

## Optimal Blowup Rates for the Minimal Energy Null Control of the Strongly Damped Abstract Wave Equation

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**Abstract.** The null controllability problem for a structurally damped abstract wave equation –often referred to in the literature as a structurally damped equation– is considered with a view towards obtaining optimal rates of blowup for the associated minimal energy function  $\mathcal{E}_{\min}(T)$ , as terminal time  $T \downarrow 0$ . Key use is made of the underlying analyticity of the semigroup generated by the elastic operator  $\mathcal{A}$ , as well as of the explicit characterization of its domain of definition. We ultimately find that the blowup rate for  $\mathcal{E}_{\min}(T)$ , as  $T$  goes to zero, depends on the extent of structural damping.

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### 1. – Introduction

With  $H$  being a Hilbert space, let linear operator  $\mathring{A} : D(\mathring{A}) \subset H \rightarrow H$  be strictly positive and self-adjoint. Moreover, let  $B \in \mathcal{L}(H)$  be positive and self-adjoint. Therewith, we consider the structurally damped and controlled abstract model

$$(1) \quad \begin{cases} v_{tt} + \mathring{A}v + \mathring{A}^{\frac{\alpha}{2}} B \mathring{A}^{\frac{\alpha}{2}} v_t = u & \text{on } (0, T) \\ [v(0), v_t(0)] = [v_0, v_1] \in D(\mathring{A}^{\frac{1}{2}}) \times H \end{cases}$$

where the parameter  $\alpha$  is in the range  $0 \leq \alpha < 1$ . Also, the “control”  $u(t)$  is a function in  $L^2(0, T; H)$ . So as it appears, this model constitutes an abstract wave equation, under the influence of the structural damping term  $\mathring{A}^{\frac{\alpha}{2}} B \mathring{A}^{\frac{\alpha}{2}} v_t$ . When  $B = \rho I$ , where parameter  $\rho > 0$ , this system is often referred to as a “structurally damped” wave equation. With  $u = 0$ , the system’s underlying generator  $\mathcal{A} : D(\mathcal{A}) \subset D(\mathring{A}^{\frac{1}{2}}) \times H \rightarrow D(\mathring{A}^{\frac{1}{2}}) \times H$  generates a strongly

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continuous semigroup of contractions  $\{e^{At}\}_{t \geq 0}$  on the space  $\mathcal{X} \equiv D(\mathring{A}^{\frac{1}{2}}) \times H$ . This result is well-known, and follows from a straightforward application of the Lumer Phillips theorem. A much deeper result in this regard is the following: When  $\alpha$  is in the range  $\frac{1}{2} \leq \alpha \leq 1$ , and under the additional assumption that  $B$  be an isomorphism on  $H$ , then the semigroup  $\{e^{At}\}_{t \geq 0}$  is *analytic* (see [3], [4]). Consequently, those controlled partial differential equations which can be described by the abstract system (1), when  $\frac{1}{2} \leq \alpha \leq 1$  and when  $B$  is boundedly invertible on  $H$ , will manifest parabolic-like dynamics.

For this model, we wish to consider the *null controllability problem*. This problem can be broadly stated as that of finding a control function  $u$ , such that the corresponding solution of (1) is brought from the initial state to rest at terminal time  $T$ . Because the abstract system (1) models parabolic-like behaviour, including an infinite speed of propagation, one should expect that if this system is indeed null controllable within the given class of control inputs  $u$ , the property should hold true in arbitrarily short time  $T > 0$ . This expectation is fully in line with what is known about the canonical parabolic controllability problem; namely the problem of controlling the heat equation, be it via boundary or interior control (see e.g., [2], [16], [19]). Denoting

$$(2) \quad \mathcal{X} = D(\mathring{A}^{\frac{1}{2}}) \times H,$$

we are accordingly led to our working definition of null controllability:

DEFINITION 1. The abstract system (1) is said to be null controllable, if for any time  $T > 0$  and arbitrary initial data  $[v_0, v_1] \in \mathcal{X}$ , there exists a control function  $u \in L^2(0, T; H)$  such that the corresponding solution  $[v, v_t]$  to (1) satisfies  $[v(T), v_t(T)] = [0, 0]$ .

When  $B = I$ , the null controllability problem for the system (1) has in fact been successfully addressed in [13], in the case that  $\mathring{A} : D(\mathring{A}) \subset H \rightarrow H$  has *compact* resolvent. Indeed, in [13] (Theorem 1.1.1 therein), it was shown that for  $\frac{1}{2} \leq \alpha < 1$ , the system (1), with  $B = I$ , is null controllable within the class of controls  $L^2(0, T; H)$ . The method of proof employed in [13] is based on spectral properties of the elastic generator, which play a critical role in the analysis.

The aim of the present paper is twofold. First, we wish to extend the null controllability result of [13] to more general models, which *do not necessarily admit of a spectral representation*. In particular, *we will dispense with the assumption on the compactness of the resolvent of  $\mathring{A}$ , and we will not necessarily assume that  $B$  is the identity operator*. A second and more important goal in this paper is *to obtain a precise, optimal, estimate for the norm of the “minimal norm steering control”, as  $T \downarrow 0$* . In turn, it is known that the rate of blowup for the minimal norm control is directly related to the “sharpest” constant  $C_T$  appearing in the “observability” inequality which is associated with null controllability (see (7) below). Since the primary intent of [13] was to first and foremost establish the null controllability for abstract analytic systems such as (1), the issue of

blowup rates of the minimal norm control was not intended to be addressed therein. On the other hand, questions related to the singularity of the minimal energy function have become of central interest in areas such as stochastic and nonlinear PDE's, examples of which include the Ornstein Uhlenbeck processes, and Kolmogorov and Hamilton Jacobi equations [6], [8], [10]. More will be said on this in what follows.

The task of finding a precise description of the rate of singularity was taken up—independently from, and essentially simultaneously to, our present effort—in the follow-up paper [23], where optimal blowup rates are obtained by the *spectral method*. Obtaining optimal blowup rates (in nonspectral situations) is also our present goal. Accordingly, we will be primarily concerned with the problem of deriving those “sharp” observability estimates which give rise to the null controllability property stated in Definition 1, *without the use of any underlying spectrality*. By contrast to [23], we will employ a special multipliers method, with a suitably selected scalar weight. Moreover, in this work we will make use of properties of the fractional powers of the elastic generator  $\mathcal{A}$ , as well as the underlying analyticity of the corresponding semigroup  $\{e^{-\mathcal{A}t}\}_{t \geq 0}$ . We believe that our proof is shorter, simpler and applicable to a more general class of problems than that considered in [23]. On the other hand, the proof of [23] does provide constructive (suboptimal) steering controls for the finite dimensional approximations of the overall infinite dimensional system. In short, the respective results and techniques of proofs in the present paper and in [23] provide complementary sets of information.

We now briefly explain our task of obtaining the optimal blowup rate for the minimal norm steering control. Assume for the time being that the null controllability property given in Definition 1 holds true for the abstract system (1), for arbitrary  $T > 0$ . Then for each fixed  $T$  and given initial data  $[v_0, v_1] \in \mathcal{X}$ , one can proceed to solve the associated optimization problem of finding a control  $u$  such that the corresponding solution  $[v, v_t]$  satisfies  $[v(T), v_t(T)] = [0, 0]$ , and moreover has its  $L^2(0, T; H)$  –measurement being minimized over all  $L^2(0, T; H)$ -controls which steer the solution to zero. Assuming the null controllability property to hold true, this optimization problem has a wellknown method of solution (see e.g., Appendix B of [12] and [14]). We denote this minimizer, or minimal norm control, as  $u_T^0(v_0, v_1)$ . With this minimizer in hand, for each fixed  $T > 0$  and initial data  $[v_0, v_1] \in \mathcal{X}$ , we have the following:

DEFINITION 2. The *minimal energy function*  $\mathcal{E}_{\min}(T)$  is defined as

$$(3) \quad \mathcal{E}_{\min}(T) \equiv \sup_{\|[v_0, v_1]\|_{\mathcal{X}}=1} \|u_T^0(v_0, v_1)\|_{L^2(0, T; H)}.$$

Given the presumed null controllability of the system (1), this function  $\mathcal{E}_{\min}(T)$  is evidently bounded on  $(0, T]$ , for any  $T$  positive. Moreover, it seems clear that this function should tend to blowup as  $T \downarrow 0$ . Capturing the precise estimate of this blowup is the very objective of this paper. The problem of

studying the order of the singularity for the minimal energy function is a rather classical one, and indeed is now well understood for finite dimensions (see [20], [22]). Concerning infinite dimensions, the recent paper [1] has addressed the null controllability problem and the related question of blowup for  $\mathcal{E}_{\min}(T)$ , in the case of 2-dimensional linear thermoelastic systems. (See also [21], wherein the estimate  $e^{1/T}$  is shown for the heat equation under boundary control.) As we have already noted, the key to determining the rate of blowup of  $\mathcal{E}_{\min}(T)$  as  $T \downarrow 0$ , is ascertaining the “best” constant  $C_T$  possible for the observability inequality associated with null controllability. Our proof below is accordingly geared toward finding such  $C_T$ .

In contrast to the spectral approach adopted in [13], [23], in order to obtain the observability inequality requisite for null controllability (see (7) below), we will start by invoking a relatively user-friendly multiplier method. However, in the course of the proof, absolutely critical use is made of intermediate results which are built, not only on the underlying analyticity of the system (1), but also on properties of the domains of fractional powers of underlying generator  $A : D(A) \subset \mathcal{X} \rightarrow \mathcal{X}$  (as explicitly defined in (8) below).

Our main result is as follows:

**THEOREM 3.** *With the operator  $\hat{A}$  as given above, assume that the operator  $B$  has the following properties: (i)  $B \in \mathcal{L}(D(\hat{A}^{\frac{\alpha}{2}}))$ ; (ii) The self-adjoint operator  $B \in \mathcal{L}(H)$  is strictly positive. Then, with  $\alpha$  in the range  $0 \leq \alpha < 1$ , the abstract system (1) is null controllable within the class of controls in  $L^2(0, T; H)$ . The minimal energy function  $\mathcal{E}_{\min}(T) = \mathcal{O}(T^{-\frac{\mu_\alpha}{2}})$ , where*

$$(4) \quad \mu_\alpha = \begin{cases} 3, & \text{if } 0 \leq \alpha \leq \frac{3}{4}; \\ \frac{\alpha}{1-\alpha}, & \text{if } \frac{3}{4} < \alpha < 1. \end{cases}$$

**REMARK 4.** We point out that Theorem 3 is optimal for  $0 \leq \alpha \leq \frac{3}{4}$ , in view of Seidman’s finite dimensional result in [20]. In fact, [20] provides, for the finite dimensional case, an explicit formula for computing the growth of the minimal norm. In fact, the growth rate for the case of finite dimensional truncations of the damped wave equation is of order  $\mathcal{O}(T^{-3/2})$ , which is precisely our result for  $\alpha \leq 3/4$ . However, for  $\alpha > \frac{3}{4}$  the controllability problem is of a purely infinite dimensional nature, with rates for  $\mathcal{E}_{\min}(T)$  which will be arbitrarily large as  $\alpha$  increases. In short, for  $\alpha \leq \frac{3}{4}$  the result is in line with the known finite dimensional theory, but for  $\alpha > \frac{3}{4}$  the infinite dimensional character of the problem dominates. The explicit estimate (4) which blows up when  $\alpha \uparrow 1$  gives the inference that the system (1) is not null controllable for  $\alpha = 1$ , as was shown outright in [13].

**REMARK 5.** The condition that  $B$  be an isomorphism on  $\mathcal{L}(H)$  (which is implied by the assumptions in Theorem 3) is made in order to guarantee

analyticity of the semigroup generated by the operator

$$A = \begin{pmatrix} 0 & I \\ -\mathring{A} & -\mathring{A}^{\frac{\alpha}{2}} B \mathring{A}^{\frac{\alpha}{2}} \end{pmatrix},$$

with  $D(A) \subset \mathcal{X} \rightarrow \mathcal{X}$ , and for  $\alpha$  in the range  $1 > \alpha \geq 1/2$  [3]. However, this latter property holds for a larger class of operators  $B$  [3] than those of isomorphisms. Accordingly, our treatment could be also extended to this class. For the sake of clarity of the exposition we do not attempt to provide the most general hypotheses imposed on the operator  $B$ .

By way of further motivating the present paper, we note that those null controllability studies of infinite dimensional systems which consider the issue of obtaining precise estimates on the norm measurements of minimal steering controls, are closely connected to current problems arising in the field of stochastic differential equations. For example, null controllability is tied to the analysis involved in deriving regularity properties for the so-called Bellman's function, a quantity associated with the minimal time control problem. In addition, null controllability is closely related to the regularity of several Markov semigroups such those which deal with Ornstein-Uhlenbeck processes and related Kolmogorov equations. In fact, it can be shown in some cases (see e.g., [5], Theorem 8.3.3) that null controllability is equivalent to the differentiability and regularizing effect of the Ornstein-Uhlenbeck process. Moreover, the regularity of solutions to the Kolmogorov equation depends on the singularity of the minimal energy function as  $T \downarrow 0$  [6], [8], [10]. In addition, for some special examples of Ornstein-Uhlenbeck semigroups, it is shown that null controllability is equivalent to the hypoellipticity condition of Hörmander (see [5], p. 112 and [15]). Also, as shown in [5], optimal estimates for the norms of controls are critical in being able to prove Liouville's property for harmonic functions of Markov processes (see p. 108 of [5]).

We note furthermore that in the deterministic case, the connection between the asymptotic behavior of the minimal energy function and the regularity of the Bellman's function (which describes the minimal time control for the given control process) is made very clear in the recent paper [9]. It is shown there that the Holderian regularity of Bellman's function, and its modulus of continuity, are determined by the singularity of  $\mathcal{E}_{\min}(T)$  when  $T \downarrow 0$ . In sum, the issue of obtaining optimal estimates of the singularity of  $\mathcal{E}_{\min}(T)$  is not only a problem of interest in the specific context of null controllability, but is also key in the solution of problems drawn from several areas of deterministic and stochastic PDE's.

Finally, we note that the question of the asymptotic behavior of the minimal energy function as  $T \rightarrow \infty$  has very recently drawn considerable attention (see [18]). In fact, this asymptotic behavior (i.e., the vanishing energy at infinity) is shown in [18] to be connected with the validity of Liouville's theorem for Ornstein-Uhlenbeck operators.

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## 2. – The needed observability inequality

In order to formulate the necessary and sufficient condition for null controllability we introduce the following adjoint system:

$$(5) \quad \begin{cases} v_{tt} + \mathring{A}v + \mathring{A}^{\frac{\alpha}{2}} B \mathring{A}^{\frac{\alpha}{2}} v_t = 0 & \text{on } (0, T) \\ [v(0), v_t(0)] = [v_0, v_1] \in \mathcal{X}. \end{cases}$$

Associated with this adjoint problem is the so-called energy of the system, given by

$$(6) \quad \mathcal{E}(t) = \frac{1}{2} \|\mathring{A}^{\frac{1}{2}} v(t)\|_H^2 + \frac{1}{2} \|v_t(t)\|_H^2.$$

It is a well-known fact from functional analysis that the validity of the given null controllability statement is equivalent to the existence of the inequality

$$(7) \quad (2\mathcal{E}(T))^{\frac{1}{2}} \leq \|[v(T), v_t(T)]\|_{\mathcal{X}} \leq C_T \|v_t\|_{L^2(0,T;H)},$$

where  $[v, v_t] \in C([0, T]; \mathcal{X})$  is the solution to the (adjoint) homogeneous problem (5).

Accordingly, we will work towards the attainment of the inequality (7), a precise estimation of the singularity of  $C_T$  as  $T \downarrow 0$ .

So as to convince the reader that the estimate for  $C_T$  in (7) also provides the estimate for the singular behaviour of the minimal energy  $\mathcal{E}_{\min}(T)$ , we recall a standard optimization argument in control theory [12], [7], [14]. For this, we introduce the following functional analytic framework.

On the Hilbert space  $\mathcal{X}$  we denote  $\mathcal{A} : D(\mathcal{A}) \subset \mathcal{X} \rightarrow \mathcal{X}$  to be

$$(8) \quad \mathcal{A} \equiv \begin{bmatrix} 0 & \mathring{I} \\ -\mathring{A} & -\mathring{A}^{\alpha/2} B \mathring{A}^{\alpha/2} \end{bmatrix}$$

$$D(\mathcal{A}) = \{[v_0, v_1] \in D(\mathring{A}^{\frac{1}{2}}) \times D(\mathring{A}^{\frac{1}{2}}) : \mathring{A}^{1-\alpha} v_0 + \mathring{A}^{-\alpha/2} B \mathring{A}^{\alpha/2} v_1 \in D(\mathring{A}^{\alpha})\}.$$

For  $\frac{1}{2} \leq \alpha \leq 1$ , it is well-known that  $\mathcal{A}$  generates an *analytic* contraction semigroup  $\{e^{At}\}_{t \geq 0}$  on  $\mathcal{X}$  which is exponentially stable (see [3]). In consequence of this analyticity, we have the estimate (see e.g., p. 70 of [17])

$$(9) \quad \|\mathcal{A}^\eta e^{At}\|_{\mathcal{L}(\mathcal{X})} \leq \frac{C_\eta}{t^\eta} \quad \text{for all } t > 0.$$

We introduce next the bounded linear operator  $L_T : L^2(0, T; U) \rightarrow \mathcal{X}$ , given by

$$L_T u \equiv \int_0^T e^{A(T-t)} \begin{pmatrix} 0 \\ u(t) \end{pmatrix} dt.$$

In these terms, null controllability is equivalent to showing the inclusion

$$e^{AT}(\mathcal{X}) \subset L_T(\mathcal{U}), \mathcal{U} \equiv L^2(0, T; U).$$

This, in turn, by surjectivity theorem [7] is equivalent to the inequality

$$(10) \quad \|e^{A^*T} \mathbf{x}\|_{\mathcal{X}} \leq C_T \|L_T^* \mathbf{x}\|_{L^2(0, T; U)} \quad \text{for all } \mathbf{x} \in \mathcal{X}.$$

We note that inequality in (10), upon specification of  $L_T^*$ , is equivalent to (7). Assuming the validity of the inequality (10), we can subsequently search for the minimal norm control which, by standard optimization argument [12] Appendix B and [14], takes the form

$$u_T^0 = -L_T^*(L_T L_T^*)^{-1} e^{AT} \mathbf{x}$$

where  $\mathbf{x} = [v_0, v_1] \in \mathcal{X}$  is the initial data of the controlled process. We note that the existence of the pseudo inverse  $\Gamma_T \equiv L_T^*(L_T L_T^*)^{-1} e^{AT}$ , and its boundedness as a mapping from  $\mathcal{X}$  into  $L^2(0, T; U)$ , results from the validity of (10).

On the other hand, as easily verified,

$$\mathcal{E}_{\min}(T) = \|\Gamma_T\|_{\mathcal{L}(\mathcal{X}, L^2(0, T; U))} \leq C_T$$

where  $C_T$  here is the same constant which appears in (10).

Thus, the constant  $C_T$  provides the estimate for the singularity of the minimal energy function  $\mathcal{E}_{\min}$ . Since the solution of (5), corresponding to initial data  $[v_0, v_1]$ , may be written as

$$(11) \quad \begin{bmatrix} v(t) \\ v_t(t) \end{bmatrix} = e^{A^*t} \begin{bmatrix} v_0 \\ -v_1 \end{bmatrix},$$

the inequality in (7) is equivalent to (10). Thus, the crux of the proof of Theorem 3 is in establishing the inequality (7), while maintaining control of the singularity of the constant  $C_T$  as  $T \downarrow 0$ .

### 3. – Technical lemmas

Using the operator theoretic notions established in Section 2, we first prove some supporting results, which will be key in what follows.

LEMMA 6. *Let  $1 \geq \alpha \geq \frac{1}{2}$ . Assume that self-adjoint operator  $B \in \mathcal{L}(H)$  is strictly positive and moreover satisfies  $B \in \mathcal{L}(D(\mathring{A}^{(k+\frac{1}{2})(1-\alpha)}))$  for some given nonnegative integer  $k \geq 1$ . Then for all integer  $n = 1, \dots, k$  and  $\theta \in [0, 1]$ , we have the continuous inclusion*

$$(12) \quad D(\mathcal{A}^{n+\theta}) \subset D(\mathring{A}^{n+\frac{1}{2}-n\alpha+\theta(1-\alpha)}) \times D(\mathring{A}^{n-\frac{1}{2}-(n-1)\alpha+\theta(1-\alpha)}).$$

PROOF OF LEMMA 6. By applying an inductive argument we will first show the following containment: if the self-adjoint operator  $B \in \mathcal{L}(H)$  is strictly positive and moreover satisfies  $B \in \mathcal{L}(D(\mathring{A}^{(k+\frac{1}{2})(1-\alpha)}))$  for some given nonnegative integer  $k \geq 1$ , then for  $n = 1, \dots, k + 1$ ,

$$(13) \quad D(\mathcal{A}^n) \subset D(\mathring{A}^{n+\frac{1}{2}-n\alpha}) \times D(\mathring{A}^{n-\frac{1}{2}-(n-1)\alpha}),$$

from which the estimate (12) will readily follow by interpolation. In fact, the interpolation property between fractional powers of domains of  $\mathring{A}$  follows from the self-adjointness of  $\mathring{A}$  and the analogous interpolation property for the domains of  $\mathcal{A}$  follows from the fact that  $\mathcal{A}$  is invertible and generates an analytic contraction semigroup [2], [14] (inasmuch as  $B \in \mathcal{L}(H)$  is an isomorphism and  $1 \geq \alpha \geq \frac{1}{2}$ ).

To this end, we have by definition that for all  $n = 1, 2, \dots$ ,

$$(14) \quad D(\mathcal{A}^n) = \left\{ [v_0, v_1] \in D(\mathcal{A}) : \mathcal{A} \begin{bmatrix} v_0 \\ v_1 \end{bmatrix} \in D(\mathcal{A}^{n-1}) \right\}.$$

To start, if  $[v_0, v_1] \in D(\mathcal{A})$ , we have from its definition in (8) that

$$\begin{aligned} v_1 &\in D(\mathring{A}^{\frac{1}{2}}); \\ \mathring{A}v_0 + \mathring{A}^{\frac{\alpha}{2}}B\mathring{A}^{\frac{\alpha}{2}}v_1 &= g \in H; \end{aligned}$$

whence, upon application of the bounded operator  $\mathring{A}^{\frac{1}{2}-\alpha}$  (as  $\alpha \geq \frac{1}{2}$ ), we obtain

$$\mathring{A}^{\frac{3}{2}-\alpha}v_0 = \mathring{A}^{\frac{1}{2}-\alpha}g - \mathring{A}^{\frac{1-\alpha}{2}}B\mathring{A}^{\frac{\alpha-1}{2}}\mathring{A}^{\frac{1}{2}}v_1 \in H$$

(note that we have used the fact that since the self-adjoint  $B \in \mathcal{L}(H)$  and is moreover in  $\mathcal{L}(D(\mathring{A}^{(k+\frac{1}{2})(1-\alpha)}))$ , then by interpolation  $B \in \mathcal{L}(D(\mathring{A}^{\frac{1}{2}(1-\alpha)}))$ , via interpolation parameter  $\theta = \frac{1-\alpha}{2k+1-\alpha}$ ). We conclude then that the containment (13) is true for  $n = 1$ .



Assume now that the containment (13) is valid for  $n = k$ . Then if  $[v_0, v_1] \in D(\mathcal{A}^{n+1})$ , we have

$$\left[ \mathring{A}v_0 + \mathring{A}^{\alpha/2} B \mathring{A}^{\alpha/2} v_1 \right] = \mathcal{A} \begin{bmatrix} v_0 \\ v_1 \end{bmatrix} \in D(\mathcal{A}^n).$$

In other words,

$$\begin{aligned} v_1 &\in D(\mathring{A}^{n+\frac{1}{2}-n\alpha}) \\ \mathring{A}v_0 + \mathring{A}^{\alpha/2} B \mathring{A}^{\alpha/2} v_1 &= g \in D(\mathring{A}^{n-\frac{1}{2}-(n-1)\alpha}). \end{aligned}$$

We have then by using regularity assumption imposed on  $B$

$$\mathring{A}^{(n+1)+\frac{1}{2}-(n+1)\alpha} v_0 = \mathring{A}^{n+\frac{1}{2}-(n+1)\alpha} g - \mathring{A}^{(n+\frac{1}{2})(1-\alpha)} B \mathring{A}^{-(n+\frac{1}{2})(1-\alpha)} \mathring{A}^{n+\frac{1}{2}-n\alpha} v_1 \in H,$$

where we have used the fact that  $B \in \mathcal{L}(D(\mathring{A}^{(k+\frac{1}{2})(1-\alpha)}))$ ; and also  $n + \frac{1}{2} - (n + 1)\alpha \leq n - \frac{1}{2} - (n - 1)\alpha$  for  $\alpha \geq \frac{1}{2}$ . Interpolating now between  $n$  and  $n + 1$ , for  $n = 1, \dots, k$  gives the asserted result.  $\square$

Now let  $\Pi : \mathcal{X} \rightarrow D(\mathring{A}^{\frac{1}{2}})$  denote the projection onto the first coordinate; i.e.,  $\Pi([v_0, v_1]) = v_0$ . With this operator in mind, we can proceed to combine Lemma 6 with the characterization of the domains of the fractional powers of  $D(\mathcal{A})$  in [4].

**COROLLARY 7.** *Let  $1 \geq \alpha \geq \frac{1}{2}$ . Assume that self-adjoint operator  $B \in \mathcal{L}(H)$  is strictly positive and satisfies  $B \in \mathcal{L}(D(\mathring{A}^{(k+\frac{1}{2})(1-\alpha)}))$  for some given nonnegative integer  $k \geq 0$ . Then for all integer  $n = 0, \dots, k$  and  $\theta \in [0, 1]$ , we have the continuous inclusion*

$$(15) \quad \Pi D(\mathcal{A}^{n+\theta}) \subset D(\mathring{A}^{n+\frac{1}{2}-n\alpha+\theta(1-\alpha)}).$$

**PROOF OF COROLLARY 7.** Lemma 6 does not provide the desired containment for  $n = 0$  and  $\theta \in [0, 1)$ . But if self-adjoint  $B \in \mathcal{L}(H)$  is strictly positive (and so an isomorphism on  $H$ ), then by Theorem 1.1 of [4] we have for  $0 \leq \theta \leq 1$ ,

$$\Pi D(\mathcal{A}^\theta) \subset D(\mathring{A}^{\frac{1}{2}+\theta(1-\alpha)})$$

(and so the condition  $B \in \mathcal{L}(D(\mathring{A}^{(k+\frac{1}{2})(1-\alpha)}))$  is irrelevant for  $n = 0$  and  $\theta \in [0, 1]$ ). Combining this with the result of Lemma 6 gives the result.  $\square$

In turn, we can use this corollary along with analyticity of the semigroup generated by  $\mathcal{A}$  in order to establish the following result:

LEMMA 8. Let  $1 \geq \alpha \geq \frac{1}{2}$ . Assume that self-adjoint operator  $B \in \mathcal{L}(H)$  is strictly positive and satisfies  $B \in \mathcal{L}(D(\mathring{A}^{(k+\frac{1}{2})(1-\alpha)}))$  for some given nonnegative integer  $k \geq 0$ . Then for all integer  $n = 0, \dots, k$  and  $\theta \in [0, 1]$ , the solution  $[v, v_t]$  of (5) satisfies the following estimate:

$$(16) \quad \|\mathring{A}^{(n+\theta)(1-\alpha)+\frac{1}{2}} v(t)\|_H \leq C_{n,\theta} \frac{1}{t^{n+\theta}} \sqrt{\mathcal{E}\left(\frac{t}{n+2}\right)} \quad \text{for all } t > 0.$$

PROOF OF LEMMA 8. Using Corollary 7, the semigroup representation of  $[v(t), v_t(t)]$  in (11), and the fact that the solution  $[v(t), v_t(t)] \in D(\mathcal{A}^{n+\theta})$  for  $t > 0$  (by virtue of the analyticity of  $\{e^{-\mathcal{A}t}\}_{t \geq 0}$ ), we have

$$(17) \quad \|\mathring{A}^{n+\frac{1}{2}-n\alpha+\theta(1-\alpha)} v(t)\|_H \leq \left\| \Pi \mathcal{A}^{n+\theta} e^{-\mathcal{A}t} \begin{bmatrix} v_0 \\ v_1 \end{bmatrix} \right\|_{\mathcal{X}}.$$

Now one can use the commutativity property of semigroups and their generators to write

$$\mathcal{A}^{n+\theta} e^{-\mathcal{A}t} \begin{bmatrix} v_0 \\ v_1 \end{bmatrix} = (\mathcal{A} e^{-\frac{\mathcal{A}t}{n+2}})^n \mathcal{A}^\theta e^{-\frac{\mathcal{A}t}{n+2}} e^{-\frac{\mathcal{A}t}{n+2}} \begin{bmatrix} v_0 \\ v_1 \end{bmatrix}.$$

Combining this relation with the analytic estimate (9) gives now

$$\begin{aligned} & \left\| \mathcal{A}^{n+\theta} e^{-\mathcal{A}t} \begin{bmatrix} v_0 \\ v_1 \end{bmatrix} \right\|_{\mathcal{X}} \\ & \leq \|\mathcal{A} e^{-\frac{\mathcal{A}t}{n+2}}\|_{\mathcal{L}(\mathcal{X})}^n \|\mathcal{A}^\theta e^{-\frac{\mathcal{A}t}{n+2}}\|_{\mathcal{L}(\mathcal{X})} \left\| e^{-\frac{\mathcal{A}t}{n+2}} \begin{bmatrix} v_0 \\ v_1 \end{bmatrix} \right\|_{\mathcal{X}} \\ & \leq C_{n,\theta} \frac{(n+2)^{n+\theta}}{t^{n+\theta}} \left\| e^{-\frac{\mathcal{A}t}{n+2}} \begin{bmatrix} v_0 \\ v_1 \end{bmatrix} \right\|_{\mathcal{X}}. \end{aligned}$$

This inequality paired with (17) will now establish the assertion.  $\square$

Next, given  $1 > \alpha \geq \frac{1}{2}$ , we write

$$\frac{\alpha - 1/2}{1 - \alpha} = \left[ \frac{\alpha - 1/2}{1 - \alpha} \right] + \theta = k + \theta$$

where  $0 < \theta < 1$  and  $[\cdot]$  denotes the integral part of a real number.

In other words,

$$(18) \quad \begin{aligned} k &= \left[ \frac{\alpha - \frac{1}{2}}{1 - \alpha} \right]; \\ \theta &= \frac{\alpha - \frac{1}{2}}{1 - \alpha} - k. \end{aligned}$$

In these terms, the exponent  $(k + \theta)(1 - \alpha) + \frac{1}{2}$  appearing in Lemma 8 (with  $n = k$  therein) can be written as

$$k + 1/2 - k\alpha + \theta(1 - \alpha) = (k + \theta) + 1/2 - \alpha(k + \theta) = \frac{\alpha - 1/2}{1 - \alpha} + 1/2 - \frac{\alpha(\alpha - 1/2)}{1 - \alpha} = \alpha.$$

Combining this choice of  $(k, \theta)$  with Lemma 8 gives now,

COROLLARY 9. Let  $k = \lceil \frac{\alpha - \frac{1}{2}}{1 - \alpha} \rceil$  and  $\theta = \frac{\alpha - \frac{1}{2}}{1 - \alpha} - k$ . Assume that a self-adjoint operator  $B \in \mathcal{L}(H)$  is strictly positive and moreover satisfies  $B \in \mathcal{L}(D(\mathring{A}^{\frac{\alpha}{2}}))$ . Then for  $\alpha \in [\frac{1}{2}, 1)$ , the solution  $[v, v_t]$  of (5) obeys the following estimate for all  $t > 0$ :

$$\|\mathring{A}^\alpha v(t)\|_H \leq \frac{C_\alpha}{t^{\frac{\alpha - 1/2}{1 - \alpha}}} \sqrt{\mathcal{E}\left(\frac{t}{k + 2}\right)}.$$

PROOF. After noting that  $(k + \frac{1}{2})(1 - \alpha) = \frac{1}{2}\alpha - \theta(1 - \alpha) \leq \frac{\alpha}{2}$ , we can consequently apply inequality (16) of Lemma 8 – taking therein  $n = k = \lceil \frac{\alpha - \frac{1}{2}}{1 - \alpha} \rceil$ , and  $\theta = \frac{\alpha - \frac{1}{2}}{1 - \alpha} - k$ , so as to obtain the desired conclusion.  $\square$

#### 4. – Proof proper of Theorem 3

In what follows, we will have need of the polynomial

$$(19) \quad h(t) \equiv t^s (T - t)^s,$$

where

$$(20) \quad s = \begin{cases} 2, & \text{if } 0 \leq \alpha \leq \frac{3}{4} \\ \frac{2\alpha - 1}{1 - \alpha}, & \text{if } \frac{3}{4} < \alpha < 1 \end{cases}$$

(so in particular,  $s > 2$  for given  $\alpha \in (\frac{3}{4}, 1)$ ). This function is to be used in a multiplier method.

To start, we multiply the equation (5) by  $h(t)v$ , and integrate in time and space so as to have

$$\int_0^T h(t)(v_{tt} + \mathring{A}v + \mathring{A}^{\alpha/2} B \mathring{A}^{\alpha/2} v_t, v)_H dt = 0.$$

An integration of parts with respect to this expression (using implicitly  $h(0) = h(T) = 0$ ) yields now the following:

$$(21) \quad \begin{aligned} \int_0^T h(t) \|\mathring{A}^{\frac{1}{2}} v\|_H^2 dt &= \int_0^T (v_t, (hv)_t)_H dt - \int_0^T h(t) (\mathring{A}^\alpha v, \mathring{A}^{-\alpha/2} B \mathring{A}^{\alpha/2} v_t)_H dt \\ &= - \int_0^T h(t) (\mathring{A}^\alpha v, \mathring{A}^{-\alpha/2} B \mathring{A}^{\alpha/2} v_t)_H dt + \int_0^T h(t) \|v_t\|_H^2 dt \\ &\quad + \int_0^T h'(t) (v_t, v)_H dt. \end{aligned}$$

(i) Now, concerning the first term on the right hand side of (21), the argument will depend on the range of  $\alpha$ . When  $\alpha > 1/2$ , critical use will be made of analyticity of the semigroup  $e^{At}$ , along with the technical lemmas presented in Section 3. We begin with the case  $\alpha \leq 1/2$ , which has a more direct argument of proof.

Note that if  $\alpha \in [0, \frac{1}{2})$ , then the dynamical operator  $\mathcal{A} : D(\mathcal{A}) \subset \mathcal{X} \rightarrow \mathcal{X}$  is no longer of analytic character (but is of Gevrey’s class for  $0 < \alpha < 1/2$ , [3]). But on the other hand,  $\mathring{A}^\alpha v$  is strictly below the level of energy for such values of  $\alpha$ , and so the Lemma 8, essentially a product of analyticity, is not needed at all for  $\alpha \in [0, \frac{1}{2}]$ . Indeed, to estimate the first term on the right hand side of (21) for  $\alpha \leq \frac{1}{2}$ , we proceed as follows:

$$\begin{aligned}
 & \int_0^T h(t)(\mathring{A}^\alpha v, \mathring{A}^{-\alpha/2} B \mathring{A}^{\alpha/2} v_t)_H dt \\
 (22) \quad &= \int_0^T h(t)(\mathring{A}^{\alpha/2} B \mathring{A}^{-\alpha/2} \mathring{A}^\alpha v, v_t)_H dt \\
 &\leq \frac{\epsilon}{2} \int_0^T h(t) \|\mathring{A}^\alpha v\|_H^2 dt + C_\epsilon \int_0^T \|v_t\|_H^2 dt \\
 &\leq \epsilon \|\mathring{A}^{\alpha-\frac{1}{2}}\|_{\mathcal{L}(H)}^2 \int_0^T h(t) \mathcal{E}(t) dt + C_\epsilon \int_0^T \|v_t\|_H^2 dt,
 \end{aligned}$$

where we have used our standing assumption  $B \in \mathcal{L}(D(\mathring{A}^{\alpha/2}))$ .

For the case  $\alpha > 1/2$ , we employ the result of Corollary 9 with positive integer  $k$  prescribed therein:

$$\begin{aligned}
 & \left| \int_0^T h(t)(\mathring{A}^\alpha v, \mathring{A}^{-\alpha/2} B \mathring{A}^{\alpha/2} v_t)_H dt \right| \\
 &= \int_0^T h(t)(\mathring{A}^{\alpha/2} B \mathring{A}^{-\alpha/2} \mathring{A}^\alpha v, v_t)_H dt \\
 &\leq C \int_0^T h(t) \|\mathring{A}^\alpha v\|_H \|v_t\|_H dt \\
 &\leq \int_0^T h(t) \frac{C}{t^{\frac{\alpha-1/2}{1-\alpha}}} \sqrt{\mathcal{E}\left(\frac{t}{k+2}\right)} \|v_t\|_H dt.
 \end{aligned}$$

This gives then

$$\begin{aligned}
 & \left| \int_0^T h(t)(\mathring{A}^\alpha v, \mathring{A}^{-\alpha/2} B \mathring{A}^{\alpha/2} v_t)_H dt \right| \\
 (23) \quad &\leq \frac{\epsilon}{k+2} \int_0^T h(t) \mathcal{E}\left(\frac{t}{k+2}\right) dt + C_{\epsilon,\alpha} \int_0^T \frac{h(t)}{t^{\frac{2\alpha-1}{1-\alpha}}} \|v_t\|_H^2 dt \\
 &\leq \frac{\epsilon}{k+2} \int_0^T h(t) \mathcal{E}\left(\frac{t}{k+2}\right) dt + C_{\epsilon,\alpha} T^{2s-\frac{2\alpha-1}{1-\alpha}} \int_0^T \|v_t\|_H^2 dt.
 \end{aligned}$$

(ii) Moreover, concerning the third term on the right hand side of (21),

$$\begin{aligned}
 & \left| \int_0^T h'(t)(v_t, v)_H dt \right| \\
 (24) \quad & \leq \int_0^T |h'(t)| \|\mathring{\mathbf{A}}^{-\frac{1}{2}}\|_{\mathcal{L}(H)} \|\mathring{\mathbf{A}}^{\frac{1}{2}} v\|_H \|v_t\|_H \frac{\sqrt{h(t)}}{\sqrt{h(t)}} dt \\
 & \leq \frac{\epsilon}{2} \int_0^T h(t) \|\mathring{\mathbf{A}}^{\frac{1}{2}} v\|_H^2 dt + C_\epsilon \int_0^T \frac{[h'(t)]^2}{h(t)} \|v_t\|_H^2 dt \\
 & \leq \epsilon \int_0^T h(t) \mathcal{E}(t) dt + C_\epsilon T^{2s-2} \int_0^T \|v_t\|_H^2 dt.
 \end{aligned}$$

Incorporating (22) (for the case  $\alpha \leq 1/2$ ), (23) (for the case  $\alpha > 1/2$ ) and (24) into (21) yields now for  $T \leq 1$ ,

$$\begin{aligned}
 & \int_0^T h(t) \|\mathring{\mathbf{A}}^{\frac{1}{2}} v\|_H^2 dt \\
 & \leq \epsilon \int_0^T h(t) \mathcal{E}(t) dt + \frac{\epsilon}{k+2} \int_0^T h(t) \mathcal{E}\left(\frac{t}{k+2}\right) dt \\
 & + C_{\epsilon,\alpha} T^{2s-\frac{2\alpha-1}{1-\alpha}} \int_0^T \|v_t\|_H^2 dt + C_\epsilon T^{2s-2} \int_0^T \|v_t\|_H^2 dt \\
 & + C_\epsilon \int_0^T h(t) \|v_t\|_H^2 dt.
 \end{aligned}$$

After adding to both sides of relation above the term  $\int_0^T h(t) \|v_t\|_H dt$  and considering  $T \leq 1$  we obtain

$$\begin{aligned}
 & \int_0^T (1-\epsilon)h(t)\mathcal{E}(t)dt \\
 & \leq \frac{\epsilon}{k+2} \int_0^T h(t)\mathcal{E}\left(\frac{t}{k+2}\right) dt \\
 & + C_{\epsilon,\alpha} T^{2s-\frac{2\alpha-1}{1-\alpha}} \int_0^T \|v_t\|_H^2 dt + C_\epsilon T^{2s-2} \int_0^T \|v_t\|_H^2 dt \\
 & \leq \epsilon \int_0^{\frac{T}{k+2}} h((k+2)t)\mathcal{E}(t)dt + C_{\epsilon,\alpha} T^{2s-\frac{2\alpha-1}{1-\alpha}} \int_0^T \|v_t\|_H^2 dt \\
 & + C_\epsilon T^{2s-2} \int_0^T \|v_t\|_H^2 dt.
 \end{aligned}$$

Hence,

$$\begin{aligned}
 (25) \quad & \int_0^{\frac{T}{k+2}} [(1-\epsilon)h(t) - \epsilon h((k+2)t)]\mathcal{E}(t)dt + (1-\epsilon) \int_{\frac{T}{k+2}}^T h(t)\mathcal{E}(t)dt \\
 & \leq C_{\epsilon,\alpha} T^{2s-\frac{2\alpha-1}{1-\alpha}} \int_0^T \|v_t\|_H^2 dt + C_\epsilon T^{2s-2} \int_0^T \|v_t\|_H^2 dt.
 \end{aligned}$$

Choosing  $\epsilon > 0$  small enough so that

$$(1 - \epsilon)h(t) - \epsilon h((k+2)t) > 0 \quad \text{on} \quad \left(0, \frac{T}{k+2}\right) \quad \text{and} \quad (1 - \epsilon) > 0$$

(e.g., we can take  $\epsilon \leq \frac{(k+1)^s}{(2+k)^s(1+(k+2)^s)}$ ), then the estimate (25) and the inherent dissipativity of the structurally damped system (5) (i.e.,  $\mathcal{E}(t) \leq \mathcal{E}(s)$  for  $0 \leq s \leq t \leq T$ ), give in combination,

$$(26) \quad \mathcal{E}(T) \int_{\frac{T}{k+2}}^T h(t) dt \leq C_{\epsilon, \alpha} T^{2s - \frac{2\alpha-1}{1-\alpha}} \int_0^T \|v_t\|_H^2 dt + C_\epsilon T^{2s-2} \int_0^T \|v_t\|_H^2 dt.$$

Now,

$$(27) \quad \begin{aligned} \int_{\frac{T}{k+2}}^T h(t) dt &= \int_{\frac{T}{k+2}}^T t^s (T-t)^s dt \\ &\geq \int_{\frac{T}{k+2}}^T \left(t - \frac{T}{k+2}\right)^s (T-t)^s dt \\ &= \left(\frac{k+1}{k+2} T\right)^{2s+1} B(s+1, s+1) \end{aligned}$$

(see e.g., [11], p. 285, 3.196 no. 3), where  $B(\cdot, \cdot)$  denotes the Beta function, defined by

$$\beta(x, y) = \int_0^1 t^{x-1} (1-t)^{y-1} dt, \quad x \geq 0, y > 0.$$

Combining this inequality with (26) gives finally

$$\mathcal{E}(T) \leq C_\alpha (T^{\frac{-\alpha}{1-\alpha}} + T^{-3}),$$

which inequality for  $T \leq 1$  can be written as

$$\mathcal{E}(T) \leq C_\alpha T^{-\mu_\alpha} \int_0^T \|v_t\|_H^2 dt,$$

where

$$\mu_\alpha = \begin{cases} 3, & \text{if } 0 \leq \alpha \leq \frac{3}{4}; \\ \frac{\alpha}{1-\alpha}, & \text{if } \frac{3}{4} < \alpha < 1. \end{cases}$$

We conclude therefore, that the abstract system (1) is null controllable, with the associated observability inequality  $C_T$  of (7) being  $\mathcal{O}(T^{-\frac{\mu_\alpha}{2}})$ . Subsequently, a standard argument (see e.g., [14], [1]) gives now that likewise, the minimal energy function  $\mathcal{E}_{\min}(T) = \mathcal{O}(T^{-\frac{\mu_\alpha}{2}})$ . This completes the proof of Theorem 3.

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