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# On $q$-Runge Pairs

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**Abstract.** We show that the converse of the aproximation theorem of Andreotti and Grauert does not hold. More precisely we construct a 4-complete open subset $D \subset \mathbb{C}^6$ (which is an analytic complement in the unit ball) such that the restriction map $H^3(\mathbb{C}^6, \mathcal{F}) \to H^3(D, \mathcal{F})$ has a dense image for every $\mathcal{F} \in Coh(\mathbb{C}^6)$ but the pair $(D, \mathbb{C}^6)$ is not a 4-Runge pair.

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## 1. – Introduction

Let $X$ be a $n$ dimensional $q$-complete complex manifold and $D \subset X$ a $q$-complete open subset. As is well- known (see e.g. [Col1]) the pair $(D, X)$ is called a $q$-Runge pair if for each compact subset $K \subset D$ there exists a $q$-convex exhaustion function $\varphi : X \to \mathbb{R}$ (which may depend on $K$) such that $K \subset \{\varphi < 0\} \subset D$. A main result of Andreotti and Grauert [A-G] asserts that if $(D, X)$ is a $q$-Runge pair then the following holds:

*) for every coherent sheaf $\mathcal{F} \in Coh(X)$ the restrictin map $H^{q-1}(X, \mathcal{F}) \to H^{q-1}(D, \mathcal{F})$ has a dense image.

One could say that a pair $(D, X)$ of $q$-complete manifolds satisfying the assumption *) is a cohomologically $q$-Runge pair. It is then naturally to ask (see [Col]) if the converse of the approximation theorem of Andreotti and Grauert holds; in other words is it true that a cohomologically $q$-Runge pair is a $q$-Runge pair? As is well- known, the answer to this question is yes if $q = 1$ or $q = n$. For $q = n$ this condition of approximation turns out to be equivalent to a purely topological property, namely $X \setminus D$ has no compact connected components (see [Co-Sil] for a complete discussion of this case even on complex spaces with

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singularities). However, for $1 < q < n$, we shall show in this short note that the answer to the above question is negative. More precisely we prove:

THEOREM 1.1. *There exists a* 3*-dimensional closed analytic subset* $A$ *contained in the unit ball* $B \subset \mathbb{C}^6$ *such that the open set* $D := B \setminus A$ *has the following properties:*

1) $D$ *is* 4*-complete*
2) *for every coherent sheaf* $\mathcal{F} \in Coh(\mathbb{C}^6)$ *the restriction map* $H^3(\mathbb{C}^6, \mathcal{F}) \to H^3(D, \mathcal{F})$ *has a dense image*
3) *the pair* $(D, \mathbb{C}^6)$ *is not a* 4*-Runge pair*

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## 2. – Preliminary results

We briefly recall some well-known definitions and results concerning $q$-convexity (see also [A-G], [Col]).

Let $U \subset \mathbb{C}^n$ be an open subset. A smooth function $\varphi \in \mathcal{C}^\infty(U, \mathbb{R})$ is called $q$-convex if its Levi form $\mathcal{L}(\varphi)$ has at least $(n - q + 1)$ strictly positive eigenvalues at any point of $U$. By means of local charts of coordinates this definition can be easily extended to complex manifolds. A complex manifold $X$ is said to be $q$-complete if there exists a smooth $q$-convex function $\varphi : X \to \mathbb{R}$ which is an exhaustion function, i.e. for every $c \in \mathbb{R}$, $\{\varphi < c\} \subset\subset X$. If $q = 1$ then, by Grauert's solution to the Levi problem [Gra], $X$ is 1-complete iff $X$ is a Stein manifold. By Andreotti and Grauert results [A-G] a $q$-complete manifold $X$ satisfies $H^i(X, \mathcal{F}) = 0$ for $i \geq q$, $\mathcal{F} \in Coh(X)$. An open subset $D \subset X$, where $D$ and $X$ are assumed to be $q$-complete, is called $q$-Runge (or equivalently $(D, X)$ is a $q$-Runge pair) if for every compact subset $K \subset D$ there exists a $\mathcal{C}^\infty$ smooth $q$-convex exhaustion function $\varphi : X \to \mathbb{R}$ (which may depend on $K$) such that $K \subset \{\varphi < 0\} \subset D$. By [A-G] a $q$-Runge pair $(D, X)$ satisfies the following approximation property: the restriction map $H^{q-1}(X, \mathcal{F}) \to H^{q-1}(D, \mathcal{F})$ has a dense image for every $\mathcal{F} \in Coh(X)$. By Morse theory it follows easily (see [Fo], [Va]) that a $q$-Runge pair $(D, X)$ satisfies the topological condition:

**) $H_{n+q-1}(X, D; \mathbb{Z})$ is torsion free and $H_{n+j}(X, D; \mathbb{Z}) = 0$ if $j \geq q$.

Unless $q = n = \dim X$ this topological condition is only necessary but not sufficient to guarantee the $q$-Runge condition. In the top degree $q = n$ the topological condition **) is equivalent to each of the following assumptions (see e.g. [Co-Sil]):

i) $H_{2n}(X, D; \mathbb{Z}) = 0$
ii) $X \setminus D$ has no compact connected components
iii) $(D, X)$ is a $n$-Runge pair
iv) $(D, X)$ is a cohomologically $n$-Runge pair

We note that the condition '$H_{n+q-1}(X, D; \mathbb{Z})$ is torsion free' will play an important role in the construction of our counter-example proving Theorem 1.1.

Another important tool needed for the proof of Theorem 1.1 are the results of W.Barth in [Ba] concerning the vanishing of the local cohomology groups $H^i_A$ with supports in an analytic set $A \subset \mathbb{C}^n$ having only one singular point $0 \in A$. In order to state the results of W.Barth we make the following notations:

Let $A$ be a closed analytic subset of $\mathbb{C}^n$ such that $0 \in A$, $A$ has pure dimension $k \geq 2$ and 0 is the only one singular point of $A$. For $\varepsilon > 0$ we denote by $B(\varepsilon)$ the open ball of radious $\varepsilon$ about the origin and by $K_\varepsilon = A \cap \partial B(\varepsilon)$. As well-known for $\varepsilon > 0$ small enough, the cohomology groups $H^i(K_\varepsilon, \mathbb{C})$ of the link $K_\varepsilon$ do not depend on $\varepsilon$. W.Barth [Ba] proves the following theorem concerning the vanishing of the cohomology groups with supports in $A$:

Let $2 \leq p \leq k$; then $H^q_A(B(\varepsilon), \mathcal{O}) = 0$ for $q \geq n - p + 2$ and the separated ${}^\sigma H^{n-p+1}(B(\varepsilon), \mathcal{O})$ (of $H^{n-p+1}(B(\varepsilon), \mathcal{O})$ ) vanishes if and only if $H^0(K_\varepsilon, \mathbb{C}) = \mathbb{C}$ and $H^j(K_\varepsilon, \mathbb{C}) = 0$ for $1 \leq j \leq p - 2$.

We shall apply this result in the following situation: $n = 6$, $p = q = 3$ and the singularity will be the vertex of the cone over the image of the Veronese embedding $\mathbb{P}^2 \hookrightarrow \mathbb{P}^5$.

## 3. – The construction of the example proving Theorem 1.1

We consider the complex space $A$, with an isolated singularity, which is obtained from $\mathbb{C}^3$ by identifying $z$ with $-z$ via the quotient map $\psi : \mathbb{C}^3 \to A$. Note that $A$ can be embedded naturally in $\mathbb{C}^6$. It is exactly the image of the map $\Phi : \mathbb{C}^3 \to \mathbb{C}^6$ given by $(\alpha, \beta, \gamma) \mapsto (\alpha^2, \alpha\beta, \alpha\gamma, \beta^2, \beta\gamma, \gamma^2)$. Observe also that $\psi : \mathbb{C}^3 \to A$ can be identified with $\Phi : \mathbb{C}^3 \to \Phi(\mathbb{C}^3)$. This is a simple computation and so it is ommitted. $A$ has the origin of $\mathbb{C}^6$ as the only one singular point. We are interested to compute the cohomology groups of the link of $A$ at 0. For this we remark that the images of the balls $B^3(0, \varepsilon) \subset \mathbb{C}^3$ by $\psi$, $\varepsilon > 0$, give a fundamental system of good neighbourhoods (in the sense of Prill [Pr]) $V_\varepsilon$ of $0 \in A$ and the boundaries $\partial V_\varepsilon$ of $V_\varepsilon$ are real projective spaces of dimension 5. In particular it follows that the link $K_\varepsilon$ of $A$ at 0 satisfies:$H^0(K_\varepsilon, \mathbb{C}) = \mathbb{C}$, $H^1(K_\varepsilon, \mathbb{C}) = 0$, $H^2(K_\varepsilon, \mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$ (see e.g. [Gre] p. 90).

Let $B^6(0, \varepsilon) \subset \mathbb{C}^6$ be open balls, $\varepsilon > 0$ small enough, $K_\varepsilon = A \cap \partial B^6(0, \varepsilon)$. We fix $\varepsilon_0 > 0$ such that the previous mentioned results of Barth holds for

$B = B^6(0, \varepsilon_0)$ and we denote $D = B \setminus A$. Then $D$ is 4-complete because $A$ can be described in $\mathbb{C}^6$ by 4 holomorphic equations (e.g. if $(x, y, z, u, v, t)$ are coordinates in $\mathbb{C}^6$ then $A$ can be described by the equations $xu = y^2, xt = z^2, ut = v^2, xut = yzv$). We compute now the homology of D. Denote $A' := A \cap B$ and $K := K_{\varepsilon_0}$. From the exact sequence

$$0 = H^i(\bar{A}', \mathbb{Z}) \to H^i(K, \mathbb{Z}) \to H^{i+1}(\bar{A}', K, \mathbb{Z}) = H_c^{i+1}(A', \mathbb{Z}) \to H^{i+1}(\bar{A}', \mathbb{Z}) = 0$$

we get $H^i(K, \mathbb{Z}) = H_c^{i+1}(A', \mathbb{Z})$ for $i \geq 1$ and by Poincaré duality we have $H_c^j(A', \mathbb{Z}) = H_{12-j}(B, D, \mathbb{Z})$. It follows that $H_8(D, \mathbb{Z}) = H_9(B, D, \mathbb{Z}) = H_c^3(A', \mathbb{Z}) = H^2(K, \mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$. Since $H_9(\mathbb{C}^6, D, \mathbb{Z}) = H_8(D, \mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$ has torsion we get, as remarked in the preliminaries that $(\mathbb{C}^6, D)$ cannot be a 4-Runge pair (the condition **) for $n = 6$ and $q = 4$). On the other hand since $H^0(K, \mathbb{C}) = \mathbb{C}$ and $H^1(K, \mathbb{C}) = 0$ the theorem of W.Barth (for $p = k = 3$ and $n = 6$) implies that $H_{A'}^j(B, \mathcal{O}) = 0$ for $j \geq 5$ and ${}^\sigma H_{A'}^4(B, \mathcal{O}) = 0$ where ${}^\sigma H^i$ denotes the separated $H^i/\overline{\{0\}}$ associated to the cohomology space $H^i$. We therefore get $H^j(D, \mathcal{O}) = 0$ for $j \geq 4$ and ${}^\sigma H^3(D, \mathcal{O}) = 0$. So the restriction map $0 = H^3(\mathbb{C}^6, \mathcal{O}) \to H^3(D, \mathcal{O})$ has a dense image. We have to check a similar property for every $\mathcal{F} \in Coh(\mathbb{C}^6)$. For this we note that over $D$ we have an exact sequence of coherent sheaves $0 \to \mathcal{K} \to \mathcal{O}^p \to \mathcal{F} \to 0$ from which we get an exact sequence of cohomology groups: $H^3(D, \mathcal{O}^p) \to H^3(D, \mathcal{F}) \to H^4(D, \mathcal{K}) = 0$ which implies that ${}^\sigma H^3(D, \mathcal{F}) = 0$, hence the resriction map $0 = H^3(\mathbb{C}^6, \mathcal{F}) \to H^3(D, \mathcal{F})$ has a dense image, as required. Theorem 1.1 is completely proved. □

REMARK 3.1. It would be interesting to know if $H^3(D, \mathcal{O}) = 0$, or more generally, if $H^3(D, \mathcal{F}) = 0$ for every $\mathcal{F} \in Coh(D)$. If the answer would be yes then one would get an example of a cohomologically 3-complete open subset $D \subset \mathbb{C}^6$ which is not 3-complete.

REMARK 3.2. Let $X$ be a $q$-complete manifold and $D \subset X$ an open subset which is $q$-complete. In [So] the pair $(D, X)$ is called $q$-Runge (we shall call it weakly $q$-Runge) if the restriction map $H^{q-1}(X, \Omega^p) \to H^{q-1}(D, \Omega^p)$ has a dense image for every $p \geq 0$, where $\Omega^p$ denotes the sheaf of germs of holomorphic $p$-forms. By the previous example provided by Theorem 1.1 we see that the conditions "$q$-Runge" and "weakly $q$-Runge" are not equivalent, we have only the implication "$q$-Runge" $\Longrightarrow$ "weakly $q$-Runge". As observed in [So] a weakly $q$-Runge pair satisfies the topological condition $H_{n+j}(X, D, \mathbb{Z}) = 0$ for $j \geq q$.

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