) \oplus H^p(\widetilde{E}, \mathbb{C}) \to H^p(\widetilde{E}, \mathbb{C}) \ \cdots$$

to prove that the cohomology of X inherits a mixed structure. By the property (F) (filtering) the mixed structure does not depend on the choice of the particular complex $\Lambda^{\cdot}_X$ and by the property (EP) (existence of pullback) it is functorial with respect to X. This is enough to get the conclusion.

In fact some more work, according to the methods of [D] (see also [A]) will show

THEOREM 27. *If X is a compact projective variety, the spectral sequence of the complex $\Gamma(X, \Lambda^{\cdot}_X)$ corresponding to the weight filtration degenerates at E_2.*

As another example we describe the mixed structure on the relative cohomology of a morphism $f : X \to Y$ between projective varieties. Let $\Lambda^{\cdot}_Y \in \mathcal{R}(Y)$. By (EP) (existence of pullback) there exists a $\Lambda^{\cdot}_X \in \mathcal{R}(X)$ and a pullback $\alpha : \Lambda^{\cdot}_Y \to f_* \Lambda^{\cdot}_X$. We consider on Y the complex $\Lambda^{\cdot}_f$ where $\Lambda^p_f = \Lambda^p_Y \oplus f_* \Lambda^{p-1}_X$ and the differential is

$$d(\omega, \eta) = (d\omega, d\eta + (-1)^p \alpha(\omega)) .$$

Then the relative cohomology $H^p(f)$ turns out to be the cohomology of the complex of the global sections of $\Lambda^{\cdot}_f$. The filtrations on $\Lambda^{\cdot}_Y$ and $\Lambda^{\cdot}_X$ induce filtrations on $\Lambda^{\cdot}_f$ and finally on $H^p(f)$. By means of the natural exact sequence

$$\cdots H^{p-1}(X, \mathbb{C}) \to H^p(f) \to H^p(Y, \mathbb{C}) \to H^p(X, \mathbb{C}) \cdots$$

it is possible to define the mixed structure on $H^p(f)$. Again by the properties (F) and (EP) the result does not depend on the choices of $\Lambda^{\cdot}_X$ and the pullback α.

5. – Logarithmic complexes

In this section we show how to modify the previous constructions in order to define complexes of forms with logarithmic poles at infinity, in the sense of Griffiths and Schmid [GS].

By a pair (of complex spaces) (X, Q) we mean the data of a complex space X and of a closed, nowhere dense complex subspace Q. We denote the complement $X \setminus Q$ by X°. More generally, if U is an open subset of X we define $U^\circ = U \setminus Q$. A morphism of pairs $f : (X, Q) \to (Y, R)$ is a morphism $f : X \to Y$ such that $f(Q) \subset R$ and $f(X^\circ) \subset Y^\circ$.

Let (X, Q) be a pair such that X is smooth and Q is a divisor with normal crossing. Let $\Omega^{\cdot}_X\langle\log Q\rangle$ be the complex of holomorphic forms with logarithmic poles along Q. We define the (smooth) logarithmic De Rham complex of the pair (X, Q) by

$$\mathcal{E}^p_X\langle\log Q\rangle = \sum_{k+q=p} \Omega^k_X\langle\log Q\rangle \otimes \mathcal{E}^{0,q}_X$$

equipped with the usual differential (the above tensor product is taken over $\mathcal{O}_X$). The sheaves $\mathcal{E}^p_X\langle\log Q\rangle$ are obviously fine.

A classical result (see [L] for a smooth divisor, and [GS] for the general divisor with normal crossing) states the following

Proposition 28. *Let (X, Q) be a pair such that X is smooth and Q is a divisor with normal crossing; Let $k : X \setminus Q \to X$ be the embedding. The natural morphism of complexes on X*

$$\mathcal{E}^{\cdot}_X\langle\log Q\rangle \to k_*\mathcal{E}^{\cdot}_{X^\circ}$$

induces isomorphisms of the cohomology sheaves:

$$\mathcal{H}^p(\mathcal{E}^{\cdot}_X\langle\log Q\rangle) \to \mathcal{R}^p j_*\mathbb{C}_{X^\circ} \tag{20}$$

in particular for every open subset U of X the natural morphisms

$$H^p(U, \mathcal{E}^{\cdot}_X\langle\log Q\rangle) \to H^p(U^\circ, \mathbb{C})$$

are isomorphisms.

For a pair (X, Q) we define a family of complexes of fine sheaves $\mathcal{R}(X\langle\log Q\rangle) = \{\Lambda^{\cdot}_X\langle\log Q\rangle\}$ and for every morphism $f : (X, Q) \to (Y, R)$ a family of morphisms of complexes between the $\Lambda^{\cdot}_Y\langle\log R\rangle \in \mathcal{R}(Y\langle\log R\rangle)$ and some of the $\Lambda^{\cdot}_X\langle\log Q\rangle \in \mathcal{R}(X\langle\log Q\rangle)$, more precisely morphisms $\Lambda^{\cdot}_Y\langle\log R\rangle \to f_*\Lambda^{\cdot}_X\langle\log Q\rangle$ which we simply denote $\Lambda^{\cdot}_Y\langle\log R\rangle \to \Lambda^{\cdot}_X\langle\log Q\rangle$ and call (*admissible*) *pullback* with the following properties.

(I) The restriction $\Lambda^{\cdot}_{X^\circ}$ of $\Lambda^{\cdot}_X\langle\log Q\rangle$ to X° belongs to $\mathcal{R}(X^\circ)$ (as defined in the previous paragraphs), and the natural morphism of complexes on X

$$\Lambda^{\cdot}_X\langle\log Q\rangle \to k_*\Lambda^{\cdot}_{X^\circ}$$

induces isomorphisms in cohomology:

$$\mathcal{H}^p(\Lambda^{\cdot}_X\langle\log Q\rangle) \to \mathcal{R}^p k_* \mathbb{C}_{X^\circ} \tag{14}$$

(II) For $p > 2\dim X$, $\Lambda^p_X\langle\log Q\rangle = 0$.

(III) If X is smooth and Q is a divisor with normal crossing the logarithmic De Rham complex $\mathcal{E}^{\cdot}_X\langle\log Q\rangle$ belongs to $\mathcal{R}(X\langle\log Q\rangle)$; for every morphism $f : (X, Q) \to (Y, R)$ with Y smooth and R a divisor with normal crossing the ordinary De Rham pullback $f^* : \mathcal{E}^{\cdot}_Y\langle\log R\rangle \to f_*\mathcal{E}^{\cdot}_X\langle\log Q\rangle$ is an admissible pullback.

(IV) Composition, existence, uniqueness and filtering of the admissible pullback are true. We leave to the reader the precise statements.

It is useful to put by definition $\Lambda^p_X\langle\log Q\rangle = 0$ at points $x \in X$ where $Q = X$ (in a neighborhood of x).

We sketch now the construction of the family $\mathcal{R}(X\langle\log Q\rangle)$ for any pair (X, Q).

STEP 1. Let X be smooth, and Q any subspace. Let

$$\begin{array}{ccc} \widetilde{Q} & \xrightarrow{i} & \widetilde{X} \\ q\downarrow & & \pi\downarrow \\ Q & \xrightarrow{j} & X \end{array} \tag{15}$$

be a proper modification,where $\widetilde{X}$ is smooth, $\widetilde{Q}$ is a divisor with normal crossing and π is an isomorphism outside $\widetilde{Q}$; we define

$$\mathcal{E}^{\cdot}_X\langle\log Q\rangle = \pi_*(\mathcal{E}^{\cdot}_{\widetilde{X}}\langle\log \widetilde{Q}\rangle)$$

which satisfies (13); the above complex, which depends on the choice of the diagram (15), will be called a logarithmic De Rham complex of the pair (X, Q); the reader should keep in mind that this gives new complexes even when Q is a divisor with normal crossing.

STEP 2. Let (X, Q) be any pair, $E \subset X$ a nowhere dense closed subspace, with $\mathrm{Sing}(X) \subset E$, and consider as usual a diagram

$$\begin{array}{ccc} \widetilde{E} & \xrightarrow{i} & \widetilde{X} \\ q\downarrow & & \pi\downarrow \\ E & \xrightarrow{j} & X \end{array}$$

Let $\widetilde{Q} = \pi^{-1}(Q)$, $M = E \cap Q$, $\widetilde{M} = \widetilde{E} \cap \widetilde{Q}$. Let $\Lambda^{\cdot}_E\langle\log M\rangle \in \mathcal{R}(E\langle\log M\rangle)$ (which exists by induction on $\dim(X)$). Then we can find $\Lambda^{\cdot}_{\widetilde{E}}\langle\log \widetilde{M}\rangle \in \mathcal{R}(\widetilde{E}\langle\log \widetilde{M}\rangle)$, a logarithmic De Rham complex $\mathcal{E}^{\cdot}_{\widetilde{X}}\langle\log \widetilde{Q}\rangle$ as in step 1), a pullback $\phi : \Lambda^{\cdot}_E\langle\log M\rangle \to \Lambda^{\cdot}_{\widetilde{E}}\langle\log \widetilde{M}\rangle$ (corresponding to q), a pullback ψ :

$\mathcal{E}^{\cdot}_{\widetilde{X}}\langle\log\widetilde{Q}\rangle \to \Lambda^{\cdot}_{\widetilde{E}}\langle\log\widetilde{M}\rangle$ (corresponding to i) with the following property: the complex

$$\Lambda^{\cdot}_{X}\langle\log Q\rangle = \pi_*\mathcal{E}^{\cdot}_{\widetilde{X}}\langle\log\widetilde{Q}\rangle \oplus j_*\Lambda^{\cdot}_{E}\langle\log M\rangle \oplus (j\circ q)_*\Lambda^{\cdot}_{\widetilde{E}}\langle\log\widetilde{M}\rangle(-1)$$

whose differential is by definition

$$d(\omega,\sigma,\theta) = (d\omega, d\sigma, d\theta + (-1)^p(\psi(\omega)-\phi(\sigma))$$

induces the isomorphism of cohomology sheaves (14).

Proof of the surjectivity in (14). Let $x \in Q$ and $\rho = (\omega,\sigma,\theta) \in \Lambda^p_X(U \setminus Q)$ be a d-closed section of $k_*\Lambda^p_{X^\circ}$ in an open neighborhood U of x. Since $d\sigma = 0$, by induction on dimensions there are $\sigma' \in \Lambda^p_E\langle\log M\rangle(U\cap E)$ and $\sigma'' \in \Lambda^{p-1}_E((U\cap E)\setminus Q)$ with $\sigma - d\sigma'' = \sigma'$ (after possibly shrinking U); replacing ρ by $\rho - d(0,\sigma'',0)$ we can suppose that $\sigma \in \Lambda^p_E\langle\log M\rangle(U\cap E)$. A similar argument (using Proposition 1) shows that we can also suppose $\omega \in \mathcal{E}^p_{\widetilde{X}}\langle\log\widetilde{Q}\rangle(U)$. Since $(-1)^p(\psi(\omega)-\phi(\sigma)) = -d\theta$, by induction (we use the injectivity on $\widetilde{E}$) there is $\theta' \in \Lambda^{p-1}_{\widetilde{E}}\langle\log\widetilde{M}\rangle(q^{-1}(U\cap E))$ such that $(-1)^p(\psi(\omega)-\phi(\sigma)) = -d\theta'$. Again by induction, $d(\theta-\theta') = 0$ implies that we can write $\theta - \theta' = d\tilde{\theta} + \theta''$, where $\theta'' \in \Lambda^{p-1}_{\widetilde{E}}\langle\log\widetilde{M}\rangle(q^{-1}(U\cap E))$ and $\tilde{\theta} \in \Lambda^{p-2}_{\widetilde{E}}(q^{-1}(U\cap E)\setminus\widetilde{Q}))$. Finally

$$(\omega,\sigma,\theta) - d(0,0,\tilde{\theta}) = (\omega,\sigma,\theta'+\theta'')$$

which ends the proof of the surjectivity in (14). The proof of the injectivity uses similar arguments and it is left to the reader.

From the construction it follows that for a given complex $\Lambda^{\cdot}_X\langle\log Q\rangle$ there is a uniquely determined family $((X_l, Q_l)), h_l)_{l\in L}$ of pairs (X_l, Q_l) (X_l smooth and Q_l a divisor with normal crossing in X_l) and proper maps of pairs h_l : $(X_l, Q_l) \to (X, Q)$ such that

$$\Lambda^p_X\langle\log Q\rangle = \oplus_l\mathcal{E}^{p-q(l)}_{X_l}\langle\log Q_l\rangle$$

where $q(l)$ is a nonnegative integer; moreover, there exist mappings h_{lm}: $(X_l, Q_l) \to (X_m, Q_m)$, commuting with h_l and h_m, such that the differential $\Lambda^p_X\langle\log Q\rangle \to \Lambda^{p+1}_X\langle\log Q\rangle$ is given by

$$d(\omega_l) = \left(d\omega_l + \sum_m \epsilon^{(p)}_{lm} h^*_{lm}\omega_m\right)$$

where $\epsilon^{(p)}_{lm}$ can take the values $0, \pm 1$.

The family $((X_l, Q_l), h_l)_{l\in L}$ will be called the **hypercovering** of (X, Q) associated to $\Lambda^{\cdot}_X\langle\log Q\rangle$, and $q(l)$ will be the **rank** of (X_l, Q_l).

6. – Further applications and comments

It is easy to see that the complex $D^{\cdot}_X$ of differential forms in sense of Grauert and Grothendieck (see the introduction and [BH]) is in a natural way a subcomplex of $\Lambda^{\cdot}_X$. More precisely for any $\Lambda^{\cdot}_X \in \mathcal{R}(X)$ there is an injective morphism of complexes

$$\eta_X : D^{\cdot}_X \to \Lambda^{\cdot}_X$$

such that for every pullback $\alpha : \Lambda^{\cdot}_Y \to \Lambda^{\cdot}_X$ corresponding to a morphism $f : X \to Y$ the following diagram

$$\begin{array}{ccc} D^{\cdot}_Y & \stackrel{\eta_Y}{\to} & \Lambda^{\cdot}_Y \\ f^* \downarrow & & \alpha \downarrow \\ D^{\cdot}_X & \stackrel{\eta_X}{\to} & \Lambda^{\cdot}_X \end{array} \tag{16}$$

commutes.

With the notations of Definition 2: $\Lambda^{\cdot}_X = \pi_* \mathcal{E}^{\cdot}_{\widetilde{X}} \oplus j_* \Lambda^{\cdot}_E \oplus (j \circ q)_* \Lambda^{\cdot}_{\widetilde{E}}(-1)$ we define

$$\eta_X(\omega) = \pi^*(\omega), \eta_E(j^*(\omega)), 0)$$

where η_E has been defined by induction, and $\pi^*(\omega) \in D^{\cdot}_{\widetilde{X}} = \mathcal{E}^{\cdot}_{\widetilde{X}}$. In other words, looking at the hypercovering $(X_l, h_l)_{l \in L}$, we have $\eta_X(\omega)_l = h_l{}^*(\omega)$ if $q(l) = 0$ and $\eta_X(\omega)_l = 0$ otherwise. Hence $\eta_X(\omega)$ lives on the 0-skeleton of the hypercovering, i.e. on the spaces X_l with $q(l) = 0$. Again by induction we prove $\psi(\pi^*(\omega)) = \phi(\eta_E(j^*(\omega)))$ which implies that η_X commutes to differentials in $D^{\cdot}_X$ and $\Lambda^{\cdot}_X$. Since π^* is injective, η_X is injective too.

Another interesting natural subcomplex of $\Lambda^{\cdot}_X$ is the complex

$$\widetilde{\Lambda}^{\cdot}_X = \{\Omega = (\omega, \sigma, 0) \in \Lambda^{\cdot}_X : \psi(\omega) = \phi(\sigma)\}$$

It is easy to prove that the forms in $\widetilde{\Lambda}^{\cdot}_X$ live on the 0-skeleton of the hypercovering $(X_l, h_l)_{l \in L}$. There is a canonical embedding of $D^{\cdot}_X$ into $\widetilde{\Lambda}^{\cdot}_X$. In [AG7] we prove the following:

– $\widetilde{\Lambda}^{\cdot}_X$ is a fine resolution of $\mathbb{C}_X$;
– every pullback $\alpha : \Lambda^{\cdot}_Y \to \Lambda^{\cdot}_X$ sends $\widetilde{\Lambda}^{\cdot}_Y$ to $\widetilde{\Lambda}^{\cdot}_X$;
– a wedge product can be defined in $\widetilde{\Lambda}^{\cdot}_X$ by the (recursive) formula

$$(\omega_1, \sigma_1, 0) \wedge (\omega_2, \sigma_2, 0) = (\omega_1 \wedge \omega_2, \sigma_1 \wedge \sigma_2, 0)$$

which induces the cup product in cohomology;

– in particular the direct sum $\bigoplus_p H^0(X, \widetilde{\Lambda}^p_X)$ becomes a differential graded algebra, which allows homotopy computations via Sullivan theory [S].

Finally, we remark that it is possible to define a complex $(S^{\cdot}_X, \partial)$ of subanalytic chains, dual to $\Lambda^{\cdot}_X$, whose coomology is the (Borel-Moore) homology of X. It is defined recursively, by $S^p_X = \pi_* \mathcal{S}^p_{\widetilde{X}} \oplus j_* S^{\cdot}_E \oplus (j \circ q)_* S^{p-1}_{\widetilde{E}}$; a form

$\Omega = (\omega, \sigma, \theta)$ can been integrated on a chain $\Gamma = (\alpha, \beta, \gamma)$ by the formula $\int_\Gamma \Omega = \int_\alpha \omega + \int_\beta \sigma + \int_\gamma \theta$. The Stokes formula $\int_\Gamma d\Omega = \int_{\partial\Gamma} \Omega$ holds, giving the duality between cohomology and homology.

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Dipartimento di Matematica
Viale Morgagni 67/A
50134 Firenze (Italy)
ancona@math.unifi.it

UFR de mathématiques
Université Pierre et Marie Curie
4, Place Jussieu
F-75005 Paris (France)
gaveau@ccr.jussieu.fr