# Decomposition of CR-Manifolds and Splitting of CR-maps 

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#### Abstract

We show the uniqueness of local and global decompositions of abstract CR-manifolds into direct products of irreducible factors, and a splitting property for their CR-diffeomorphisms into direct products with respect to these decompositions. The assumptions on the manifolds are finite non-degeneracy and finite-type on a dense subset. In the real-analytic case, these are the standard assumptions that appear in many other questions. In the smooth case, the assumptions cannot be weakened by replacing "dense" with "open" as is demonstrated by an example. An application to the cancellation problem is also given. The proof is based on the development of methods of [BER99b], [BRZ00], [KZ01] and the use of "approximate infinitesimal automorphisms" introduced in this paper.


Mathematics Subject Classification (2000): 32V05 (primary), 32V35, 32G07 (secondary).

## 1. - Introduction

Decompositions of various types of manifolds into direct products of submanifolds play an important role in their study. For instance, for semisimple Lie groups and for symmetric spaces, such decompositions are crucial for the classification. In Riemannian geometry such a decomposition is known as de Rham decomposition (see [KN96]). In all these cases the corresponding decomposition is unique unless there are present so-called "flat factors" whose classification is simple. Most geometric and functional-theoretic questions for the manifolds then are reduced to the irreducible factors.

In this paper we study local decompositions of germs of (abstract) CRmanifolds into irreducible factors as well as their global analogues and establish their uniqueness. Here the role of "flat factors" is played by Levi-flat directions, where, in general, "higher order Levi forms" have to be taken into account. The simplest example is given by the Levi-flat manifold $M=\mathbb{R} \times \mathbb{C}$ where

The second author was supported by postdoctoral fellowship program from Korea Science and Engineering Foundation (KOSEF).
Pervenuto alla Redazione il 18 marzo 2002.
the choice of the direction of $\mathbb{R}$ is obviously not unique. There are different known nondegeneracy conditions to exclude such phenomenon, most of them are usually formulated for real-analytic CR-manifolds. Those that seem to be the easiest to transfer to the smooth case and also the easiest to compute are the condition of finite nondegeneracy (see [H83], [BHR96], [E98]) and of finite type (in the sense of Kohn [K72] and Bloom-Graham [BG77]). We refer to Section 2 for main definitions and mention here only that finite nondegeneracy and finite type are implied respectively by the nondegeneracy of the Levi form and by the condition that the span of all Levi form values is of maximal possible dimension.

Our main result states that, in order to have the unique decomposition property, it is sufficient to require finite nondegeneracy and finite type only on a dense subset. In this paper "smooth" will always mean $C^{\infty}$. A germ of a smooth CR-manifold is called irreducible, if it is not CR-diffeomorphic to a direct product of two germs of smooth CR-manifolds of positive dimension. We prove:

Thorem 1.1. Let $(M, p)$ be a germ of a smooth CR-manifold which is finitely nondegenerate and of finite type on a dense subset. Then, up to permutations, there exists a unique decomposition

$$
(M, p) \cong\left(M_{1}, p_{1}\right) \times \cdots \times\left(M_{m}, p_{m}\right)
$$

where each germ $\left(M_{j}, p_{j}\right)$ is irreducible. Furthermore, if $f$ is a (germ of a) smooth local CR-diffeomorphism between $(M, p)$ and another (germ of a) smooth CRmanifold $\left(M^{\prime}, p^{\prime}\right)$ and if $\left(M^{\prime}, p^{\prime}\right) \cong\left(M_{1}^{\prime}, p_{1}^{\prime}\right) \times \cdots \times\left(M_{m^{\prime}}^{\prime}, p_{m^{\prime}}\right)$ is the corresponding decomposition into irreducible factors, then $m=m^{\prime}$ and, after a permutation of the factors $\left(M_{j}^{\prime}, p_{j}^{\prime}\right), f$ factors as a direct product of the form $f=f^{1} \times \cdots \times f^{m}$, where $f^{j}:\left(M_{j}, p_{j}\right) \rightarrow\left(M_{j}^{\prime}, p_{j}^{\prime}\right)$ are (germs of) local CR-diffeomorphisms for $j=$ $1, \ldots, m$.

If $M$ is real-analytic, the assumption of finite nondegeneracy on a dense subset in Theorem 1.1 is equivalent to holomorphic nondegeneracy of its local analytic CR-embedding, i.e. to the nonexistence of holomorphic local oneparameter automorphism group (see [BER96], [BER99a]). This assumption is optimal for points in general position in the following sense. If there is no dense subset where $M$ is finitely nondegenerate, then the existence of a local one-parameter automorphism group implies that at a point of general position $M$ is locally CR-isomorphic to a product of $\mathbb{C}$ and another CR-manifold. In this case it is easy to see that the decomposition in Theorem 1.1 at such a point is never unique.

If, on the other hand, there is no dense subset where $M$ is of finite type, the situation is reduced to the finite type case by considering the CR-orbits (see [N66], [S73], [BER99a]).

In case $M$ is real-analytic, the assumptions in Theorem 1.1 are clearly equivalent to $M$ being finitely nondegenerate and of finite type at some sequence of points converging to $p$. If $M$ is merely smooth, the second condition is
essentially weaker and is not sufficient for the conclusion of Theorem 1.1 to hold as Example 2.1 below shows.

Our next result is the following global version of Theorem 1.1. Here we call a CR-manifold (globally) irreducible if it is not CR-diffeomorphic to a direct product of two smooth CR-manifolds of positive dimension.

Theorem 1.2. Let $M$ be a smooth CR-manifold which is finitely nondegenerate and of finite type on a dense subset. Then, up to permutations, there exists a unique decomposition

$$
M \cong M_{1} \times \cdots \times M_{m}
$$

where each $M_{j}$ is irreducible. Furthermore, if $f$ is a smooth CR-diffeomorphism between $M$ and another smooth CR-manifold $M^{\prime}$ and if and $M^{\prime} \cong M_{1}^{\prime} \times \cdots \times M_{m^{\prime}}^{\prime}$ is the corresponding decomposition, then $m=m^{\prime}$ and after a permutation of factors of $M^{\prime}, f$ factors as a direct product of the form $f=f^{1} \times \cdots \times f^{m}$, where $f^{j}: M_{j} \rightarrow M_{j}^{\prime}$ are smooth CR-diffeomorphisms.

Again, also here Example 2.1 shows that the assumptions cannot be weakened by replacing a dense subset by an open subset. As immediate applications of Theorems 1.1 and 1.2 we obtain the following cancellation result:

Corollary 1.3. Let $M_{1}, M_{2}$ and $S$ be CR-manifolds that are finitely nondegenerate and of finite type on their dense subsets. If $M_{1} \times S$ and $M_{2} \times S$ are CR-diffeomorphic, then $M_{1}$ and $M_{2}$ are also CR-diffeomorphic. Furthermore, iffor some points $p_{1} \in M_{1}, p_{2} \in M_{2}, s \in S,\left(M_{1}, p_{1}\right) \times(S, s)$ and $\left(M_{2}, p_{2}\right) \times(S, s)$ are CR-diffeomorphic, then also $\left(M_{1}, p_{1}\right)$ and $\left(M_{2}, p_{2}\right)$ are CR-diffeomorphic.

A key ingredient of the proofs of Theorems 1.1 and 1.2 consists of establishing a rigidity property for local CR-diffeomorphisms (Proposition 4.1) that roughly states that, under the assumptions of Theorem 1.1, any smooth family of local diffeomorphisms that is CR in all arguments, is necessarily constant. The proof of this fact is based on a realization of the space of infinitesimal CR-automorphisms as a totally real subspace in a suitable jet space. For (not infinitesimal) CR-automorphisms of real-analytic CR-manifolds fixing a reference point, such a realization has been obtained by Baouendi-EbenfeltRothschild [BER99b, Theorem 4]. The method of [BER99b] is based on the local complexification of real-analytic CR-manifolds that may not exist for general abstract CR-manifolds as in our case. In [KZ01] the second and the third authors proposed a method of approximate local complexification used to obtain jet parametrizations of local CR-automorphisms and even of local automorphisms that are CR only up to some finite order. Using this method we can reduce the problem to the real-analytic case but, as in [KZ01], after such a reduction we have to consider not only infinitesimal CR-diffeomorphisms of the corresponding submanifold $\widetilde{M} \subset \mathbb{C}^{N}$ but also more general holomorphic vector fields that preserve $\widetilde{M}$ only up to finite order (at the reference point). We call these vector fields "approximate infinitesimal automorphisms". In contrast to usual infinitesimal automorphisms, the local flow of an approximate infinitesimal automorphism may not consist even of approximate automorphisms since they
may not send the reference point into a point of $\widetilde{M}$. Thus we cannot reduce the problem to (not infinitesimal) automorphisms and instead adapt the technique of [BER99b], [BRZ00], [KZ01] directly to our case.

An outline of the paper is as follows. In Section 2 we review basic facts and definitions for CR-manifolds and give an example showing that the assumption of finite nondegeneracy and of finite type in Theorems 1.1 and 1.2 cannot be replaced by the same assumptions on a sequence of points. In Section 3 we define approximate infinitesimal automorphisms and establish their totally real realizations in jet spaces. In Section 4 we prove the rigidity property for local CR-automorphisms mentioned above. Finally, Section 5 and Section 6 are devoted to the proofs of Theorems 1.1 and 1.2. The arguments in these parts are partially inspired by [U81].

## 2. - Preliminaries and an example

Recall that an (abstract) smooth CR-manifold is a smooth manifold $M$ together with an involutive subbundle $T^{0,1} M$ of the complexified tangent bundle $T M \otimes \mathbb{C}$ such that $T^{0,1} M \cap T^{1,0} M=0$, where $T^{1,0} M=\overline{T^{0,1} M}$. (Involutivity means here that Lie brackets of vector fields in $T^{0,1} M$ are again in $T^{0,1} M$.) Instead of prescribing $T^{0,1} M$ one can also consider a real subbundle $T^{c} M$ of $T M$ together with a complex structure $J$ on $T_{p}^{c} M$ for each $p \in M$ depending smoothly on $p$. The relation between $T^{0,1} M$ and $\left(T^{c} M, J\right)$ is given by $T^{0,1} M=$ $\left\{\xi+i J \xi: \xi \in T^{c} M\right\}$. The reader is referred to the books [B91], [BER99a] for basic properties of CR-manifolds.

A CR-manifold $M$ is said to be of finite type at a point $p$ (in the sense of Kohn [K72] and Bloom-Graham [BG77]) if all vector fields in $T^{0,1} M$ and $T^{1,0} M$ span together with their commutators the maximal possible space $T_{p} M \otimes \mathbb{C}$. The type $v$ of $M$ at $p$ is the minimal length of commutators needed to span the maximal space. In this case we say that $M$ is of type $v$ at $p$ or $(M, p)$ is of type $\nu$.

A CR-manifold $M$ is called finitelynondegenerate at $p$ (see [H83], [BHR96], [E98] and also [BER99a, Section 11.1]) if, for some integer $k \geq 1$,

$$
\begin{align*}
& \operatorname{span}_{\mathbb{C}}\left\{\mathcal{T}_{L_{s}}\left(\ldots \mathcal{T}_{L_{2}}\left(\mathcal{T}_{L_{1}} \theta\right) \ldots\right)(p): 0 \leq s\right. \\
& \left.\quad \leq k, L_{j} \in \Gamma\left(T^{0,1} M\right), \theta \in \Gamma\left(T^{* 0} M\right)\right\}=T_{p}^{* 1,0} M, \tag{2.1}
\end{align*}
$$

where $T^{* 0} M$ and $T^{* 1,0} M$ denote the bundles of complex 1-forms that vanish on $T^{c} M \times \mathbb{C}$ and on $T^{0,1} M$ respectively and $\mathcal{T}_{L}$ is the Lie derivative along $L$. Recall that for any $(0,1)$ vector field $L$, the Lie derivative $\mathcal{T}_{L}$ leaves the space $\Gamma\left(T^{* 1,0} M\right)$ invariant and is given there by $\mathcal{T}_{L} \omega=i_{L} d \omega$, where $i_{L}$ denotes the contraction. If the number $k$ is minimal with the above property, we say that $M$ is $k$-nondegenerate at $p$ or $(M, p)$ is $k$-nondegenerate.

The following example shows that the conclusion of Theorem 1.1 may not hold if $(M, p)$ is only assumed to be finitely nondegenerate and of finite type at a sequence of points converging to $p$.

EXAMPLE 2.1. Let $M_{0} \subset \mathbb{C}_{z, w}^{2}$ be given by $\operatorname{Im} w=\lambda(\operatorname{Re} w) z \bar{z}$, where $\lambda(x)$ is a smooth function on $\mathbb{R}$ that is zero for $x \leq 0$ and positive for $x>0$ and let $M_{1} \subset \mathbb{C}_{z, w}^{2}$ be the quadric $\operatorname{lm} w=z \bar{z}$. Then $M:=M_{0} \times M_{1}$ is finitely nondegenerate (even Levi-nondegenerate) and of finite type at every point $(0, x, a, b) \in M$ with $x>0$. However, the obvious decomposition of $(M, 0)$ as $\left(M_{0}, 0\right) \times\left(M_{1}, 0\right)$ is not unique. Indeed, let $\varphi$ be a smooth real function on $\mathbb{R}$ that is one for $x \geq 0$ and greater than one for $x<0$. Then the map

$$
\left(z_{0}, w_{0}, z_{1}, w_{1}\right) \mapsto\left(z_{0}, w_{0}, \varphi\left(\operatorname{Re} w_{0}\right) z_{1},\left(\varphi\left(\operatorname{Re} w_{0}\right)\right)^{2} w_{1}\right)
$$

defines a CR-automorphism of $M$ that does not preserve the given splitting $M=M_{0} \times M_{1}$.

## 3. - Approximate infinitesimal automorphisms

We begin by considering a germ $(M, p)$ of a generic real-analytic submanifold in $\mathbb{C}^{N}$ with a vector-valued defining function $r=\left(r^{1}, \ldots, r^{d}\right)$. Recall that an infinitesimal automorphism of $(M, p)$ in $\mathbb{C}^{N}$ is a germ at $p$ of a holomorphic vector field $L$ on $\mathbb{C}^{N}$ such that $\operatorname{Re} L$ is tangent to $M$, i.e. $\operatorname{Re} L r=0$ on $M$. More generally, we introduce the notion of an approximate infinitesimal automorphisms of a given order $k$. By definition, an approximate infinitesimal automorphism of $(M, p)$ oforder $k$ is a germ at $p$ of a holomorphic vector field $L$ on $\mathbb{C}^{N}$ satisfying

$$
\operatorname{Re} \operatorname{Lr}(x)=o\left(|x-p|^{k}\right) \quad \text { as } \quad x \in M \rightarrow p
$$

We denoted by $\mathrm{aut}^{k}(M, p)$ the vector space of all approximate infinitesimal automorphisms of $(M, p)$ of order $k$.

For a germ of a holomorphic map $f$ at $p \in \mathbb{C}^{N}$, denote by $j_{p}^{k} f$ its $k$-jet at $p$. Also denote by $J_{p}^{k}\left(\mathbb{C}^{N}, \mathbb{C}^{N}\right)$ the $k$-th jet space of holomorphic self maps of $\mathbb{C}^{N}$ at $p$. Our goal in this section is to prove the following property that may be of independent interest:

Proposition 3.1. Let $(M, p)$ be a germ of a real-analytic generic submanifold in $\mathbb{C}^{N}$ of codimension d. Suppose $(M, p)$ is l-nondegenerate and of type $v$. Then for any $k \geq(2 d(v-1)+2)(2 d+3) l$, the image of aut ${ }^{k}(M, p)$ under the jet evaluation map

$$
j_{p}^{(2 d+3) l}: \operatorname{aut}^{k}(M, p) \rightarrow J_{p}^{(2 d+3) l}\left(\mathbb{C}^{N}, \mathbb{C}^{N}\right)
$$

is a totally real linear subspace of $J_{p}^{(2 d+3) l}\left(\mathbb{C}^{N}, \mathbb{C}^{N}\right)$.

Proof. Without loss of generality we may assume $p=0$. Let $L:=$ $\sum_{j} \xi^{j} \frac{\partial}{\partial z_{j}} \in \operatorname{aut}^{k}(M, p)$ be an approximate infinitesimal automorphism and let $\left\{\theta_{t}: t \in(-\varepsilon, \varepsilon)\right\}$ be its local flow defined in a neighborhood of 0 in $\mathbb{C}^{N}$. Since $L$ is holomorphic, $\theta_{t}(z)$ is holomorphic in $z$ and real-analytic in $t$. Consider the power series expansion

$$
\theta_{t}(z)=\sum_{j \geq 0} \Theta_{j}(z) t^{j}
$$

of $\theta_{t}(z)$ with respect to $t$, where $\Theta_{j}: \mathbb{C}^{N} \rightarrow \mathbb{C}^{N}$ is the germ of a holomorphic map at 0 . Then $\Theta_{0}=$ id and $\Theta_{1}=\left(\xi^{1}, \ldots, \xi^{N}\right)$.

Now let $r(z, \bar{z})=\left(r^{1}(z, \bar{z}), \ldots, r^{d}(z, \bar{z})\right)$ be a real-analytic defining function of $(M, p)$ and let

$$
h(z, \bar{z}, t):=r\left(\theta_{t}(z), \overline{\theta_{t}(z)}\right)
$$

Consider the power series expansion

$$
h(z, \bar{z}, t)=\sum_{j \geq 0} h_{j}(z, \bar{z}) t^{j}
$$

of $h$ with respect to $t$. Since

$$
r\left(\theta_{t}(z), \overline{\theta_{t}(z)}\right)=r(z, \bar{z})+2 t \operatorname{Re} \operatorname{Lr}(z, \bar{z})+o(|t|) \quad \text { as } \quad t \rightarrow 0
$$

we have $h_{0} \equiv 0$ on $M$ and $h_{1}(z, \bar{z})=o\left(|z|^{k}\right)$ as $z \rightarrow 0$ in $M$ by the assumptions. Then, by the standard complexification argument, we obtain on the complexification $\mathcal{M}:=\{(z, \zeta) \in U \times \bar{U}: r(z, \zeta)=0\}$ of $M$, where $U$ is a sufficiently small neighborhood of 0 in $\mathbb{C}^{N}$,

$$
\begin{equation*}
r\left(\theta_{t}(z), \bar{\theta}_{t}(\zeta)\right)=h(z, \zeta, t)=\sum_{j \geq 1} h_{j}(z, \zeta) t^{j} \tag{3.1}
\end{equation*}
$$

such that

$$
h_{1}(z, \zeta)=o\left(|(z, \zeta)|^{k}\right) \quad \text { as } \quad(z, \zeta) \in \mathcal{M} \rightarrow 0
$$

Choose a linear basis of real-analytic $(0,1)$ vector fields $L_{1}, \ldots, L_{n}$, on $M$ near 0 , where $n=N-d$. By a slight abuse of notation we write the same letters for their complexifications on $\mathcal{M}$. For a multi-index $\alpha=\left(\alpha_{1}, \ldots, \alpha_{n}\right)$, write $|\alpha|:=\alpha_{1}+\cdots+\alpha_{n}$ and $L^{\alpha}:=L_{1}^{\alpha_{1}} \cdots L_{n}^{\alpha_{n}}$. Since $M$ is $l$-nondegenerate and $\theta_{t}$ is invertible at 0 , we have near 0 the following span condition (see [BER99a, Proposition 11.2.4])

$$
\operatorname{span}\left\{L^{\alpha} r_{z}^{m}\left(\theta_{t}(z), \overline{\theta_{t}(z)}\right): 1 \leq m \leq d,|\alpha| \leq l\right\}=\mathbb{C}^{N}
$$

where $r_{z}^{m}:=\left(\frac{\partial r^{m}}{\partial z_{1}}, \ldots, \frac{\partial r^{m}}{\partial z_{N}}\right) \in \mathbb{C}^{N}$ is the gradient of the $m$ th component of $r$.

Then by applying the operators $L^{\alpha}$ for $|\alpha| \leq l$ to (3.1) and using the implicit function theorem, we obtain as in [KZ01, Section 4.1] the basic reflection identity

$$
\begin{equation*}
\theta_{t}(z)=\Psi\left(z, \zeta, j_{\zeta}^{l} \bar{\theta}_{t}\right)+h^{0}(z, \zeta, t), \quad(z, \zeta) \in \mathcal{M} \tag{3.2}
\end{equation*}
$$

where $\Psi$ is a holomorphic map defined in a neighborhood of $\left(0,0, j_{0}^{l}\right.$ id) in $\mathbb{C}^{N} \times J^{l}\left(\mathbb{C}^{N}, \mathbb{C}^{N}\right)$ which is independent of $\theta_{t}$, and $h^{0}=\sum_{j \geq 1} h_{j}^{0}(z, \zeta) t^{j}$ is a holomorphic map (depending on $\theta_{t}$ ) defined in a neighborhood of 0 in $\mathbb{C}^{N} \times$ $\mathbb{C}^{N} \times(-\varepsilon, \varepsilon)$ such that

$$
h_{1}^{0}(z, \zeta)=o\left(|(z, \zeta)|^{k-l}\right) \quad \text { as } \quad(z, \zeta) \in \mathcal{M} \rightarrow 0
$$

Moreover, differentiating (3.2) in $z$, we obtain for $\tau \leq k-l$,

$$
\begin{equation*}
j_{z}^{\tau} \theta_{t}=\Psi^{\tau}\left(z, \zeta, j_{\zeta}^{\tau+l} \bar{\theta}_{t}\right)+h^{\tau}(z, \zeta, t), \quad(z, \zeta) \in \mathcal{M} \tag{3.3}
\end{equation*}
$$

where $h^{\tau}(z, \zeta, t)=\sum_{j \geq 1} h_{j}^{\tau}(z, \zeta) t^{j}$ is a holomorphic map satisfying

$$
h_{1}^{\tau}(z, \zeta)=o\left(|(z, \zeta)|^{k-(l+\tau)}\right) \quad \text { as } \quad(z, \zeta) \in \mathcal{M} \rightarrow 0
$$

For every positive integer $\mu$, define the iterated complexification $\mathcal{M}^{\mu}$ of order $\mu$ as follows (see [Z97], [Z99], [KZ01]). Let $\mathcal{M}^{\mu}$ be the connected component of $\left\{\left(\zeta^{\mu}, \ldots, \zeta^{1}, \zeta^{0}\right) \in \mathbb{C}^{(\mu+1) N}: r_{j}\left(\zeta^{j}, \zeta^{j-1}\right)=0, j=1, \ldots, \mu\right\}$ containing 0 , where

$$
r_{j}\left(\zeta^{j}, \zeta^{j-1}\right):= \begin{cases}r\left(\zeta^{j}, \zeta^{j-1}\right) & \text { if } j \text { is odd } \\ \bar{r}\left(\zeta^{j}, \zeta^{j-1}\right) & \text { if } j \text { is even }\end{cases}
$$

Then by iterating (3.3) $2 d+3$ times and evaluating at $\zeta^{0}=0$, we obtain

$$
\theta_{t}(z)=\hat{\Psi}\left(z, \mathcal{B}, j_{0}^{(2 d+3) l} \bar{\theta}_{t}\right)+\hat{h}(z, \mathcal{B}, t), \quad(z, \mathcal{B}) \in \mathcal{M}^{2 d+3} \cap\left\{\zeta^{0}=0\right\}
$$

where $\hat{\Psi}$ is independent of $\theta_{t}, \mathcal{B}=\left(\zeta^{2 d+2}, \ldots, \zeta^{0}\right)$ and $\hat{h}(z, \mathcal{B}, t)=\sum_{j \geq 1} \hat{h}_{j}(z, \mathcal{B}) t^{j}$ is such that

$$
\hat{h}_{1}(z, \mathcal{B})=o\left(|(z, \mathcal{B})|^{k-(2 d+3) l}\right), \quad(z, \mathcal{B}) \in \mathcal{M}^{2 d+3} \cap\left\{\zeta^{0}=0\right\} \rightarrow 0
$$

Since $M$ is of type $v$, by similar arguments as in [KZ01, Secion 4.2], we obtain a singular jet parametrization

$$
\theta_{t}(z)=\check{\Psi}\left(\lambda, \frac{z}{\lambda^{m}}, j_{0}^{(2 d+3) l} \bar{\theta}_{t}\right)+\check{h}\left(\lambda, \frac{z}{\lambda^{m}}, t\right)
$$

for some integer $m \leq 2 d(v-1)$, where $\lambda \in \mathbb{C}, \check{\Psi}$ is a holomorphic map in a neighborhood of $\left(0,0, j_{0}^{(2 d+3) l}\right.$ id) in $\mathbb{C} \times \mathbb{C}^{N} \times J_{0}^{(2 d+3) l}\left(\mathbb{C}^{N}, \mathbb{C}^{N}\right)$ independent
of $\theta_{t}$ and $\check{h}(\lambda, \tilde{z}, t)=\sum_{j \geq 1} \check{h}_{j}(\lambda, \tilde{z}) t^{j}$ is a holomorphic map in a neighborhood of 0 in $\mathbb{C} \times \mathbb{C}^{N} \times(-\varepsilon, \varepsilon)$ such that

$$
\check{h}_{1}(\lambda, \tilde{z})=o\left(|(\lambda, \tilde{z})|^{k-(2 d+3) l}\right) \quad \text { as } \quad(\lambda, \tilde{z}) \rightarrow 0
$$

Then by [KZ01, Lemma 4.4], we have

$$
\begin{equation*}
\theta_{t}(z)=\Phi\left(z, j_{0}^{(2 d+3) l} \bar{\theta}_{t}\right)+\tilde{h}(z, t), \tag{3.4}
\end{equation*}
$$

where $\Phi$ is a holomorphic map in a neighborhood of $\left(0, j^{(2 d+3) l}\right.$ id) in $\mathbb{C}^{N} \times$ $J_{0}^{(2 d+3) l}\left(\mathbb{C}^{N}, \mathbb{C}^{N}\right)$ and $\tilde{h}(z, t)=\sum_{j \geq 1} \tilde{h}_{j}(z) t^{j}$ is a holomorphic map in a neighborhood of 0 in $\mathbb{C}^{N} \times(-\varepsilon, \varepsilon)$ such that

$$
\tilde{h}_{1}(z)=o\left(|z|^{\frac{k-(2 d+3) l}{m+1}}\right) \quad \text { as } \quad z \in \mathbb{C}^{N} \rightarrow 0
$$

We differentiate (3.4) in $t$ at $t=0$ to obtain a jet parametrization

$$
\begin{equation*}
\xi(z)=\hat{\Phi}\left(z, j_{0}^{(2 d+3) l} \bar{\xi}\right)+o\left(|z|^{\frac{k-(2 d+3) l}{m+1}}\right) \quad \text { as } \quad z \in \mathbb{C}^{N} \rightarrow 0 \tag{3.5}
\end{equation*}
$$

where $\xi:=\left(\xi^{1}, \ldots, \xi^{N}\right)$ denote the components of the original infinitesimal automorphism $L$. Since $m \leq 2 d(v-1)$ and

$$
k \geq(2 d(v-1)+2)(2 d+3) l \geq(m+2)(2 d+3) l
$$

differentiation of (3.5) $(2 d+3) l$ times in $z$ at $z=0$ yields

$$
j_{0}^{(2 d+3) l} \xi=\check{\Phi}\left(j_{0}^{(2 d+3) l} \bar{\xi}\right)
$$

where $\check{\Phi}$ is a holomorphic map in a neighborhood of $j_{0}^{(2 d+3) l}$ id in $J_{0}^{(2 d+3) l}$ $\left(\mathbb{C}^{N}, \mathbb{C}^{N}\right)$. Hence we have $\zeta=\check{\Phi}(\bar{\zeta})$ for any $\zeta \in j_{0}^{(2 d+3) l}\left(\operatorname{aut}^{k}(M, p)\right)$ and therefore $j_{0}^{(2 d+3) l}\left(\operatorname{aut}^{k}(M, p)\right) \subset J_{0}^{(2 d+3) l}\left(\mathbb{C}^{N}, \mathbb{C}^{N}\right)$ is totally real.

## 4. - Rigidity properties of CR-families of automorphisms

Our next step in proving Theorems 1.1 and 1.2 consists of establishing rigidity properties for local CR-families of automorphisms given as germs of smooth CR-maps $\varphi:(S, a) \times(M, p) \rightarrow(M, p)$, where $(S, a)$ and $(M, p)$ are CR-manifolds. By rigidity here we mean the following property:

Proposition 4.1. Let ( $S, a$ ) and ( $M, p$ ) be germs of smooth CR-manifolds that are finitely nondegenerate and offinite type on dense subsets. If $\varphi:(S, a) \times(M, p) \rightarrow$ $(M, p)$ is a germ of a smooth CR-map such that $\varphi(a, \cdot)=\mathrm{id}$, then $\varphi(s, \cdot)=$ id for all $s \in S$ sufficiently close to $a$.

In fact, it will follow from the proof that the same conclusion holds under the weaker assumption that $(S, a)$ is only minimal (in the sense of Tu manov [T88]) on a dense subset.

Proof. Let $d$ be the CR-codimension of $(M, p)$, i.e. the codimension of the complex tangent space $T_{p}^{c} M$ in $T_{p} M$. We first assume that $M$ is finitely nondegenerate and of finite type at $p$ (and not only on a dense subset). Choose $l$ such that $M$ is $l$-nondegenerate at $p$. It is shown in [KZ01] that, for any invertible jet $(p, \lambda) \in J^{2(d+1) l}(M, M)$, there exists a $J^{2(d+1) l+1}(M, M)$-valued smooth function $\Phi$ defined in a neighborhood of $(p, \lambda)$ such that all germs of smooth CR-diffeomorphisms $f: M \rightarrow M$ at any $q \in M$ with $\left(q, j_{q}^{2(d+1) l} f\right)$ sufficiently close to $(p, \lambda)$, satisfy a complete differential system

$$
j_{x}^{2(d+1) l+1} f=\Phi\left(x, j_{x}^{2(d+1) l} f\right)
$$

for $x$ sufficiently close to $q$.
Now choose $X \in \operatorname{aut}(M, p)$ and let $\left\{\theta_{t}: t \in(-\varepsilon, \varepsilon)\right\}$ be its local flow. Then there is a neighborhood $U$ of $p$ in $M$ such that for all $t \in(-\varepsilon, \varepsilon), \theta_{t}$ is well-defined in $U$ and satisfies

$$
\begin{equation*}
j_{x}^{2(d+1) l+1} \theta_{t}=\Phi\left(x, j_{x}^{2(d+1) l} \theta_{t}\right) \tag{4.1}
\end{equation*}
$$

for all $x \in U$. By differentiating (4.1) in $t$ at $t=0$, we obtain a complete differential system

$$
j_{x}^{2(d+1) l+1} X=\Psi\left(x, j_{x}^{2(d+1) l} X\right), x \in U
$$

for $X$, where $\Psi$ is a $J^{2(d+1) l+1}(M, T M)$-valued smooth function defined in a neighborhood of $\left(p, j_{p}^{2(d+1) l} X\right)$ in the space $J^{2(d+1) l}(M, T M)$ of $2(d+1) l$-jets of vector fields on $M$. As a consequence, we obtain finite jet determination for infinitesimal CR-automorphisms: if $X, Y \in \operatorname{aut}(M, p)$ and $j_{p}^{2(d+1) l} X=$ $j_{p}^{2(d+1) l} Y$, then $X \equiv Y$.

Let $\varphi:(S, a) \times(M, p) \rightarrow(M, p)$ be a germ of a CR-map satisfying the assumptions of the Proposition 4.1. For any (1,0) vector field $L$ on $S$ in a neighborhood of $a$ such that $L(a) \neq 0$, define one-parameter families $\left\{\theta_{t}: t \in\right.$ $(-\varepsilon, \varepsilon)\}$ and $\left\{\eta_{t}: t \in(-\varepsilon, \varepsilon)\right\}$ of local CR-diffeomorphisms of $M$ by

$$
\theta_{t}(x):=\varphi\left(\sigma_{1}(t), x\right), \quad \eta_{t}(x):=\varphi\left(\sigma_{2}(t), x\right),
$$

where $\sigma_{1}$ and $\sigma_{2}$ are the integral curves of $\operatorname{Re} L$ and $\operatorname{Im} L$, respectively such that $\sigma_{1}(0)=\sigma_{2}(0)=a$. Therefore $\dot{\theta}_{0}$ and $\dot{\eta}_{0}$ are infinitesimal CR-automorphisms, where the dot denotes the derivative in $t$. Moreover, by the definition of $\theta_{t}$ and $\eta_{t}$, we have $\dot{\theta}_{0}, \dot{\eta}_{0} \in \Gamma\left(M, T^{c} M\right)$. On the other hand,

$$
\dot{\eta}_{0}=\varphi_{*}\left(\dot{\sigma}_{2}(0), x\right)=\varphi_{*}(\operatorname{lm} L(a), x)=J \varphi_{*}(\operatorname{Re} L(a), x)=J \dot{\theta}_{0}
$$

where $J$ is the complex structure on $T^{c} M$. Hence we obtain an infinitesimal automorphism $X:=\dot{\theta}_{0} \in \operatorname{aut}(M, p) \cap \Gamma\left(M, T^{c} M\right)$ such that also $J X \in \operatorname{aut}(M, p)$.

Now set $k:=(2 d(v-1)+2)(2 d+3) l$ as in Proposition 3.1, where $v$ is the type of $M$ at $p$. By [KZ01, Proposition 3.1], we can choose a neighborhood $U$ of $p$ and a smooth embedding $\psi: U \rightarrow \mathbb{C}^{n+d}, n=\operatorname{dim}_{C R} M$, which is CR of order $k+1$ at $p$, i.e. for any $(0,1)$ vector field $L$ on $M$ defined near $p$, $L \psi(x)=o\left(|x-p|^{k}\right)$ as $x \rightarrow p$, and such that $\psi(p)=0$ and $\psi(U)$ is a generic real-analytic submanifold of codimension $d$.

We write $\psi_{*}(X)=\operatorname{Re} \sum_{j=1}^{n+d} \xi^{j} \frac{\partial}{\partial z_{j}}$. Since $\psi$ is CR of order $k+1$ at $p$, it follows that each $\xi^{j}$ is CR of order $k$ at 0 . Therefore we can choose a holomorphic vector field $\sum_{j=1}^{n+d} \widetilde{\xi}^{j} \frac{\partial}{\partial z_{j}} \in \operatorname{aut}^{k}(\psi(M), 0)$ such that $j_{0}^{k}\left(\xi^{1}, \ldots, \xi^{n+d}\right)=$ $j_{0}^{k}\left(\widetilde{\xi}^{1}, \ldots, \widetilde{\xi}^{n+d}\right)$. Define $j:$ aut $(M, p) \rightarrow J_{0}^{(2 d+3) l}\left(\mathbb{C}^{n+d}, \mathbb{C}^{n+d}\right)$ by

$$
j(X):=j_{0}^{(2 d+3) l}\left(\widetilde{\xi}^{1}, \ldots, \widetilde{\xi}^{n+d}\right)
$$

Then by the finite jet determination in aut $(M, p)$ mentioned above and by Proposition 3.1, $j$ is injective and the image $j(\operatorname{aut}(M, p)) \subset J_{0}^{(2 d+3) l}\left(\mathbb{C}^{n+d}, \mathbb{C}^{n+d}\right)$ is a totally real linear subspace. Moreover, since $\psi$ is CR of order $k+1$ at $p$, for any $X \in \operatorname{aut}(M, p) \cap \Gamma_{p}\left(M, T^{c} M\right)$ we have

$$
\varphi_{*}(J X)=J \varphi_{*}(X)+o\left(|x-p|^{k}\right), \quad x \in M \rightarrow p
$$

and therefore $j(J X)=J(j(X))$ since $J X \in \operatorname{aut}(M, p)$. Since $j(\operatorname{aut}(M, p))$ is totally real and $j$ is injective, this implies $X \equiv 0$. In the above notation this means $\dot{\theta}_{0}(x) \equiv 0$ or equivalently $\varphi_{*}(\operatorname{Re} L(a), x)=0$. A similar argument applied to $\varphi_{b}(s, x):=\varphi_{b}^{-1}(\varphi(s, x))$ for $\varphi_{b}:=\varphi(b, \cdot)$ and $b \in S$ sufficiently close to $a$ shows that $\varphi_{*}(\operatorname{Re} L(b), x)=0$. Since $L$ is an arbitrary $(1,0)$ vector field on $S$, it follows that also $\varphi_{*}(L, x)=\varphi_{*}(\bar{L}, x)=0$. Since $(S, a)$ is of finite type, this implies that $\varphi(\cdot, x)$ is constant for every $x \in M$ (sufficiently close to $p$ ). (If $(S, a)$ is merely minimal, the same conclusion follows by observing that $\varphi(\cdot, x)$ is constant along CR-curves on $S$.) Hence we obtain the required conclusion in the case $M$ is finitely nondegenerate and of finite type at $p$.

To prove the statement in the general case, suppose that the conclusion does not hold for a germ of a smooth CR-map $\varphi:(S, a) \times(M, p) \rightarrow(M, p)$. Then the partial derivative $\partial_{s} \varphi(s, x)$ does not vanish at points arbitrary close to $(a, p)$. On the other hand, the above argument implies that the derivative is zero near all minimal points of $S$ that are sufficiently close to $a$. By the assumption, the minimal points are dense, and hence we reach a contradiction. The proof is complete.

## 5. - Local splitting of CR-maps; proof of Theorem 1.1

In this section we prove Theorem 1.1. Let $(M, p)$ be as in Theorem 1.1 and fix a decomposition into irreducible factors

$$
(M, p) \cong\left(M_{1}, p_{1}\right) \times \cdots \times\left(M_{m}, p_{m}\right)
$$

It is clear that such a decomposition always exists but a priori may not be unique. Let ( $M^{\prime}, p^{\prime}$ ) be another CR-manifold and

$$
\begin{equation*}
\left(M^{\prime}, p^{\prime}\right) \cong\left(M_{1}^{\prime}, p_{1}^{\prime}\right) \times \cdots \times\left(M_{m^{\prime}}^{\prime}, p_{m^{\prime}}^{\prime}\right) \tag{5.1}
\end{equation*}
$$

be a corresponding decomposition. Define

$$
(\widetilde{M}, \tilde{p}):=\left(M_{1}, p_{1}\right) \times \cdots \times\left(M_{m-1}, p_{m-1}\right)
$$

Since $M$ is finitely nondegenerate and of finite type on a dense subset, it follows directly from the definition that the same holds for $\widetilde{M}$ and $M_{m}$.

Now let $f=\left(f^{1}, \ldots, f^{m^{\prime}}\right)$ be a germ of a smooth CR-diffeomorphism between $(M, p)$ and $\left(M^{\prime}, p^{\prime}\right)$, where $f^{j}$ is the $j$ th component with respect to the decomposition (5.1). We fix connected representatives for all germs of CR-manifolds and denote them by the same letters. We may then assume that $f$ maps $M=M_{1} \times \cdots \times M_{m}$ diffeomorphically onto an open connected subset $U^{\prime} \subset M^{\prime}=M_{1}^{\prime} \times \cdots \times M_{m^{\prime}}^{\prime}$. For an open subset $U_{m} \subset M_{m}$, define the subsets $U_{j}^{\prime}:=f^{j}\left(\{\tilde{p}\} \times U_{m}\right) \subset M_{j}^{\prime}$. We also write $f^{-1}=\left(g^{1}, \ldots, g^{m}\right): U^{\prime} \rightarrow M$.

Lemma 5.1. If $U_{m} \subset M_{m}$ is a sufficiently small neighborhood of $p_{m}$, one has $\Pi_{j=1}^{m^{\prime}} U_{j}^{\prime} \subset f\left(\{\tilde{p}\} \times M_{m}\right)$.

Proof. By the construction,

$$
f\left(\{\tilde{p}\} \times U_{m}\right)=\left\{\left(f^{1}(\tilde{p}, v), \cdots, f^{m^{\prime}}(\tilde{p}, v)\right): v \in U_{m}\right\} \subset \Pi_{j=1}^{m^{\prime}} U_{j}^{\prime}
$$

Let

$$
\pi: \underbrace{\left(M_{m}, p_{m}\right) \times \cdots \times\left(M_{m}, p_{m}\right)}_{m^{\prime}} \times(\widetilde{M}, \tilde{p}) \rightarrow(\widetilde{M}, \tilde{p})
$$

be the germ given by

$$
\begin{aligned}
& \pi\left(v^{1}, \ldots, v^{m^{\prime}}, z\right): \\
& =\left(g^{1}\left(f^{1}\left(z, v^{1}\right), \ldots, f^{m^{\prime}}\left(z, v^{m^{\prime}}\right)\right), \ldots, g^{m-1}\left(f^{1}\left(z, v^{1}\right), \ldots, f^{m^{\prime}}\left(z, v^{m^{\prime}}\right)\right)\right),
\end{aligned}
$$

where $z \in \widetilde{M}$ and $v^{j} \in M_{m}$ for $1 \leq j \leq m^{\prime}$. Then $\pi(v, \ldots, v, z) \equiv z$ holds for $(z, v) \in M$ near $p$. By Proposition 4.1, we conclude that $\pi\left(v^{1}, \ldots, v^{m^{\prime}}, z\right) \equiv z$ for $\left(v^{1}, \ldots, v^{m^{\prime}}\right) \in M_{m} \times \cdots \times M_{m}$ near $\left(p_{m}, \ldots, p_{m}\right)$ and for $z \in \widetilde{M}$ near $\tilde{p}$. This implies $f^{-1}\left(f^{1}\left(\tilde{p}, v^{1}\right), \ldots, f^{m^{\prime}}\left(\tilde{p}, v^{m^{\prime}}\right)\right) \in\{\tilde{p}\} \times M_{m}$ for all $v^{1}, \ldots, v^{m^{\prime}} \in$ $M_{m}$ near $p_{m}$ and the lemma follows.

We need the following standard lemma proven here for the convenience of the reader.

Lemma 5.2. Let $S \subset \mathbb{R}^{n_{1}} \times \mathbb{R}^{n_{2}}$ be a smooth submanifold. Denote by $\pi_{j}: S \rightarrow$ $\mathbb{R}^{n_{j}}, j=1,2$, the canonical projections. Suppose that, for an open subset $U \subset S$, $\pi_{1}(U) \times \pi_{2}(U) \subset S$. Then for any $p \in U$ and any sufficiently small neighborhood $\Omega$ of p in $U, \pi_{j}(\Omega)$ is a submanifold of $\mathbb{R}^{n_{j}}$ for $j=1,2$, and $\Omega$ is open in $\pi_{1}(\Omega) \times \pi_{2}(\Omega)$.

Proof. The inclusion $\pi_{1}(U) \times \pi_{2}(U) \subset S$ implies $d \pi_{1}\left(T_{p} S\right) \times d \pi_{2}\left(T_{p} S\right) \subset$ $T_{p} S$ for every $p \in U$ and therefore $d \pi_{1}\left(T_{p} S\right) \times d \pi_{2}\left(T_{p} S\right)=T_{p} S$. By the semicontinuity of the dimension of each $d \pi_{j}\left(T_{p} S\right), j=1,2$, with respect to $p$, we conclude that both dimensions are constant and therefore both $\pi_{1}$ and $\pi_{2}$ are of constant rank on $U$. The required statement follows from the rank theorem.

In our situation each $U_{j}^{\prime}$ is a subset of $M_{j}^{\prime}$ for all $j=1, \ldots, m^{\prime}$, such that the product $U_{1}^{\prime} \times \cdots \times U_{m^{\prime}}^{\prime}$ is contained in a smooth submanifold that is locally CR-equivalent to $M_{m}$ by Lemma 5.1. Hence it follows from Lemma 5.2 that, if the neighborhood $U_{m}$ of $p_{m}$ in $M_{m}$ is chosen sufficiently small, each $U_{j}^{\prime}$ is a smooth submanifold of $M^{\prime}$. Moreover, $U_{m}$ is a CR-submanifold of $M$ in the sense that its tangent subspace intersects the complex tangent space of $M$ along complex subspaces of constant dimension. Then, $U_{1}^{\prime} \times \cdots \times U_{m^{\prime}}^{\prime}$ is a CRsubmanifold of $M^{\prime}$ by Lemma 5.1 and hence each $U_{j}^{\prime} \subset M_{j}^{\prime}$ is a CR-submanifold. We obtain a decomposition of $\left(M_{m}, p_{m}\right) \cong\left(U_{1}^{\prime}, p_{1}^{\prime}\right) \times \cdots \times\left(U_{m^{\prime}}^{\prime}, p_{m^{\prime}}^{\prime}\right)$ into a product of germs of smooth CR-submanifolds. Since, however, $\left(M_{m}, p_{m}\right)$ was chosen to be irreducible, it is CR-equivalent to ( $U_{j}^{\prime}, p_{j}^{\prime}$ ) for some $j \in$ $\left\{1, \ldots, m^{\prime}\right\}$. Without loss of generality, $j=m^{\prime}$. Then the other factors $U_{j}^{\prime}$ are zero-dimensional. We conclude that $f^{1}(\tilde{p}, \cdot), \ldots, f^{m^{\prime}-1}(\tilde{p}, \cdot)$ are constant near $p_{m}$ and that $f^{m^{\prime}}(\tilde{p}, \cdot):\left(M_{m}, p_{m}\right) \rightarrow\left(M_{m^{\prime}}^{\prime}, p_{m^{\prime}}^{\prime}\right)$ defines a CR-equivalence. By Proposition 4.1, we obtain:

Lemma 5.3. For $z \in \widetilde{M}$ sufficiently close to $\tilde{p}$, one has $f^{m^{\prime}}(z, \cdot) \equiv f^{m^{\prime}}(\tilde{p}, \cdot)$.
Lemma 5.4. Set $\widetilde{f}:=\left(f^{1}, \ldots, f^{m^{\prime}-1}\right), \widetilde{M^{\prime}}:=\prod_{j=1}^{m^{\prime}-1} M_{j}^{\prime}$ and $\tilde{p}^{\prime}:=$ $\left(p_{1}^{\prime}, \ldots, p_{m^{\prime}-1}^{\prime}\right) \in \widetilde{M^{\prime}}$. Then
(i) $\tilde{f}\left(\cdot, p_{m}\right):(\widetilde{M}, \tilde{p}) \rightarrow\left(\widetilde{M}^{\prime}, \tilde{p}^{\prime}\right)$ is a CR-diffeomorphism;
(ii) for any $z_{m} \in M_{m}$ sufficiently close to $p_{m}$, one has $\tilde{f}\left(\cdot, z_{m}\right) \equiv \tilde{f}\left(\cdot, p_{m}\right)$.

From Lemmata 5.3 and 5.4 we conclude that $f$ splits into the product $f=$ $\widetilde{h} \times h^{m}$ with $\widetilde{h}:=\widetilde{f}\left(\cdot, p_{m}\right):(\widetilde{M}, \tilde{p}) \rightarrow\left(\widetilde{M}^{\prime}, \tilde{p}^{\prime}\right)$ and $h^{m}:=f^{m^{\prime}}(\widetilde{p}, \cdot):\left(M_{m}, p_{m}\right) \rightarrow$ $\left(M_{m^{\prime}}^{\prime}, p_{m^{\prime}}^{\prime}\right)$. The proof of Theorem 1.1 is completed by induction on $m$.

## 6. - Global splitting of CR-maps; proof of Theorem 1.2

We now turn to the proof of the global decomposition result stated in Theorem 1.2. Let $M$ and $f: M \rightarrow M^{\prime}$ be as in Corollary 1.2. Without loss of generality we may assume that $M$ is connected. Fix decompositions

$$
M \cong M_{1} \times \cdots \times M_{m}, \quad M^{\prime} \cong M_{1}^{\prime} \times \cdots \times M_{m^{\prime}}^{\prime}
$$

into irreducible factors and write $f=\left(f^{1}, \ldots, f^{m^{\prime}}\right)$, where $f^{i}$ is the $i$-th component of $f$.

For any point $\left(p_{1}, \ldots, p_{m}\right) \in M, p_{i} \in M_{i}$, decompose

$$
\left(M_{i}, p_{i}\right)=\left(M_{i, 1}, p_{i, 1}\right) \times \cdots \times\left(M_{i, s_{i}}, p_{i, s_{i}}\right)
$$

and

$$
\left(M_{i}^{\prime}, p_{i}^{\prime}\right)=\left(M_{i, 1}^{\prime}, p_{i, 1}^{\prime}\right) \times \cdots \times\left(M_{i, r_{i}}^{\prime}, p_{i, r_{i}}^{\prime}\right)
$$

into local irreducible factors, where $p_{i}^{\prime}:=f^{i}(p)$ and let $\pi_{j, r}^{\prime}$ be the canonical projection of $M_{j}^{\prime}$ to $M_{j, r}^{\prime}$ defined in a small neighborhood of $p_{j}^{\prime}$, where $M_{j, r}^{\prime}$ is a representative of $\left(M_{j, r}^{\prime}, p_{j, r}^{\prime}\right)$.

Now assume that

$$
\operatorname{dim} M_{m}=\max \left(\operatorname{dim} M_{1}, \ldots, \operatorname{dim} M_{m}, \operatorname{dim} M_{1}^{\prime}, \ldots, \operatorname{dim} M_{m^{\prime}}\right)
$$

Fix $\tilde{p}:=\left(p_{1}, \ldots, p_{m-1}\right) \in \widetilde{M}:=M_{1} \times \cdots \times M_{m-1}$. By Theorem 1.1, the germ $f:(M, p) \rightarrow\left(M^{\prime}, p^{\prime}\right)$ can be written as a product of germs of CRdiffeomorphisms $f^{i, s}:\left(M_{i, s}, p_{i, s}\right) \rightarrow\left(M_{j_{i, s}, r_{i, s}}^{\prime}, p_{j_{i, s}, r_{i, s}}^{\prime}\right)$. Hence there exists arbitrarily small connected open neighborhood $\Omega$ of $p_{m}$ in $M_{m}$ such that

$$
f(\{\tilde{p}\} \times \Omega)=f^{1}(\{\tilde{p}\} \times \Omega) \times \cdots \times f^{m^{\prime}}(\{\tilde{p}\} \times \Omega)
$$

Moreover we can choose $\Omega$ so that for each $j, j=1, \ldots, m^{\prime}, h^{j, r}:=$ $\pi_{j, r}^{\prime}\left(f^{j}(\tilde{p}, \cdot)\right)$ is well-defined in $\Omega$,

$$
f^{j}(\{\tilde{p}\} \times \Omega)=h^{j, 1}(\Omega) \times \cdots \times h^{j, r_{j}}(\Omega)
$$

and there exists a subset $\mathcal{A}_{j} \subset\left\{1, \ldots, r_{j}\right\}$ such that $h^{j, r}$ is constant if $r \in \mathcal{A}_{j}$ and of maximal rank at every point of $\Omega$ if $r \notin \mathcal{A}_{j}$.

We claim that

$$
\begin{equation*}
f\left(\{\tilde{p}\} \times M_{m}\right)=f^{1}\left(\{\tilde{p}\} \times M_{m}\right) \times \cdots \times f^{m^{\prime}}\left(\{\tilde{p}\} \times M_{m}\right) . \tag{6.1}
\end{equation*}
$$

Indeed, set

$$
\begin{align*}
& \mathfrak{G}:=\{\left(x_{1}, \ldots, x_{m^{\prime}}\right) \in \underbrace{M_{m} \times \cdots \times M_{m}}_{m^{\prime}}:\left(f^{1}\left(\tilde{p}, x_{1}\right), \ldots,\right.  \tag{6.2}\\
& \left.\left.f^{m^{\prime}}\left(\tilde{p}, x_{m^{\prime}}\right)\right) \in f\left(\{\tilde{p}\} \times M_{m}\right)\right\}
\end{align*}
$$

For any $x \in M_{m}$, we can choose a neighborhood $\Omega_{x} \subset M_{m}$ of $x$ such that

$$
f\left(\{\tilde{p}\} \times \Omega_{x}\right)=f^{1}\left(\{\tilde{p}\} \times \Omega_{x}\right) \times \cdots \times f^{m^{\prime}}\left(\{\tilde{p}\} \times \Omega_{x}\right) .
$$

Therefore $(\underbrace{x, \ldots, x}_{m^{\prime}})$ is an interior point of $\mathfrak{G}$. Let $\mathfrak{G}_{x}$ be the maximal connected open set consisting of interior points of $\mathfrak{G}$ containing $(x, \ldots, x)$ and let $\overline{\mathfrak{G}}_{x}$ be its closure in $M_{m} \times \cdots \times M_{m}$. Then by continuity of $f, \overline{\mathfrak{G}}_{x}$ is again a subset of $\mathfrak{G}$.

Choose any $\left(x_{1}, \ldots, x_{m^{\prime}}\right) \in \overline{\mathfrak{G}}_{x}$. Then by (6.2), there is a point $p_{m} \in M_{m}$ such that $f\left(\tilde{p}, p_{m}\right)=\left(f^{1}\left(\tilde{p}, x_{1}\right), \ldots, f^{m^{\prime}}\left(\tilde{p}, x_{m^{\prime}}\right)\right)$. Let $\Omega$ and $\mathcal{A}_{1}$ be as above and choose a connected neighborhood $\Omega_{1}$ of $x_{1}$ in $M_{m}$ such that in $\Omega_{1}, h^{1, r}$ is well-defined for all $r=1, \ldots, r_{1}$,

$$
f^{1}\left(\{\tilde{p}\} \times \Omega_{1}\right)=h^{1,1}\left(\{\tilde{p}\} \times \Omega_{1}\right) \times \cdots \times h^{1, r_{1}}\left(\{\tilde{p}\} \times \Omega_{1}\right)
$$

and there exists a subset $\mathcal{B} \subset\left\{1, \ldots, r_{1}\right\}$ such that $h^{1, r}$ is constant if $r \in \mathcal{B}$ and of maximal rank at every point of $\Omega_{1}$ otherwise.

Since $\left(x_{1}, \ldots, x_{m^{\prime}}\right) \in \overline{\mathfrak{G}}_{x}$ and $f\left(\{\tilde{p}\} \times M_{m}\right) \subset M^{\prime}$ is a locally closed submanifold, there exist an open subset $V \subset \Omega_{1}$ and a point $\left(y_{2}, \ldots, y_{m^{\prime}}\right) \in$ $\underbrace{M_{m} \times \cdots \times M_{m}}_{m^{\prime}-1}$ arbitrarily close to $\left(x_{2}, \ldots, x_{m^{\prime}}\right)$ such that

$$
f^{1}(\{\tilde{p}\} \times V) \times f^{2}\left(\tilde{p}, y_{2}\right) \times \cdots \times f^{m^{\prime}}\left(\tilde{p}, y_{m}\right) \subset f(\{\tilde{p}\} \times \Omega) .
$$

Then $h^{1, r}$ is constant in $V$ for $r \in \mathcal{A}_{1}$. Since $V$ is an open subset of $\Omega_{1}$ and $h^{1, r}$ is of maximal rank at every point of $\Omega_{1}$ if $r \notin \mathcal{B}$, this implies $\mathcal{A}_{1} \subset \mathcal{B}$. Therefore $h^{1, r}(y)=h^{1, r}\left(p_{m}\right)$ for all $y \in \Omega_{1}$ if $r \in \mathcal{A}_{1}$ and hence $f^{1}\left(\{\tilde{p}\} \times \Omega_{1}\right) \subset f^{1}(\{\tilde{p}\} \times \Omega)$ if $\Omega_{1}$ is sufficiently small.

By the same argument as above we can choose neighborhoods $\Omega_{j} \subset M_{m}$ of $x_{j}, j=2, \ldots, m^{\prime}$, such that $f^{j}\left(\{\tilde{p}\} \times \Omega_{j}\right) \subset f^{j}(\{\tilde{p}\} \times \Omega)$ or equivalently

$$
f^{1}\left(\{\tilde{p}\} \times \Omega_{1}\right) \times \cdots \times f^{m^{\prime}}\left(\{\tilde{p}\} \times \Omega_{m^{\prime}}\right) \subset f(\{\tilde{p}\} \times \Omega) .
$$

Then $\left(x_{1}, \ldots, x_{m^{\prime}}\right)$ is an interior point of $\overline{\mathfrak{G}}_{x}$. Since $M_{m} \times \cdots \times M_{m}$ is a connected set, this implies $\overline{\mathfrak{G}}_{x}=M_{m} \times \cdots \times M_{m}$.

Now we have $f\left(\{\tilde{p}\} \times M_{m}\right)=f^{1}\left(\{\tilde{p}\} \times M_{m}\right) \times \cdots \times f^{m^{\prime}}\left(\{\tilde{p}\} \times M_{m}\right)$. Since, by the local splitting property of $f$ proven in Theorem 1.1, each $f^{j}(\tilde{p}, \cdot)$ is of constant rank, this implies that $f^{j}\left(\{\tilde{p}\} \times M_{m}\right)$ is a closed CR-submanifold of $M_{j}^{\prime}$. Since $M_{m}$ is irreducible in the sense of Corollary 1.2, we may assume that $f^{j}\left(\{\tilde{p}\} \times M_{m}\right)$ is of zero dimension if $j \neq m^{\prime}$ and $M_{m}$ is CR-diffeomorphic to $f^{m^{\prime}}\left(\{\tilde{p}\} \times M_{m}\right)$. Since $M_{m}$ is of maximal dimension among the irreducible factors of $M$ and $M^{\prime}$, this implies $f^{m^{\prime}}\left(\{\tilde{p}\} \times M_{m}\right)$ is an open subset of $M_{m^{\prime}}^{\prime}$. Since $f^{m^{\prime}}\left(\{\tilde{p}\} \times M_{m}\right)$ is also closed in $M_{m^{\prime}}^{\prime}$ and $M_{m^{\prime}}^{\prime}$ is connected, we have $f^{m^{\prime}}\left(\{\tilde{p}\} \times M_{m}\right)=M_{m^{\prime}}^{\prime}$. Then by local splitting property of $f$ we can show that for any $(q, x) \in M, q \in \widetilde{M}, x \in M_{m}$, sufficiently close to $\left(\tilde{p}, p_{m}\right)$, we have $\tilde{f}(q, x)=\tilde{f}\left(q, p_{m}\right)$ and $f^{m^{\prime}}(q, x)=f^{m^{\prime}}(\tilde{p}, x)$, where $\tilde{f}:=\left(f^{1}, \ldots, f^{m^{\prime}-1}\right)$.

Since $\left(\tilde{p}, p_{m}\right)$ is arbitrary and $M$ is connected, $f$ can be written as a product of two CR-diffeomorphisms $\hat{f}: \widetilde{M} \rightarrow \widetilde{M^{\prime}}$, and $\hat{g}: M_{m} \rightarrow M_{m^{\prime}}^{\prime}$, where $\widetilde{M^{\prime}}:=M_{1}^{\prime} \times \cdots \times M_{m^{\prime}-1}^{\prime}$. The proof of Theorem 1.2 is completed by induction on $m$.

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