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Local and Canonical Heights of Subvarieties

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Abstract. Classical results of Weil, Néron and Tate are generalized to local heights of subvarieties with respect to hermitian pseudo-divisors. The local heights are well-defined if the intersection of supports is empty. In the archimedean case, the metrics are hermitian and the local heights are defined by a refined version of the $*$-product of Gillet-Soulé developped on compact varieties without assuming regularity. In the non-archimedean case, the local heights are intersection numbers using methods from rigid and formal geometry to handle non-discrete valuations. To include canonical metrics of line bundles algebraically equivalent to 0, a local Chow cohomology is introduced on formal models over the valuation ring. Using Tate's limit argument, canonical local heights of subvarieties on an abelian variety are obtained with respect to any pseudo-divisors. By integration over an M-field, we deduce corresponding results for global heights of subvarieties.

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1. – Introduction

In diophantine geometry, one of the most important problems is to control the number of rational points. A basic tool to prove finiteness statements or to describe the distribution of infinitely many points is the height. Weil [We] pointed out that heights may be decomposed into a sum of local heights and most of the properties already occur locally. For a meromorphic section s of a metrized line bundle, the local height of a point P with respect to the place v is given by $\lambda_{\mathrm{div}(s)}(P, v) = -\log \|s(P)\|_v$. For a complete variety, Weil's theorem says that local (resp. global) heights are determined by the divisor (resp. its rational equivalence class) up to bounded functions. Néron [Ne] has noticed that on an abelian variety or with respect to divisors algebraically equivalent to 0, there are canonical local heights, determined by the divisor up to an additive constant. The ambiguity disappears for the global Néron-Tate heights.

Nesterenko and Philippon [Ph1] define the height of a subvariety Y of $\mathbb{P}^n$ over a number field K as the height of the Chow form. Faltings [Fa2] gave a

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new definition of heights of subvarieties using the arithmetic intersection theory of Gillet-Soulé [GS2]. Faltings' height is the arithmetic analogue of the degree in algebraic geometry, it is defined as an arithmetic intersection number with $t+1$ arithmetic divisors $\widehat{\operatorname{div}}(s_i)$ for generic global sections s_i of $\overline{O}_{\mathbb{P}^n}(1)$.

On an abelian variety A over K, there is also a Néron-Tate height of t-dimensional cycles with respect to $L_0, \dots, L_t \in \operatorname{Pic}(A)$. For $L_0 = \dots = L_t$ even and projective normal, it was introduced in [Ph2]. In [Gu1], Weil's theorem was generalized to heights of subvarieties and with Tate's limit argument, the Néron-Tate height was obtained for $L_0, \dots, L_t$ even. On the other hand, if one line bundle is odd, then the Néron-Tate height is a special case of the height pairing of Beilinson [Bei] and Bloch [Bl]. This gives also canonical heights on a smooth variety over K if at least one line bundle is algebraically equivalent to 0.

Gillet-Soulé [GS1] proved an arithmetic Hilbert-Samuel formula and Bost-Gillet-Soulé [BoGS] gave an arithmetic Bézout theorem for the height of subvarieties. A beautiful application of heights of subvarieties is Zhang's proof [Zh3] of the Bogomolov conjecture. This was generalized by Moriwaki [Mor] from number fields to finitely generated fields over $\mathbb{Q}$.

Using Faltings' definition, the local height of a t-dimensional subvariety Y is given by the $*$-product of the $[\log \|s_i\|^{-2}]$ on Y in the archimedean case and by the intersection number of the $\operatorname{div}(s_i)$ on a model of Y in the case of a discrete valuation. For a non-archimedean absolute value on any field K, the valuation ring K° is not noetherian and hence the intersection theory on models over K° from algebraic geometry is not available. Then one has to work on admissible formal K°-models $\mathfrak{X}$ using the theory of Bosch-Lütkebohmert (cf. [BL2], [BL3]) initiated by Raynaud. Using valuations, an intersection theory with Cartier divisors on $\mathfrak{X}$ was given in [Gu3] leading to the desired local heights in the non-archimedean case.

Classically, one has considered heights of points over number fields or function fields. It was pointed out by Vojta [Vo] that there are striking similarities of the height of points to the characteristic function in Nevanlinna theory leading to far reaching conjectures.

In [Gu2], the notion of M-fields was introduced including all these cases and also the finitely generated fields over $\mathbb{Q}$ considered by Moriwaki [Mor]. By integrating local heights over M, global heights of subvarieties were obtained satisfying Weil's theorem leading to a generalization of the first main theorem of Nevanlinna theory to higher dimensions.

In the present article, a first goal is to define local heights with respect to pseudo-divisors. Let X be a proper scheme over a field K with an absolute value. Local heights are invariant under base change, so we may assume without loss of generality that K is complete and algebraically closed. A pseudo-divisor on X is a triple $D = (L, Y, s)$ where L is a line bundle, Y is a closed subset of X and s is a nowhere vanishing section of $L|_{X \setminus Y}$ (cf. [Fu], 2.2). The concept of pseudo-divisors has two advantages over Cartier divisors. First, every line bundle gives rise to a pseudo-divisor and second, the pull-back of pseudo-divisors is always well-defined. In the complex case, Section 2 treats a refined $*$-product of hermitian pseudo-divisors with Green currents for cycles on X without any smoothness assumptions on X. In Section 3, this leads to a local height with respect to hermitian pseudo-divisors $\hat{D}_0, \dots, \hat{D}_t$, well-defined on t-dimensional cycles Z of X with $|D_0| \cap \dots \cap |D_t| \cap |Z| = \emptyset$. Previously, only Cartier divisors were considered and to apply arithmetic intersection theory, one had to assume that X is smooth and that all partial intersections formed with subsets of $D_0, \dots, D_t, Z$ are proper.

For a non-trivial non-archimedean complete absolute value on K with valuation ring K°, we study generically proper intersections with Cartier divisors on admissible formal K°-models in Section 4. For algebraic generic fibre, Section 5 gives a refined intersection theory leading to the same conclusions for local heights as in the complex case.

The main part of the article is dedicated to develop a theory of canonical local heights of cycles with similar properties as in the zero-dimensional case. There are two cases where they occur, first if one line bundle is algebraically equivalent to 0 and second if one deals with abelian varieties. For an archimedean absolute value, it is well known that the canonical metrics are the smooth hermitian metrics with harmonic curvature form.

For a non-archimedean complete absolute value, the canonical metrics on line bundles algebraically equivalent to 0 can not be interpreted in terms of formal K°-models of the line bundle (cf. Example 7.20). In Section 6, we develop a local Chow cohomology theory on admissible formal models over K° in the style of Fulton ([Fu], Chapter 17). These groups formalize the refined intersection theoretic properties of Chern classes. To get rid of particular K°-models, the projective limit over all admissible formal K°-models is considered giving rise to a local Arakelov-Chow group in the style of non-archimedean Arakelov theory of Bloch-Gillet-Soulé [BlGS] and corresponding local Arakelov-Chow cohomology groups. Every formal K°-model of a line bundle L gives rise to a so-called formal metric on L determing completely the divisoral operation on the local Arakelov-Chow groups. Section 7 handles admissible metrics which are locally equal to the tensor product of positive real powers of formal metrics. For a divisor D on a complete variety with an admissible metric on $O(D)$, we can define the proper intersection product $\widehat{D}.\alpha$ in the local Arakelov-Chow group only for α without vertical components of codimension ≥ 1 in the special fibres because there is no reduction of $\widehat{D}$ to the special fibres. If X is a semistable curve, this is just enough to get a local Arakelov-Chow cohomology class.

In Section 8, we introduce admissible first Chern classes in the Arakelov-Chow cohomology groups which may be seen as cohomological generalizations of K°-models of L and include also the above local Arakelov-Chow cohomology classes on semistable curves. They give rise to associated metrics on L called cohomological metrics. In Section 9, a theory of local heights with respect to such admissible first Arakelov-Chern classes is given and it is shown that the dependence of local heights on the admissible first Arakelov-Chern classes is given by the associated metrics. For a line bundle algebraically equivalent to 0, we get our desired canonical local heights by noting that a canonical metric is cohomological.

Section 10 extends the theory of local heights allowing uniforming limits under semipositivity assumptions. By Tate's limit argument, we get canonical local heights also on abelian varieties agreeing with the above for odd line bundles. In Section 11, global heights of subvarieties Y over an M-field K are obtained by integrating local heights of Y over M. If the product formula is satisfied, then the dependence of global heights on pseudo-divisors $D_0, \dots, D_t$ is given by the isomorphism classes of $O(D_0), \dots, O(D_t)$. If one line bundle is algebraically equivalent to zero or if the underlying space is an abelian variety, then we get global canonical heights including all previously considered cases.

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2. – Refined $*$-product with hermitian pseudo-divisors

In this section, we consider complex compact algebraic varieties $X, X', \dots$ endowed with its analytic structures. By the GAGA-principle, all closed analytic subsets, cycles, morphisms, line bundles and sections are algebraic.

2.1. First, we introduce differential forms on singular spaces as in Bloom-Herrera [BH]. Locally, an open subset U of X is a closed analytic subset of an open complex ball. *Smooth differential forms* on U are locally given by restriction of smooth forms defined on such balls. They are identified if they agree on U_{reg}. We denote the space of smooth differential forms on U by $A^*(U)$. Pull-back with respect to morphisms and d, d^c are defined on $A^*(U)$ by extending the forms locally to a ball as above and then using the constructions for complex manifolds.

2.2. A *current* of dimension r on X is a linear functional T on $A^r(X)$ given locally by a current T_B on an open ball B as above such that $T_B(\omega) = T(\omega|_{X\cap B})$ for every compactly supported $\omega \in A^r(B)$. The space of currents is denoted by $D_*(X)$ graded by dimension. Bigrading, differentiation and push-forward, the current of integration δ_Y over an irreducible closed analytic subset Y (extended linearly to cycles) and the current $[\eta]$ associated to an integrable form η are defined as in the smooth case. For details, we refer to [Ki].

Definition 2.3. A *Green current* for a cycle Z of dimension t is $g_Z \in D_{t+1,t+1}(X)$ with

$$dd^c g_Z = [\omega_Z] - \delta_Z$$

for a smooth differential form ω_Z on X. By linearity, we extend the definition to all cycles.

EXAMPLE 2.4. Let L be a holomorphic line bundle on X with a smooth hermitian metric $\| \ \|$. Then $\overline{L} = (L, \| \ \|)$ will be called a *hermitian line bundle*. We call a meromorphic section s of L *invertible*, if there is an open dense subset U of X such that s restricts to a nowhere vanishing holomorphic section of L on U. The Poincaré-Lelong equation gives

$$dd^c[\log \|s\|^{-2}] = [c_1(\overline{L})] - \delta_{\mathrm{div}(s)}$$

with $c_1(\overline{L})$ the Chern form of $\overline{L}$ and $\delta_{\mathrm{div}(s)}$ the current of integration over the Weil divisor associated to the Cartier divisor div(s) (cf. [GH], p. 388, and use resolution of singularities).

DEFINITION 2.5. A *hermitian pseudo-divisor* $\hat{D}$ on X is a triple $(\overline{L}, Y, s)$ where $\overline{L}$ is a hermitian line bundle on X, Y is a closed analytic subset of X and s is a nowhere vanishing section of L on $X \setminus Y$. Then $D := (O(D), |D|, s_d) := (L, Y, s)$ is a pseudo-divisor on X. Two hermitian pseudo-divisors on X will be identified if the support $|D|$ is the same and if there is an isometry between the hermitian line bundles mapping one section to the other.

The *sum* and the *pull-back* of hermitian pseudo-divisors are defined by

$$\hat{D}+\hat{E} := (\overline{O}(D)\otimes\overline{O}(E), |D|\cup|E|, s_D\otimes s_E)\ ,\ \varphi^*\hat{D} := (\varphi^*\overline{O}(D), \varphi^{-1}|D|, \varphi^* s_D).$$

PROPOSITION 2.6. *If $\varphi : X' \to X$ is a morphism and $g_{Z'}$ is a Green currrent for a cycle Z' on X' such that $\varphi_*[\omega_{Z'}]$ is the current associated to a smooth differential form on X* (e.g. *this holds if X and φ are smooth*), *then $\varphi_*(g_{Z'})$ is a Green current for $\varphi_*(Z')$.*

PROOF. This follows from $dd^c\varphi_*(g_{Z'}) = \varphi_*(dd^c g_{Z'}) = \varphi_*[\omega_{Z'}] - \varphi_*\delta_{Z'}$ and $\varphi_*\delta_{Z'} = \delta_{\varphi_*(Z')}$. If φ and X are smooth, then integration along the fibres shows that $\varphi_*[\omega_{Z'}]$ is smooth. □

EXAMPLE 2.7. Let Y be an irreducible closed subvariety of X and let $f \in K(Y)^\times$. We view div(f) as a cycle on X using $i : Y \hookrightarrow X$. By Example 2.4 and Proposition 2.6, $i_*[\log |f|^{-2}]$ is a Green current for div(f) on X also denoted by $[\log |f|^{-2}]$.

A K_1*-chain* $\mathbf{f} = \sum f_W$ on X is a formal sum of $f_W \in K(W)^\times$ where W ranges over all irreducible closed subvarieties of X and $f_W \neq 1$ only for finitely many f_W. By linearity, we define the cycle div($\mathbf{f}$) with Green current $[\log |\mathbf{f}|^{-2}]$. For a morphism $\varphi : X \to X'$, we have $\varphi_*[\log |\mathbf{f}|] = [\log |\varphi_*\mathbf{f}|]$ where $\varphi_*\mathbf{f} := \sum N_{K(W)/K(\varphi W)}(f_W)$. To prove it, we may restrict to the finite generically étale case where the norm is the sum over the fibre points (cf. [GS2], 3.6).

DEFINITION 2.8. Let $\hat{D} = (\overline{L}, |D|, s)$ be a hermitian pseudo-divisor and let g_Z be a Green current for the cycle Z on X. In the definition of the $*$*-product*

$$\hat{D} * g_Z := \hat{D} \wedge \delta_Z + c_1(\overline{L}) \wedge g_Z,$$

the second summand is the current $\eta \mapsto g_Z(c_1(\overline{L}) \wedge \eta)$. To define the first summand, by linearity, we may assume Z prime. If $|Z| \not\subset |D|$, then let $s_Z := s|_Z$ and if $Z \subset |D|$, then we choose any non-zero meromorphic section s_Z of $L|_Z$. We set $(\hat{D} \wedge \delta_Z)(\eta) := \int_Z \log \|s_Z\|^{-2}\eta$.

REMARK 2.9. The currents $\hat{D} * \delta_Z$ and $\hat{D} \wedge g_Z$ are defined up to $[\log |\mathbf{f}|^{-2}]$ for K_1-chains $\mathbf{f}$ on $|D| \cap |Z|$. If $|D|$ intersects $|Z|$ properly, then they are well-defined currents.

If $\overline{L}$ is a hermitian line bundle with an invertible meromorphic section s, then we write $\log \|s\|^{-2}$ instead of the associated pseudo-divisor $(\overline{L}, |\operatorname{div}(s)|, s)$. Then for X smooth and $\operatorname{div}(s)$ intersecting Z properly, $\log \|s\|^{-2} * g_Z$ is the $*$-product of Gillet-Soulé [GS2].

PROPOSITION 2.10. *$\hat{D} * g_Z$ is a Green current for a cycle representing $D.Z \in CH(|D| \cap |Z|)$.*

PROOF. Note first that $dd^c(c_1(\overline{L}) \wedge g_Z) = [c_1(\overline{L}) \wedge \omega_Z] - c_1(\overline{L}) \wedge \delta_Z$. Thus it is enough to show

$$dd^c(\hat{D} \wedge \delta_Z) = c_1(\overline{L}) \wedge \delta_Z - \delta_{Z'}$$

for a representative Z' of $D.Z \in CH(|D| \cap |Z|)$. We may assume Z prime. Then the claim follows from the Poincaré-Lelong equation for s_Z on Z and with Z' the Weil divisor of s_Z. □

PROPOSITION 2.11. *Let $\varphi : X' \to X$ be a morphism, let $\hat{D}$ be a hermitian pseudo-divisor on X and let $g_{Z'}$ be a Green current for a cycle Z' on X' such that $\varphi_*[\omega_{Z'}]$ is a smooth differential form on X. Then up to K_1-chains on $|D| \cap \varphi(|Z'|)$, we have the projection formula*

$$\varphi_*(\varphi^*\hat{D} * g_{Z'}) = \hat{D} * \varphi_* g_{Z'}.$$

PROOF. Let $\hat{D} = (\overline{L}, Y, s)$. The obvious identity $\varphi_*(c_1(\varphi^*\overline{L}) \wedge g_{Z'}) = c_1(\overline{L}) \wedge \varphi_* g_{Z'}$ of currents on X shows that it is enough to prove

$$\varphi_*(\varphi^*\hat{D} \wedge \delta_{Z'}) = \hat{D} \wedge \delta_{\varphi_* Z'} \tag{1}$$

up to $[\log |\mathbf{f}|^{-2}]$ for a K_1-chain $\mathbf{f}$ on $|D| \cap \varphi(|Z'|)$. We may assume Z' prime, $Z' = X'$, φ surjective and $Y \neq X$. We have to prove $\varphi_*[\log \varphi^*\|s\|^{-2}] = \deg \varphi \cdot [\log \|s\|^{-2}]$. The equidimensional case is clear from integration along the fibres because φ is generically smooth. If $\dim(X') > \dim(X)$, the formula is obvious since the degree is 0 by definition. □

2.12. Note that (1) holds without assuming that $\varphi_*[\omega_{Z'}]$ is the current associated to a smooth differential form on X. We introduce an equivalence $S \equiv T$ between currents on X if $S - T$ may be written as a sum of a d- and a d^c-boundary.

PROPOSITION 2.13. *Let $\hat{D}$, $\hat{E}$ be hermitian pseudo-divisors and let g_Z be a Green current for a cycle Z on X. Up to K_1-chains on $|D| \cap |E| \cap |Z|$, we have*

$$\hat{D} * (\hat{E} * g_Z) \equiv \hat{E} * (\hat{D} * g_Z).$$

PROOF. Let $\hat{D} = (\overline{L}, |D|, s)$ and $\hat{E} = (\overline{L'}, |E|, s')$. It is enough to show

$$\hat{D} \wedge \delta_{E.Z} + c_1(\overline{L}) \wedge (\hat{E} \wedge \delta_Z) \equiv \hat{E} \wedge \delta_{D.Z} + c_1(\overline{L'}) \wedge (\hat{D} \wedge \delta_Z) \tag{2}$$

up to K_1-chains on $|D| \cap |E| \cap |Z|$ choosing the appropriate representatives $D.Z$ and $E.Z$ from Proposition 2.10. To prove (2), we may assume Z prime. Using the notation of Definition 2.8, (2) is the push-forward of the commutativity law

$$\log \|s_Z\|^{-2} * [\log \|s'_Z\|^{-2}] \equiv \log \|s'_Z\|^{-2} * [\log \|s_Z\|^{-2}]$$

with respect to $Z \overset{i}{\hookrightarrow} X$. So we may assume $Z = X$ and that D, D' are Cartier divisors.

Let $\pi : X' \to X$ be a birational morphism of irreducible compact algebraic varieties. Then we have $\pi_*[\log \pi^* \|s\|^{-2}] = \log \|s\|^{-2}, \pi_*[c_1(\pi^* \overline{L})] = [c_1(\overline{L})]$. By projection formula, it is enough to prove commutativity on X'. With Hironaka's resolution of singularities and Chow's lemma, we reduce to a smooth projective variety where either the intersection of D and D' is proper or $D = D'$. The first case follows from the commutativity law of the $*$-product for logarithmic Green forms ([GS2], Corollary 2.2.9) and the latter case is trivial. □

PROPOSITION 2.14. *Let g_Z be a Green current and $\hat{D}$ be a hermitian pseudo-divisor on X.*

(a) *If f is an invertible meromorphic function on X, then $\log |f|^{-2} * g_Z = [\log |\mathbf{h}|^{-2}]$ for a suitable K_1-chain $\mathbf{h}$ on $|Z|$.*
(b) *If $\mathbf{f}$ is a K_1-chain on a closed subvariety Y of X, then there is a K_1-chain $\mathbf{h}$ on $|D| \cap |Y|$ with $\hat{D} * [\log |\mathbf{f}|^{-2}] \equiv [\log |\mathbf{h}|^{-2}]$.*

PROOF. Claim (a) is trivial and (b) follows from (a) and commutativity. □

REMARK 2.15. The refined $*$-product can not be extended to reduced complex spaces. The problem is that a holomorphic line bundle restricted to an irreducible closed analytic subset may have no invertible meromorphic section. This cruicial property holds however on irreducible compact complex Moishezon spaces ([Moi], Theorem 4) and all results of this section remain valid. For the proof of Proposition 2.13, one uses that a complex Moishezon space is birationally covered by an irreducible compact algebraic variety ([Moi], Theorem 7).

The refined $*$-product may also be generalized to non-compact complex algebraic varieties if one considers only algebraic objects and proper push-forwards. The proofs remain the same.

3. – Local heights on complex algebraic varieties

Let X, X' be a complex compact algebraic varieties endowed with the complex analytic structures (or more generally complex Moishezon spaces using Remark 2.15).

3.1. Let $i : Z \to X$ be the embedding of a prime cycle. On Z, 0 may be viewed as a Green current 0_Z for Z with $\omega_Z = 1$. For hermitian pseudo-divisors $\hat{D}_1, \ldots, \hat{D}_k$ on X, we define

$$\hat{D}_1 * \cdots * \hat{D}_k \wedge \delta_Z := i_*(i^*\hat{D}_1 * \cdots * i^*\hat{D}_k * 0_Z)$$

as a current on X, well-defined up to K_1-chains on $|D_1| \cap \cdots \cap |D_k| \cap |Z|$. If Z is an arbitrary cycle, then we proceed by linearity.

Note that the pseudo-divisors act from the left, hence there is no associativity law but the products have to be read from the right. The following is our substitute for associativity.

Proposition 3.2. *For a Green current g_Z, we have up to K_1-chains on $|D_1| \cap \cdots \cap |D_k| \cap |Z|$:*

$$\hat{D}_1 * \cdots * \hat{D}_k * g_Z = \hat{D}_1 * \cdots * \hat{D}_k \wedge \delta_Z + c_1(\overline{O}(D_1)) \wedge \cdots \wedge c_1(\overline{O}(D_k)) \wedge g_Z.$$

Proof. The proof follows easily by using induction on k, applying Proposition 2.10 to $\hat{D}_k * g_Z$ and then proceeding on Z. We leave the details to the reader. □

Definition 3.3. Let $\hat{D}_0, \ldots, \hat{D}_t$ be hermitian pseudo-divisors and let Z be a cycle on X of dimension t. We say that the *local height* $\lambda(Z)$ of Z with respect to $\hat{D}_0, \ldots, \hat{D}_t$ is *well-defined* if $|D_0| \cap \cdots \cap |D_t| \cap |Z| = \emptyset$ and we set

$$\lambda(Z) := \lambda_{\hat{D}_0,\ldots,\hat{D}_t}(Z) := (\hat{D}_0 * \cdots * \hat{D}_t \wedge \delta_Z)(1/2).$$

Proposition 3.4. *The local height is multilinear and symmetric in the variables $\hat{D}_0, \ldots, \hat{D}_t$, and linear in Z, under the hypothesis that all terms are well-defined.*

This follows from Section 2. From Proposition 3.2, we also obtain:

Proposition 3.5. *Let Z be a prime cycle with $|D_0| \cap \cdots \cap |D_t| \cap |Z| = \emptyset$. For D_t, let $s_{t,Z}$ be as in* 2.8 *with Weil-divisor considered as a cycle Y on X. We have the induction formula*

$$\lambda_{\hat{D}_0,\ldots,\hat{D}_t}(Z) = \lambda_{\hat{D}_0,\ldots,\hat{D}_{t-1}}(Y) - \int_Z \log \|s_{t,Z}\| \cdot c_1(\overline{O}(D_0)) \wedge \cdots \wedge c_1(\overline{O}(D_{t-1})).$$

Proposition 3.6. *For a morphism $\varphi : X' \to X$, hermitian pseudo-divisors $\hat{D}_0, \ldots, \hat{D}_t$ on X and a t-dimensional cycle Z' on X' with $|D_0| \cap \cdots \cap |D_t| \cap \varphi(|Z'|) = \emptyset$, we have functoriality*

$$\lambda_{\varphi^*\hat{D}_0,\ldots,\varphi^*\hat{D}_t}(Z') = \lambda_{\hat{D}_0,\ldots,\hat{D}_t}(\varphi_* Z').$$

PROOF. We proceed by induction on t. The claim is obvious for $t = 0$. To prove the claim for $t \geq 1$, we may assume Z' prime, $Z' = X'$ and φ surjective. By Proposition 3.5, we have

$$\lambda_{\varphi^*\hat{D}_0,\dots,\varphi^*\hat{D}_t}(Z') = \lambda_{\varphi^*\hat{D}_0,\dots,\varphi^*\hat{D}_{t-1}}(Y') - \int_{Z'} \log \|\varphi^* s_{t,X}\| \cdot c_1(\varphi^*\overline{O}(D_0)) \wedge \cdots \wedge c_1(\varphi^*\overline{O}(D_{t-1}))$$

where Y' is the Weil divisor associated to $\varphi^* s_{t,X}$. By induction hypothesis applied to Y', by projection formula and by the transformation formula of integrals, this is equal to

$$\lambda_{\hat{D}_0,\dots,\hat{D}_{t-1}}(\mathrm{div}(s_{t,X})) - \int_{\varphi_* Z'} \log \|s_{t,X}\| \cdot c_1(\overline{O}(D_0)) \wedge \cdots \wedge c_1(\overline{O}(D_{t-1})).$$

By induction formula again, we get the claim. □

PROPOSITION 3.7. *Let $\hat{D}_0, \dots, \hat{D}_t$ be hermitian pseudo-divisors on X such that the local height $\lambda(Z)$ of the t-dimensional cycle Z is well-defined. We assume $c_1(\overline{O}(D_0)) = 0$. Let Y be a representative of $D_1 \dots D_t.Z \in CH_0(|D_1| \cap \cdots \cap |D_t| \cap |Z|)$. Then $\lambda(Z) = -\log \|s_{D_0}(Y)\|$ where the right hand side is defined by linearity in the components of Y.*

PROOF. Let $s_0 := s_{D_0}$. First, we check that $\log \|s_0(Y)\|$ is well-defined. Since $\lambda(Z)$ is well-defined, we get $|D_0| \cap |Y| = \emptyset$, i.e. $\log \|s_0(Y)\|$ makes sense. If Y' is another representative of $D_1 \dots D_t.Z$, then there is a K_1-chain $\mathbf{f}$ on $|D_1| \cap \cdots \cap |D_t| \cap |Z|$ with $\mathrm{div}(\mathbf{f}) = Y - Y'$, thus

$$\log \|s_0(Y')\| - \log \|s_0(Y)\| = (\hat{D}_0 \wedge \delta_{\mathrm{div}(\mathbf{f})}) \left(\frac{1}{2}\right) = (\hat{D}_0 * [\log |\mathbf{f}|^{-2}]) \left(\frac{1}{2}\right).$$

By Proposition 2.14, the $*$-product is from a K_1-chain on $|D_0| \cap \cdots \cap |D_t| \cap |Z| = \emptyset$, hence we have $\log \|s_0(Y)\| = \log \|s_0(Y')\|$ proving independence of the representative.

To prove the formula for $\lambda(Z)$, we may assume Z prime and $Z = X$. We may assume that Y has the Green current $g_Y := \hat{D}_1 * \cdots * \hat{D}_t * 0_Z$. Then the claim follows from

$$\lambda(Z) = (\hat{D}_0 * g_Y) \left(\frac{1}{2}\right) = (\hat{D}_0 \wedge \delta_Y) \left(\frac{1}{2}\right) = -\log \|s_{0,Z}(Y)\|. \qquad \square$$

PROPOSITION 3.8. *Let $\lambda(Z)$ be the local height with respect to the hermitian pseudo-divisors $\hat{D}_0, \dots, \hat{D}_t$. Replacing the metric $\|\ \|$ on $O(D_0)$ by another smooth hermitian metric $\|\ \|'$, we get the local height $\lambda'(Z)$. Then $\rho := \log(\|s_{D_0}\|'/\|s_{D_0}\|$ extends to a C^∞-function on X and*

$$\lambda(Z) - \lambda'(Z) = \int_Z \rho \cdot c_1(\overline{O}(D_1)) \wedge \cdots \wedge c_1(\overline{O}(D_t)).$$

PROOF. This follows from Proposition 3.4 and induction formula 3.5. □

DEFINITION 3.9. Let $\overline{L}$ be a hermitian line bundle on X. For compact manifolds, the notion of semipositive curvature form is well-known. For singular varieties, the metric is said to have *semipositive curvature form* if for every smooth variety X' and every morphism $\varphi : X' \to X$, the hermitian line bundle $\varphi^*\overline{L}$ has semipositive curvature form. This will be used in Section 10.

4. – Operations of divisors on admissible formal schemes

Let K be an algebraically closed field with a non-trivial non-archimedean complete absolute value $|\ |$ and valuation ring K°. The basic reference for rigid analytic varieties is [BGR] and for admissible formal schemes is [BL2], [BL3]. For intersection products of Cartier divisors on rigid analytic varieties or admissible formal schemes, we refer to [Gu3].

4.1. We recall facts from formal geometry (cf. [Gu3] for details). There is a functor $\mathfrak{X} \mapsto \mathfrak{X}^{\text{f-an}}$ from the category of admissible formal schemes over K° to the category of formal analytic varieties and a functor from the latter category to the category of rigid analytic varieties. The composition of these functors is called the *generic fibre* $\mathfrak{X}^{\text{an}}$. The *special fibre* $\tilde{\mathfrak{X}}$ is a scheme locally of finite type over the residue field $\tilde{K}$. For a formal analytic variety $\mathfrak{Y}$, the *reduction* $\tilde{\mathfrak{Y}}$ is a reduced scheme locally of finite type over $\tilde{K}$. There is a natural finite morphism $i : (\mathfrak{X}^{\text{f-an}})^\sim \to \tilde{\mathfrak{X}}$. Let Y be a closed analytic subvariety of $\mathfrak{X}^{\text{an}}$. There is a unique admissible closed subscheme $\bar{Y}$ of $\mathfrak{X}$ with generic fibre Y.

There is also a functor $\mathfrak{Y} \mapsto \mathfrak{Y}^{\text{f-sch}}$ from the category of formal analytic varieties to the category of admissible formal schemes over K° given locally by $\mathrm{Spf}\mathcal{A} \mapsto \mathrm{Spf}\mathcal{A}^\circ$ for a K-affinoid algebra $\mathcal{A}$ with $\mathcal{A}^\circ := \{a \in \mathcal{A} \mid |a|_{\sup} \leq 1\}$. This induces an equivalence of the category of reduced formal analytic varieties to the category of admissible formal schemes with reduced special fibre, and then i is an isomorphism. This will be frequently used to switch from the admissible to the formal analytic point of view.

EXAMPLE 4.2. Let $\mathfrak{X}$ be a scheme flat and locally of finite type over K°. We claim that the formal completion $\mathfrak{X}^{\text{f-sch}}$ along the special fibre (more precisely with respect to the ideal $\langle \pi \rangle$ for $\pi \in K^\circ, |\pi| < 1$) is an admissible formal scheme over K°. Locally, the scheme $\mathfrak{X}$ is isomorphic to $\mathrm{Spec}A$ for a flat K°-algebra of finite type. Then the completion $\hat{A}$ of A with respect to the $\langle \pi \rangle$-adic topology is an admissible K°-algebra (use [BL2], Proposition 1.1, Lemma 1.6). The formal completion $\mathfrak{X}^{\text{f-sch}}$ is locally given by $\mathrm{Spec}\hat{A}$ proving that $\mathfrak{X}^{\text{f-sch}}$ is admissible. Note that $(\mathfrak{X}^{\text{f-sch}})^{\text{an}}$ is an analytic subdomain of $(\mathfrak{X} \otimes_{K^\circ} K)^{\text{an}}$ whose points are the K°-integral points of $\mathfrak{X}$. The special fibres of $\mathfrak{X}$ and $\mathfrak{X}^{\text{f-sch}}$ are the same.

4.3. Let $\mathfrak{X}$ be an admissible formal scheme over K° and let G be a subgroup of $\mathbb{R}$ containing $\log|K^\times|$. The group of cycles on $\tilde{\mathfrak{X}}$ with coefficients in G is denoted by $Z(\tilde{\mathfrak{X}}, G)$. The group of *cycles* on $\mathfrak{X}$ is defined by $Z(\mathfrak{X}, G) := Z(\mathfrak{X}^{\mathrm{an}}) \oplus Z(\tilde{\mathfrak{X}}, G)$ and graded by relative dimension over $\mathrm{Spf} K^\circ$. Note that the *horizontal part* $\mathfrak{Z}^{\mathrm{an}}$ of $\mathfrak{Z} \in Z(X, G)$ has $\mathbb{Z}$-coefficients and the *vertical part* $\mathfrak{Z}_v$ has G-coefficients. Proper push-forward and flat pull-back are defined componentwise (cf. [Gu3], Section 3). For a closed subset V of $\tilde{\mathfrak{X}}$, let $R(V, G)$ be the G-submodule of $Z(\tilde{\mathfrak{X}}, G)$ generated by cycles rationally equivalent to 0 on V.

The concept of *Cartier divisors*, *line bundles* and *meromorphic sections* is defined on any ringed space (cf. [EGA IV], 21.1-2), hence it makes sense on $\mathfrak{X}$. The support $|D|$ of a Cartier divisor D has a horizontal part $|D^{\mathrm{an}}|$, where D^{an} is the corresponding Cartier divisor on $\mathfrak{X}^{\mathrm{an}}$, and a vertical part given as the smallest closed subset V of $\tilde{\mathfrak{X}}$ such that $D = 0$ on $\mathfrak{X} \setminus V$.

For $\mathfrak{Z} \in Z(\mathfrak{X}, G)$ such that D^{an} intersects $\mathfrak{Z}^{\mathrm{an}}$ properly in $\mathfrak{X}^{\mathrm{an}}$, we have a *generically proper intersection product* $D.\mathfrak{Z}$, well-defined in $Z(\mathfrak{X}, G)/R(|D| \cap |\mathfrak{Z}_v|)$ (cf. [Gu3], Section 4).

PROPOSITION 4.4. *For admissible formal schemes over K°, the following properties hold.*

(a) *For φ proper and ψ flat, we form the Cartesian diagram below.*

$$\begin{array}{ccc} \mathfrak{X}_2' & \xrightarrow{\psi'} & \mathfrak{X}_2 \\ \downarrow{\scriptstyle \varphi'} & & \downarrow{\scriptstyle \varphi} \\ \mathfrak{X}_1' & \xrightarrow{\psi} & \mathfrak{X}_1 \end{array}$$

Then φ' is proper, ψ' is flat and the fibre-square rule $\psi^* \circ \varphi_* = \varphi'_* \circ (\psi')^*$ holds on $Z(\mathfrak{X}_2, G)$.

(b) *Let $\varphi : \mathfrak{X}' \to \mathfrak{X}$ be a proper morphism, let D be a Cartier divisor on $\mathfrak{X}$ such that $\varphi^*(D)$ is a well-defined Cartier divisor on $\mathfrak{X}'$ and let $\mathfrak{Z}'$ be a prime cycle on $\mathfrak{X}'$ such that D intersects $\varphi(\mathfrak{Z}')$ properly in the generic fibre. Then we have the projection formula*

$$\varphi_*(\varphi^*(D).\mathfrak{Z}') = D.\varphi_*(\mathfrak{Z}') \mod R(|D| \cap \tilde{\varphi}|\mathfrak{Z}'_v|, G).$$

(c) *Let D, E be Cartier divisors on $\mathfrak{X}$ intersecting the cycle $\mathfrak{Z}$ properly in the generic fibre. Let $|\tilde{\mathfrak{Z}}| := \bigcup_Y (\bar{Y})^\sim \cup |\mathfrak{Z}_v|$ where Y ranges over the horizontal components of $\mathfrak{Z}$. Then*

$$D.E.\mathfrak{Z} = E.D.\mathfrak{Z} \mod R(|D| \cap |E| \cap |\tilde{\mathfrak{Z}}|, G).$$

(d) *Let $\varphi : \mathfrak{X}' \to \mathfrak{X}$ be a flat morphism, let D be a Cartier divisor on $\mathfrak{X}$ such that $\varphi^*(D)$ is a well-defined Cartier-divisor on $\mathfrak{X}'$ and let $\mathfrak{Z}$ be a cycle on $\mathfrak{X}$*

intersecting D properly in the generic fibre $\mathfrak{X}^{\mathrm{an}}$. Then $\varphi^*(D)$ intersects $\varphi^*(\mathfrak{Z})$ properly in the generic fibre and

$$\varphi^*(D).\varphi^*(\mathfrak{Z}) = \varphi^*(D.\mathfrak{Z}) \quad \mathrm{mod}\ R(\varphi^{-1}(|D|) \cap \tilde{\varphi}^{-1}(|\mathfrak{Z}_v|), G).$$

PROOF. The fibre square rule (a) follows immediately form the corresponding statements for the generic fibres ([Gu3], Proposition 2.12) and for the special fibre ([Fu], Proposition 1.7). Claims (b) and (c) are easily deduced from [Gu3], Proposition 4.5 and Theorem 5.9.

It remains to prove (d). By [Fu], Proposition 2.3(d), the result holds for $\mathfrak{Z}$ vertical. Hence it is enough to prove the claim for $Z = \mathfrak{Z}$ horizontal and prime. For the horizontal parts, the claim follows from [Gu3], Proposition 2.10. Using the fibre square rule, we may assume $Z = \mathfrak{X}^{\mathrm{an}}$ and that $\mathfrak{X}$ is the formal scheme associated to a formal analytic variety. It is enough to prove $\varphi^* \operatorname{cyc}(D) = \operatorname{cyc}(\varphi^* D)$ by checking the multiplicities in the irreducible components of $\tilde{\mathfrak{X}}'$. By passing to formal open subspaces, we may assume that $\mathfrak{X}$ and $\mathfrak{X}'$ are formal affine, that D is given by $a \in \mathcal{O}_{\mathfrak{X}}(\mathfrak{X})$ and that $\tilde{\mathfrak{X}}$ and $\tilde{\mathfrak{X}}'$ are both irreducible. On the left, the multiplicity in $\operatorname{cyc}(\widetilde{\mathfrak{X}'})$ is equal to $-\log|a(\tilde{\mathfrak{X}})|$ and on the right, it is $-\sum_{V'} \log|a(V')|\,[\tilde{K}(V') : \tilde{K}(\tilde{\mathfrak{X}}'_{\mathrm{red}})]$ where V' is ranging over all irreducible components of the special fibre of $(\mathfrak{X}')^{\text{f-an}}$ (cf. [Gu3], 3.10). By flatness of $\tilde{\varphi}$ and finiteness of the canonical map $\left((\mathfrak{X}')^{\text{f-an}}\right)^{\sim} \to \tilde{\mathfrak{X}}'$, we conclude that every V' is mapped onto $\tilde{\mathfrak{X}}$ and hence $|a(V')| = |a(\tilde{\mathfrak{X}})|$. Then (d) follows from the following result:

LEMMA 4.5. *Let $\mathfrak{X}$ be an admissible formal scheme over K°, let* $\operatorname{cyc}(\mathfrak{X}^{\mathrm{an}}) = \sum_j m_j X_j$ *be the decomposition of the generic fibre into irreducible components X_j and let $\mathfrak{X}_j$ be the formal analytic structure on X_j induced by $\iota_j : \mathfrak{X}_j \hookrightarrow \mathfrak{X}^{\text{f-an}}$. Then we have*

$$\operatorname{cyc}(\tilde{\mathfrak{X}}) = \sum_j m_j\, (\tilde{\iota}_j)_*(\operatorname{cyc}(\tilde{\mathfrak{X}}_j)). \tag{3}$$

PROOF. Again, we may assume that $\mathfrak{X} = \operatorname{Spf} A$ is formal affine and that $\tilde{\mathfrak{X}}$ is irreducible. Using noetherian normalization with respect to $\tilde{\mathfrak{X}}_{\mathrm{red}}$, we may reduce the problem to the zero-dimensional case with base field the completion $\hat{Q}$ of the field of fractions of a Tate-algebra (cf. [Gu3], Lemma 5.6). Note that $\hat{Q}$ is not algebraically closed but stable ([BGR], Theorem 5.3.2/1, Proposition 3.6.2/3) and the value group $|\hat{Q}^\times| = |K^\times|$ is still divisible. By [Fu], Lemma A1.3, the multiplicity of $\operatorname{cyc}(\tilde{\mathfrak{X}})$ in $\tilde{\mathfrak{X}}_{\mathrm{red}}$ is

$$\ell(\tilde{A}) = \dim_{\tilde{Q}} \tilde{A} / [\tilde{Q}(\tilde{\mathfrak{X}}_{\mathrm{red}}) : \tilde{Q}]$$

where ℓ denotes the length of the local artinian ring $\tilde{A} := A \otimes_{K^\circ} \tilde{K}$ (using $\tilde{\mathfrak{X}}$ irreducible and zero-dimensional). Note that $\mathcal{A} = A \otimes_{\hat{Q}^\circ} \hat{Q}$ is a finite dimensional $\hat{Q}$-algebra and hence $\mathcal{A} \cong \prod \mathcal{A}_{\wp}$ where $\wp$ ranges over Spec$\mathcal{A}$. By [Fu], Lemma A1.3, the multiplicity $\ell(\mathcal{A}_{\wp})$ of $\mathrm{cyc}(\mathfrak{X}^{\mathrm{an}})$ in $\wp$ is equal to $\dim_{\hat{Q}}(\mathcal{A}_{\wp})/[\hat{Q}(\wp) : \hat{Q}]$. Note that $\hat{Q}(\wp) = (\mathcal{A}_{\wp})_{\mathrm{red}}$ is a finite dimensional field extension of $\hat{Q}$. By stability and divisibility, we have $[\hat{Q}(\wp) : \hat{Q}] = [\tilde{Q}(\wp) : \tilde{Q}]$. We conclude that

$$\sum_{\wp} \ell(\mathcal{A}_{\wp})[\tilde{Q}(\wp) : \tilde{Q}(\tilde{\mathfrak{X}}_{\mathrm{red}})] = \sum_{\wp} \frac{\dim_{\hat{Q}}(\mathcal{A}_{\wp})}{[\tilde{Q}(\tilde{\mathfrak{X}}_{\mathrm{red}}) : \tilde{Q}]} = \frac{\dim_{\hat{Q}}(\mathcal{A})}{[\tilde{Q}(\tilde{\mathfrak{X}}_{\mathrm{red}}) : \tilde{Q}]}$$

is the multiplicity of the right hand side of (3) in $\tilde{\mathfrak{X}}_{\mathrm{red}}$. Hence (3) follows from:

LEMMA 4.6. *Let A be an admissible algebra over the complete valuation ring Q° of height* 1. *If $\tilde{A}$ is a finite dimensional $\tilde{Q}$-algebra*, *then A is a free Q°-algebra of rank* $\dim_{\tilde{Q}}(\tilde{A})$.

PROOF. Let us choose $b_1, \ldots, b_r \in A$ such that $\tilde{b}_1, \ldots, \tilde{b}_r$ form a $\tilde{Q}$-basis of $\tilde{A}$. As a quotient of a Tate algebra, A is topologically generated by $\xi_1, \ldots, \xi_n \in A$ as a Q°-algebra. There is an element $\pi \in Q$, $|\pi| < 1$, such that $\xi_1, \ldots, \xi_n$ and every $b_i b_j$ has the form $\sum_j \lambda_j b_j (\mathrm{mod}\, \pi A)$ for $\lambda_j \in Q^\circ$. Any $a \in A$ may be written as a polynomial in $\xi_1, \ldots, \xi_n$ with coefficients in K° and hence as a K°-linear combination of $b_1, \ldots, b_r$, always up to πA. This proves

$$A = Q^\circ b_1 + \cdots + Q^\circ b_r + \pi^n A$$

for $n = 1$ and then by induction for all $n \in \mathbb{N}$. Since A is complete and separated with respect to the π-adic topology ([BL2], Proposition 1.1), we conclude that $b_1, \ldots, b_r$ generate A as a Q°-module. Obviously, they are linearly independent over Q° proving the claim. □

REMARK 4.7. This ends also the proofs of Lemma 4.5 and hence of Proposition 4.4. Note that the argument for Lemma 4.6 works for any Q°-algebra of topological finite presentation.

5. – Refined intersections on models

Let K be an algebraically closed field with a non-trivial non-archimedean complete absolute value $|\ |$. Vertical cycles have coefficients in a subgroup G of $\mathbb{R}$ containing $\log |K^\times|$.

DEFINITION 5.1. A K°-*model* of a rigid analytic variety X over K is an admissible formal scheme $\mathfrak{X}$ over K° with generic fibre $\mathfrak{X}^{\mathrm{an}} = X$. We denote the isomorphism classes of K°-models of X by M_X. A line bundle $\mathcal{L}$ on $\mathfrak{X}$ is said to be a K°-model of L if $\mathcal{L}^{\mathrm{an}} = L$.

PROPOSITION 5.2. *For quasi-compact and quasi-separated rigid analytic varieties over K, the following properties hold*:

(a) *They have always a K°-model and the same holds for every line bundle.*
(b) *If $\varphi : X' \to X$ is a morphism and $\mathfrak{X}$ is a K°-model of X, then there is a K°-model $\mathfrak{X}'$ of X' and a morphism $\mathfrak{X}' \to \mathfrak{X}$ extending φ. The extension of φ is always unique.*
(c) *The set M_X is partially ordered by defining $\mathfrak{X}' \geq \mathfrak{X}$ if and only if the identity on X extends to a morphism $\mathfrak{X}' \to \mathfrak{X}$. Then M_X is a directed set.*
(d) *If $\varphi : X' \to X$ is a flat morphism extending to a morphism $\mathfrak{X}'_0 \to \mathfrak{X}_0$ of K°-models, then there is $\mathfrak{X}_1 \in M_X$ with $\mathfrak{X}_1 \geq \mathfrak{X}_0$ such that the projection $\mathfrak{X}'_1 := \mathfrak{X}_1 \times_{\mathfrak{X}_0} \mathfrak{X}'_0 \to \mathfrak{X}_1$ is a flat extension of φ* (*meaning the fibre product in the category of admissible formal schemes*).
(e) *If Y is a closed analytic subvariety of X extending to a morphism $\mathfrak{Y} \to \mathfrak{X}$ of K°-models, then there is $\mathfrak{X}' \in M_X$ with a closed subscheme $\mathfrak{Y}' \in M_Y$ such that $\mathfrak{X}' \geq \mathfrak{X}$ and $\mathfrak{Y}' \geq \mathfrak{Y}$.*

PROOF. The claims (a)-(c) follow from a theorem of Raynaud which is proved by Bosch and Lütkebohmert in [BL2], Theorem 4.1. For the second claim in (a), we refer to [Gu3], Lemma 7.6. Claim (d) is proved in [BL3], Theorem 5.2, and (e) is part of [BL3], Corollary 5.3. □

REMARK 5.3. Let X be the rigid analytic variety associated to a proper scheme over K. Note that the GAGA-principle holds on X, i.e. every line bundle L on X is algebraic, every meromorphic section of L is algebraic, every analytic subvariety is induced by a closed algebraic subscheme of X and every analytic morphism of proper schemes over K is algebraic (cf. [Ko]; one can use also [Ber], Proposition 3.4.11). Hence the algebraic and the analytic cycle groups on X are the same. By [Fu], 2.3, we get a refined intersection product $D.Z \in CH(|D| \cap |Z|)$ for a pseudo-divisor D and a cycle Z on X.

5.4. Let $\mathfrak{X}$ be a K°-model of X. A *support* on $\mathfrak{X}$ is a pair $S := (Y, V)$ where the horizontal part Y is a closed analytic subset of X and the vertical part V is a closed subset of $\tilde{\mathfrak{X}}$ with $(\bar{Y})^\sim \subset V$. Componentwise, it makes sense to consider range and inverse images of supports, as well as intersections. If D is a Cartier divisor, then $|D|$ from 4.3 is a support on $\mathfrak{X}$. For a cycle $\mathfrak{Z}$ on $\mathfrak{X}$, the support $|\mathfrak{Z}|$ is defined as the union of all $((Y, (\bar{Y})^\sim)$ and $(\emptyset, V)$ with Y (resp. V) ranging over all horizontal (resp. vertical) components of $\mathfrak{Z}$.

DEFINITION 5.5. For a support $S = (Y, V)$ on $\mathfrak{X}$, $R(S, G)$ is the subgroup of $Z(\mathfrak{X}, G)$ generated by $\mathrm{div}_{\mathfrak{X}}(f)$ and by $R(V, G)$, where f ranges over all non-zero rational functions on closed irreducible subsets of Y and $\mathrm{div}_{\mathfrak{X}}(f)$ is the Weil divisor considered as a cycle on $\mathfrak{X}$. The *local Chow group of $\mathfrak{X}$ with support in S* is defined by

$$CH_*^S(\mathfrak{X}, G) := \{\mathfrak{Z} \in Z(\mathfrak{X}, G) \mid |\mathfrak{Z}| \subset S\}/R(S, G)$$

graded by relative dimension. If $S = \tilde{\mathfrak{X}}$, then we use the notation $CH_*^{\mathrm{fin}}(\mathfrak{X}, G)$.

LEMMA 5.6. *Let $\varphi : \mathfrak{X}' \to \mathfrak{X}$ be a proper morphism of K°-models such that φ^{an} is induced by a surjective morphism of integral proper schemes over K. For a non-zero rational function f' on X' and for the norm $N_{K(X')/K(X)}$ with respect to function fields, we have*

$$\varphi_*(\mathrm{div}_{\mathfrak{X}'}(f')) = \mathrm{div}_{\mathfrak{X}}(N_{K(X')/K(X)}(f')). \tag{4}$$

PROOF. To prove (4), we have only to check the vertical parts, the equality of the horizontal parts is just [Fu], Proposition 1.4. Note that the irreducible components of $\tilde{\mathfrak{X}}$ have the same dimension as X. So we may assume that $\dim X = \dim X'$, otherwise both sides of (4) are zero. Using 4.1, it is enough to prove (4) for formal analytic varieties $\mathfrak{X}$, $\mathfrak{X}'$. The Stein factorization $\varphi = \varphi'' \circ \varphi'$ gives a finite map φ'' and a proper morphism φ' of formal analytic varieties which is an isomorphism outside of $\pi^{-1}(S)$, where π is the reduction and S is a closed lower dimensional subset of $\tilde{\mathfrak{X}}$ (cf. [Gu3], proof of Proposition 4.5). For φ', the vertical parts of (4) also agree since φ' is an isomorphism outside $\pi^{-1}(S)$ and the irreducible components of $\widetilde{\mathfrak{X}'}$ lying over S do not contribute to the left hand side of (4) by dimensionality reasons. So we may assume that $\varphi = \varphi''$ is finite. By passing to formal open affinoid subspaces, we may assume that $\mathfrak{X} = \mathrm{Spf}\mathcal{A}$ and hence $\mathfrak{X}' = \mathrm{Spf}\mathcal{A}'$ for K-affinoid algebras $\mathcal{A}$, $\mathcal{A}'$. (Note that the norm doesn't change because of $K(X') \otimes_{K(X)} Q(\mathcal{A}) \cong Q(\mathcal{A}')$ using finiteness of X' over X and that $\mathcal{A}$, $\mathcal{A}'$ are integral domains with quotient fields $Q(\mathcal{A})$, $Q(\mathcal{A}')$.)

We check the multiplicities in an irreducible component W of $\tilde{\mathfrak{X}}$ and, by passing again to formal open affinoid subspaces, we may assume that $W = \tilde{\mathfrak{X}}$ and $f' \in \mathcal{A}'$. We have to check

$$\sum_V [\tilde{K}(V) : \tilde{K}(W)] \log |f'(V)| = \log |N_{Q(\mathcal{A}')/Q(\mathcal{A})}(f')(W)| \tag{5}$$

where V ranges over all irreducible components of $\widetilde{\mathfrak{X}'}$. Note that $|a(W)|$ is the supremum norm on $\mathcal{A}$, which extends uniquely to an absolute value $|\ |_W$ on $Q(\mathcal{A})$. Similarly, we have absolute values $|\ |_V$ on $Q(\mathcal{A}')$ for every irreducible component V of $\widetilde{\mathfrak{X}'}$. By [Gu3], Lemma 3.19, they are just the extensions of $|\ |_W$ to absolute values on $Q(\mathcal{A}')$. Let $e_{V/W}$ be the ramification index and let $f_{V/W}$ be the residue degree. By [Gu3], Lemma 3.17, the classical formula

$$\sum_V e_{V/W} f_{V/W} = [Q(\mathcal{A}') : Q(\mathcal{A})] \tag{6}$$

holds. Let $\hat{Q}(\mathcal{A})$ be the completion of $Q(\mathcal{A})$ with respect to $|\ |_W$. By [Ja], Theorem 9.13, we have $Q(\mathcal{A}') \otimes_{Q(\mathcal{A})} \hat{Q}(\mathcal{A}) \cong \prod_V R_V$ for finite dimensional local $\hat{Q}(\mathcal{A})$-algebras R_V with residue fields isomorphic to the completion $\hat{K}_V$ of $Q(\mathcal{A}')$ with respect to $|\ |_V$. Using $e_{V/W} f_{V/W} \leq [\hat{K}_V : \hat{Q}(\mathcal{A})]$, formula (6) implies $R_V \cong \hat{K}_V$ and equality holds. All value groups involved are equal to $|K^\times|$, hence $e_{V/W} = 1$. By [Gu3], Lemma 3.18, the residue fields of $|\ |_W, |\ |_V$ are isomorphic to $\tilde{K}(W)$ and $\tilde{K}(V)$, respectively, and therefore $[\tilde{K}(V) : \tilde{K}(W)] = f_{V/W} = [\hat{K}_V : \hat{Q}(\mathcal{A})]$. By [Ja], Theorem 9.8, (5) follows now from

$$|N_{Q(\mathcal{A}')/Q(\mathcal{A})}(f')|_W = \prod_V |N_{\hat{K}_V/\hat{Q}(\mathcal{A})}(f')|_W = \prod_V |f'|_V^{[\hat{K}_V:\hat{Q}(\mathcal{A})]}$$
$$= \prod_V |f'|_V^{[\tilde{K}(V):\tilde{K}(W)]}. \qquad \square$$

REMARK 5.7. Let $\varphi : \mathfrak{X}' \to \mathfrak{X}$ be a morphism of admissible formal schemes with proper algebraic generic fibres. If S' is a support on $\mathfrak{X}'$ with $\varphi(S')$ contained in the support S of $\mathfrak{X}$, then push-forward induces a map $\varphi_* : CH_*^{S'}(\mathfrak{X}', G) \to CH_*^{S}(\mathfrak{X}, G)$. Lemma 5.6 handles rational functions on horizontal prime cycles and the vertical case follows from [Fu], Proposition 1.4.

If $\varphi : \mathfrak{X}' \to X$ is a flat morphism, then we have $\varphi^* : CH_*^{S}(\mathfrak{X}, G) \to CH_*^{\varphi^{-1}S}(\mathfrak{X}', G)$ induced by pull-back of cycles. This follows easily from Proposition 4.4(d).

5.8. A *pseudo-divisor* on $\mathfrak{X}$ is a triple $D = (\mathcal{L}, S, s)$ where $\mathcal{L}$ is a line bundle on $\mathfrak{X}$, $S = (Y, V)$ is a support on $\mathfrak{X}$ and s is an invertible meromorphic section of $\mathcal{L}|_{\mathfrak{X}\setminus\bar{Y}}$ with $|\operatorname{div}(s)| \subset V$. We use the notation $O(D) := \mathcal{L}, |D| := S$ and $s_D := s$. Two pseudo-divisors D, E will be identified if $|D| = |E|$ and there is an isomophism of $O(D)$ onto $O(E)$ carrying s_D to s_E. Every Cartier divisor D may be identified with its associated pseudo-divisor $(O(D), |D|, s_D)$.

PROPOSITION 5.9. *In the category of admissible formal schemes over K° with proper algebraic generic fibres, there is, for a pseudo-divisor D and for a support S on $\mathfrak{X}$, a unique refined intersection product $CH_*^S(\mathfrak{X}, G) \longrightarrow CH_*^{|D|\cap S}(\mathfrak{X}, G)$, $\alpha \mapsto D.\alpha$, with the properties*:

(a) *If D is a Cartier divisor intersecting a cycle $\mathfrak{Z}$ properly in the generic fibre, then $D.\mathfrak{Z}$ is the generically proper intersection product from* 4.3.
(b) *The refined intersection product is bilinear using the union of supports.*
(c) *If $\varphi : \mathfrak{X}' \to \mathfrak{X}$ is a morphism and $\alpha' \in CH_*^{S'}(\mathfrak{X}', G)$ for a support S' with $\varphi(S') \subset S$, then*

$$\varphi_*(\varphi^* D.\alpha') = D.\varphi_*(\alpha') \in CH_*^{|D|\cap S}(\mathfrak{X}, G) \qquad (\textit{projection formula}).$$

(d) $D.(E.\alpha) = E.(D.\alpha) \in CH_*^{|D|\cap|E|\cap S}(\mathfrak{X}, G)$ (*commutativity for pseudo-divisors D, E*).
(e) *If $\varphi : \mathfrak{X}' \to \mathfrak{X}$ is a flat morphism, then*

$$\varphi^*(D.\alpha) = \varphi^* D.\varphi^*\alpha \in CH_*^{\varphi^{-1}(|D|\cap S)}(\mathfrak{X}', G).$$

PROOF. By linearity in the components of $\mathfrak{Z} \in Z(\mathfrak{X}, G)$, it is easy to define $D.\mathfrak{Z}$ in the local Chow group with support in $|D| \cap |\mathfrak{Z}|$ by reducing to (a). This proves also uniqueness from (a)-(c). It is clear that the definition satisfies (a) and (b). Moreover, (c) and (e) follow from Proposition 4.4. However, that our definition passes to rational equivalence is only clear after proving the commutativity of Cartier divisors below. Moreover, this lemma implies (d). The arguments are similar as in [Fu], 2.4. □

LEMMA 5.10. *Let D and D' be Cartier divisors on $\mathfrak{X}$. Then we have*

$$D.\operatorname{cyc}(D') = D'.\operatorname{cyc}(D) \in CH_*^{|D|\cap|D'|}(\mathfrak{X}, G).$$

PROOF. If $|D|$ intersects $|D'|$ properly in the generic fibre X, then this follows from Proposition 4.4. To reduce to this special case, we proceed as in the proof of [Fu], Theorem 2.4. Fulton's idea is to consider blow ups $\pi^{\mathrm{an}} : X' \to X$ with suitable centers Y such that one can prove the claim for $\pi^{\mathrm{an}*}D^{\mathrm{an}}, \pi^{\mathrm{an}*}D'^{\mathrm{an}}$ and then to use projection formula. The only new ingedient here is to construct an extension $\pi : \mathfrak{X}' \to \mathfrak{X}$ to a K°-model $\mathfrak{X}'$ such that the exceptional divisor E^{an} is the generic fibre of a Cartier divisor E on $\mathfrak{X}'$ with $\pi(|E|) = \overline{|Y|}$.

By Proposition 2.5, π^{an} extends to a morphism of K°-models. The closure F of E^{an} in $\mathfrak{X}'$ is given by a coherent ideal sheaf $\mathcal{J}$ on $\mathfrak{X}'$ ([Gu3], Proposition 3.3). Replacing $\mathfrak{X}'$ by an admissible formal blowing up as in the proof of [Gu3], Lemma 7.6, we may assume that $\mathcal{J}$ is an invertible ideal sheaf. Note that we may choose the center of the admissible formal blowing up in $\tilde{F}$, hence if we set $\mathcal{E} := \mathcal{J}$ and if we consider $s_{E^{\mathrm{an}}}$ as a section of $\mathcal{E}$, then the support of $E := \operatorname{div}_{\mathcal{E}}(s_{E^{\mathrm{an}}})$ is contained in $\pi^{-1}(\overline{|Y|})$. □

5.11. Let X be a rigid analytic variety over K with a line bundle L. A metric on L is called *formal* if there is an admissible open covering $\{U_i\}_{i\in I}$ trivializing L such that if $s \in L(U_i)$ corresponds to a regular function γ_i on U_i, then we have $\|s(x)\| = |\gamma_i(x)|$ for all $x \in U_i$.

A K°-model $\mathcal{L}$ of L gives rise to a formal metric $\| \ \|_{\mathcal{L}}$ on L by using a trivialization $\{\mathcal{U}_i\}_{i\in I}$ of $\mathcal{L}$ and the above formula to define the metric over $U_i := \mathcal{U}_i^{\mathrm{an}}$. If X is reduced, quasi-compact and quasi-separated, every formal metric arises this way ([Gu3], Proposition 7.5).

A *formal pseudo-divisor* $\hat{D}$ on X is defined as in 2.5 with a formal metric on $O(D)$.

5.12. For a K°-model $\mathfrak{X}$ of a quasi-separated and quasi-compact rigid analytic variety X, let $\tilde{Z}(\mathfrak{X}, G) := Z(\mathfrak{X}, G)/R(\tilde{\mathfrak{X}}, G)$. The *Arakelov-cycle group* of X is defined by

$$\hat{Z}(X, G) := \varprojlim_{\mathfrak{X}\in M_{X_{\mathrm{red}}}} \tilde{Z}(\mathfrak{X}, G)$$

where the inverse limit is with respect to push-forward of the morphisms extending the identity. An element $\alpha \in \hat{Z}(X, G)$ will be described by a family $(\alpha_{\mathfrak{X}})_{\mathfrak{X}\in M_{X_{\mathrm{red}}}}$ where $\alpha_{\mathfrak{X}} \in \tilde{Z}(\mathfrak{X}, G)$. Note that the horizontal part of all $\alpha_{\mathfrak{X}}$ agree, it is a cycle α^{an} on X.

Definition 5.13. Up to now, we suppose that X is proper and algebraic. For a non-zero rational function f on an irreducible closed subvariety W of X, we set $\widehat{\mathrm{div}}(f) := (\mathrm{div}_{\mathfrak{X}}(f)) \in \hat{Z}(X, G)$. For a closed analytic subset Y of X, let $\hat{R}(Y, G)$ be the subgroup of $\hat{Z}(X, G)$ generated by all such $\widehat{\mathrm{div}}(f)$ with $W \subset Y$. The *local Arakelov-Chow group with support in* Y is defined by

$$\widehat{CH}_*^Y(X, G) := \{\alpha \in \hat{Z}(X, G) \mid |\alpha^{\mathrm{an}}| \subset Y\}/R(Y, G).$$

graded by relative dimension. For $Y = \emptyset$, we use the notation $\widehat{CH}_*^{\mathrm{fin}}(X, G)$.

Remark 5.14. Let $\varphi : X' \to X$ be a proper morphism over K and let Y' be a closed analytic subset of X' with $\varphi(Y') \subset Y$. Then it is clear that push-forward may be defined componentwise on Arakelov cycles and descends to a graded homomorphism of local Arakelov-Chow groups with supports in Y' resp. Y also denoted by φ_*. Note however that flat pull-back is neither well-defined on Arakelov-cycle groups nor on Arakelov-Chow groups.

5.15. There is a unique refined intersection product $\hat{D}.\alpha$ of a formal pseudo-divisor $\hat{D}$ and $\alpha \in \widehat{CH}_*^Y(X, G)$ with properties similar as in Proposition 5.9(a)-(d). For proper intersection (a), we claim that $(\hat{D}.\alpha)_{\mathfrak{X}} = \mathrm{div}_{\mathcal{L}}(s_D).\alpha_{\mathfrak{X}}$ for every K°-model $(\mathfrak{X}, \mathcal{L})$ of $(X_{\mathrm{red}}, O(D)_{\mathrm{red}})$ with $\| \ \|_{\mathcal{L}}$ equal to the formal metric of $\hat{D}$ (cf. [Gu3], Proposition 8.4). We omit the details.

6. – Local Chow cohomology on models

Let K and G be as before. All spaces denoted by fractur letters $\mathfrak{X}, \mathfrak{X}', \ldots$ are assumed to be admissible formal schemes over the valuation ring K°. Their generic fibres $X, X', \ldots$ are assumed to be proper algebraic over K. By [Gu3], Remark 3.14, $\mathfrak{X}, \mathfrak{X}', \ldots$ are proper over K°.

Definition 6.1. Let $S = (Y, V)$ be a support on $\mathfrak{X}$ (cf. 5.4) and let $p \in \mathbb{Z}$. A *Chow cohomology class* $c \in CH_S^p(\mathfrak{X}, G)$ is a family of homomorphisms

$$CH_k^{S'}(\mathfrak{X}', G) \to CH_{k-p}^{S' \cap \psi^{-1}S}(\mathfrak{X}', G), \quad \alpha' \mapsto c \cap_\psi \alpha',$$

for all $k \in \mathbb{N}$, for all morphisms $\psi : \mathfrak{X}' \to \mathfrak{X}$ and for all supports S' on $\mathfrak{X}'$, satisfying the axioms:

(C1) If $\varphi : \mathfrak{X}'' \to \mathfrak{X}'$ is a proper morphism and $\alpha'' \in CH_k^{S''}(\mathfrak{X}'', G)$ with $\varphi(S'') \subset S'$, then

$$\varphi_*(c \cap_{\psi\circ\varphi} \alpha'') = c \cap_\psi \varphi_*(\alpha'') \in CH_{k-p}^{S' \cap \psi^{-1}S}(\mathfrak{X}', G).$$

(C2) If $\varphi : \mathfrak{X}'' \to \mathfrak{X}'$ is a flat morphism of relative dimension d and $\alpha' \in CH_k^{S'}(\mathfrak{X}', G)$, then

$$\varphi^*(c \cap_\psi \alpha') = c \cap_{\psi\circ\varphi} \varphi^*(\alpha') \in CH_{k+d-p}^{\varphi^{-1}S' \cap \varphi^{-1}\psi^{-1}S}(\mathfrak{X}'', G).$$

(C3) If D' is a pseudo-divisor on $\mathfrak{X}'$ and $\alpha' \in CH_k^{S'}(\mathfrak{X}', G)$, then

$$D'.(c \cap_\psi \alpha') = c \cap_\psi (D'.\alpha') \in CH_{k-p-1}^{S' \cap |D'| \cap \psi^{-1}S}(\mathfrak{X}', G).$$

(C4) The operation c on vertical cycles is induced by a cohomology class $\tilde{c} \in CH_V^p(\tilde{\mathfrak{X}}, G)$.

REMARK 6.2. Working in the category of algebraic schemes over a field and using axioms (C1)-(C3), we get the local Chow cohomology groups of [Fu], Example 17.3.1. Using local Chow groups with coefficients, the same procedure leads to local Chow cohomology groups with coefficients for algebraic schemes defining $CH_V^p(\tilde{\mathfrak{X}}, G)$ in (C4). Let $CH_{\mathrm{fin}}^*(\mathfrak{X}, G) := CH_{\tilde{\mathfrak{X}}}^*(\mathfrak{X}, G)$.

EXAMPLE 6.3. A pseudo-divisor D on $\mathfrak{X}$ induces $c_1(D) \in CH_{|D|}^1(\mathfrak{X}, G)$ by $c_1(D) \cap_\psi \alpha' := \psi^* D.\alpha'$. The axioms (C1)-(C4) follow easily from the properties of refined intersection theory.

6.4. We define *cup product* $c \cup c'$ and *pull-back* $\varphi^*(c)$ with respect to a morphism $\varphi : \mathfrak{Y} \to \mathfrak{X}$ formally completely analogous to [Fu], 17.2. Cup product is associative, pull-back is functorial and compatible with cup product.

DEFINITION 6.5. Let $\varphi : \mathfrak{X} \to \mathfrak{Y}$ be a flat morphism of relative dimension d and let S be a support on $\mathfrak{X}$. For $c \in CH_S^p(\mathfrak{X}, G)$, the *push-forward* $\varphi_*(c) \in CH_{\varphi S}^{p-d}(\mathfrak{Y}, G)$ is defined by

$$\varphi_*(c) \cap_\psi \alpha' := \varphi'_*(c \cap_{\psi'} \varphi'^*(\alpha')) \in CH_{k+d-p}^{\varphi' S' \cap \psi^{-1}\varphi S}(\mathfrak{Y}', G)$$

for all $k \in \mathbb{Z}$, for all morphisms $\psi : \mathfrak{Y}' \to \mathfrak{Y}$, for all supports S' on $\mathfrak{X}'$ and for all $\alpha' \in CH_k^{S'}(\mathfrak{Y}', G)$ using the base changes $\varphi' : \mathfrak{X}' \to \mathfrak{Y}'$, $\psi' : \mathfrak{X}' \to \mathfrak{X}$ of φ, ψ. The axioms follow easily from the fibre square rule 4.4 and Proposition 5.9. The flat push-forward is functorial.

PROPOSITION 6.6.

(a) $\psi^*\varphi_*(c) = \varphi'_*\psi'^*(c) \in CH_{\psi^{-1}\varphi S}^{p-d}(\mathfrak{Y}', G)$.

(b) *For* $c' \in CH_T^q(\mathfrak{Y}, G)$, *we have the projection formulas*

$$\varphi_*(c) \cup c' = \varphi_*(c \cup \varphi^*(c')), \qquad c' \cup \varphi_*(c) = \varphi_*(\varphi^*(c') \cup c).$$

PROOF. The fibre square rule is immediate from the definitions and implies that it is enough to prove (b) for ψ the identity. For $\alpha \in CH_k^{S'}(\mathfrak{X}, G)$, we have

$$(\varphi_*(c) \cup c') \cap \alpha = \varphi_*(c \cap \varphi^*(c') \cap \varphi^*(\alpha)) = \varphi_*(c \cup \varphi^*(c')) \cap \alpha$$

and similarly, we prove the other formula. □

PROPOSITION 6.7. *We have* $CH_S^p(\mathfrak{X}, G) = 0$ *for* $p < 0$ *or* $p > \dim(X) + 1$.

PROOF. The corresponding statement in algebraic geometry ([Fu], Example 17.3.3) and (C4) yield that $c \in CH_S^p(\mathfrak{X}, G)$ operates trivially on vertical cycles. In the horizontal case, we may extend the flattening theorem of Raynaud and Gruson to suitable K°-models by Proposition 5.2 and the claim follows similarly as in [Fu], Example 17.3.3. $\square$

EXAMPLE 6.8. Note that $\mathrm{Spf} K^\circ$ has a horizontal cycle 1 and a vertical cycle v, hence

$$CH_k^S(\mathrm{Spf}K^\circ, G) = \begin{cases} \mathbb{Z} \cdot 1 & \text{if } k = 0 \text{ and } S = 1 \cup v, \\ G \cdot v & \text{if } k = -1 \text{ and } S = v, \\ 0 & \text{otherwise.} \end{cases}$$

Because every morphism to $\mathrm{Spf}K^\circ$ is flat, we deduce easily, for all supports S and all $p \in \mathbb{Z}$, that $CH_S^p(\mathfrak{X}, G) \cong CH_{-p}^S(\mathfrak{X}, G)$.

PROPOSITION 6.9. *Let X be a proper scheme over K endowed with its rigid analytic structure. Let $\mathfrak{X}$ be a K°-model of X_{red}, $S = (Y, V)$ a support on $\mathfrak{X}$ and $c_{\mathfrak{X}} \in CH_S^p(\mathfrak{X}, G)$. For a proper morphism $\psi : X' \to X$ and a closed analytic subset Y' of X', there is a unique operation*

$$\widehat{CH}_k^{Y'}(X', G) \to \widehat{CH}_{k-p}^{Y' \cap \psi^{-1} Y}(X', G), \quad \alpha' \mapsto \hat{c} \cap_\psi \alpha',$$

such that for $\mathfrak{X}' \in M_{X_{\mathrm{red}}}$ with an extension $\bar{\psi} : \mathfrak{X}' \to \mathfrak{X}$ of ψ, we have $(\hat{c} \cap_\psi \alpha')_{\mathfrak{X}'} = c_{\mathfrak{X}} \cap_{\bar{\psi}} \alpha'_{\mathfrak{X}'}$.

PROOF. We use the last equality to define the operation of $\hat{c}$ on all such $\mathfrak{X}'$. By (C1), this is compatible with push-forward, thus $\hat{c}$ is well-defined and uniqueness is obvious. $\square$

DEFINITION 6.10. We call $\hat{c}$ the *local Arakelov-Chow cohomology class on X with support in Y* induced by $c_{\mathfrak{X}}$. The local Arakelov-Chow cohomology classes with support in Y (for varying K°-models $\mathfrak{X}$) form a group denoted by $\widehat{CH}_Y^p(X, G)$. By Proposition 6.7, it is zero for $p \notin \{0, \ldots, \dim X + 1\}$. For $Y = \emptyset$, we use the notation $\widehat{CH}_{\mathrm{fin}}^p(X, G)$.

6.11. Let $\hat{c} \in \widehat{CH}_Y^p(X, G)$ and $\hat{c}' \in \widehat{CH}_Z^q(X, G)$ be induced by $c_{\mathfrak{X}}$ and $c'_{\mathfrak{X}'}$. To define the *cup product*, we may assume that $\mathfrak{X} = \mathfrak{X}'$ (Proposition 5.2) and then let $\hat{c} \cup \hat{c}' \in \widehat{CH}_{Y \cap Z}^{p+q}(X, G)$ be induced by $c_{\mathfrak{X}} \cup c'_{\mathfrak{X}}$. To define the *pull-back* with respect to the morphism $\psi : X' \to X$ of proper schemes over K, we choose any extension $\bar{\psi} : \mathfrak{X}' \to \mathfrak{X}$ and then $\psi^*(\hat{c}) \in \widehat{CH}_{\psi^{-1}Y}^p(X', G)$ is induced by $\bar{\psi}^*(c_{\mathfrak{X}})$. Clearly, these definitions do not depend on the choice of $c_{\mathfrak{X}}$, $c'_{\mathfrak{X}'}$ and the extension $\bar{\psi}$. Again, the cup product is associative and compatible with pull-back.

PROPOSITION 6.12. *If* $\varphi : X'' \to X'$ *is a morphism and* $\alpha'' \in \widehat{CH}_k^{Y''}(X'', G)$ *for a closed subset* Y'' *of* X'', *then*

$$\varphi_*(\hat{c} \cap_{\psi\circ\varphi} \alpha'') = \hat{c} \cap_\psi \varphi_*(\alpha'') \in \widehat{CH}_{k-p}^{\varphi Y'' \cap \psi^{-1} Y}(X', G).$$

PROOF. This follows immediately from (C1) for $c_{\mathfrak{X}}$. □

EXAMPLE 6.13. Let $\hat{D}$ be a formal pseudo-divisor on X. Then there is $\hat{c}_1(D) \in \widehat{CH}^1_{|D|}(X, G)$ given by the refined intersection product $\hat{c}_1(D) \cap_\psi \alpha' = \psi^* \hat{D}.\alpha'$ of 5.15. It is induced by any pseudo-divisor $\mathcal{D}$ on a K°-model $\mathfrak{X}$ of X_{red} with generic fibre equal to $D|_{X_{\mathrm{red}}}$ such that $\| \ \|_{O(\mathcal{D})}$ is the metric of $\overline{O}(D)_{\mathrm{red}}$ (see 5.11).

EXAMPLE 6.14. Since SpfK° is the only K°-model of Spf$K$, Example 6.8 shows

$$\widehat{CH}^Y_{-p}(\mathrm{Spf}K, G) \cong \widehat{CH}^p_Y(\mathrm{Spf}K, G) \cong \begin{cases} \mathbb{Z} & \text{if } p = 0 \text{ and } Y = Spf K, \\ G & \text{if } p = 1 \text{ and } Y = \emptyset, \\ 0 & \text{otherwise.} \end{cases}$$

7. – Admissible metrics

In this section, K denotes an algebraically closed field with a non-trivial non-archimedean complete absolute value $| \ |$. Let us fix a subfield G of $\mathbb{R}$ containing the value group $\log |K^\times|$.

DEFINITION 7.1. Let $\mathfrak{X}$ be an admissible formal scheme over K° with generic fibre X. A metric $\| \ \|$ on the trivial bundle O_X is called $\mathfrak{X}$-*admissible* if there is an open covering $\{\mathcal{U}\}_{\mathcal{U} \in I}$ of $\mathfrak{X}$, invertible analytic functions $\gamma_j \in \mathcal{O}_X(\mathcal{U}^{\mathrm{an}})^\times$ and $\lambda_j \in G$ $(j = 1, \dots, r)$ satisfying

$$\|1(x)\| = |\gamma_1(x)|^{\lambda_1} \cdots |\gamma_r(x)|^{\lambda_r} \tag{7}$$

for all $x \in \mathcal{U}^{\mathrm{an}}$. A metric $\| \ \|$ on a line bundle L on X is called $\mathfrak{X}$-admissible if there is a K°-model $\mathcal{L}$ of L on $\mathfrak{X}$ such that $\| \ \| / \| \ \|_{\mathcal{L}}$ is a $\mathfrak{X}$-admissible metric on O_X.

REMARK 7.2. If L is a line bundle on a quasi-compact and quasi-separated rigid analytic variety X over K, then a metric on L is called *admissible* if it is $\mathfrak{X}$-admissible for a K°-model $\mathfrak{X}$ of X. The pull-back and the tensor product of admissible metrics are admissible.

EXAMPLE 7.3. For a semistable K°-model $\mathcal{C}$ of an irreducible smooth projective curve C over K, we will describe the $\mathcal{C}$-admissible metrics on O_C. Since the special fibre of $\mathcal{C}$ is reduced, we may view $\mathcal{C}$ as a formal analytic variety (cf. 4.1).

The vertices of the *intersection graph* $G(\mathcal{C})$ correspond to the irreducible components of $\tilde{\mathcal{C}}$. For each double point $\tilde{x}$ of $\tilde{\mathcal{C}}$, we define an edge of $G(\mathcal{C})$ by connecting the vertices correspnding to the irreducible components through $\tilde{x}$. Let $\pi : C \to \tilde{\mathcal{C}}$ be the reduction map. Then the *formal fibre* $\pi^{-1}(\tilde{x})$ is an open annulus $\{\zeta \in k^\times | r < |\zeta| < 1\}$ of height $0 < r < 1$ ([BL1], Proposition 2.3). If we require that the edge corresponding to the duoble point $\tilde{x}$ is isometric to a closed real interval of length $-\log r$ (or to a circle if there is only one irreducible component through $\pi(x)$, then $G(\mathcal{C})$ will be metrized graph.

There is a canonical map $p : C \to G(\mathcal{C})$. If $x \in C$ has regular reduction $\pi(x) \in \tilde{\mathcal{C}}$, then $p(x)$ is the vertex corresponding to the irreducible component of $\tilde{\mathcal{C}}$ containing $\pi(x)$. If $\pi(x)$ is an ordinary double point $\tilde{x}$ in $\tilde{\mathcal{C}}$, then we identify the edge corresponding to $\tilde{x}$ with $[0, -\log r]$ (or the corresponding loop) and we set $p(x) := -\log |\zeta(x)|$.

PROPOSITION 7.4. *For a $\mathcal{C}$-admissible metric $\| \ \|$ on O_C, there is a unique $f : G(\mathcal{C}) \to \mathbb{R}$ with $f \circ p = -\log \|1\|$. This gives a bijection between $\mathcal{C}$-admissible metrics on O_C and continuous real functions on $G(\mathcal{C})$ which are linear on the edges with slopes and constant terms in G.*

PROOF. Let $\tilde{\mathcal{U}}$ be an irreducible open affine subscheme of $\tilde{\mathcal{C}}$. Then $\mathcal{U} := \pi^{-1}(\tilde{\mathcal{U}})$ is a formal open affinoid subspace of $\mathcal{C}$. We may assume that the $\mathcal{C}$-admissible metric $\| \ \|$ is given on $\mathcal{U}$ by (7). Because $\tilde{\mathcal{U}}$ is integral, the maximum norm on $\mathcal{U}$ is multiplicative ([BGR], Proposition 6.2.3/5). Thus the absolute values $|\gamma_i(x)|$ are constant on $\mathcal{U}$ and hence the same holds for $\|1(x)\|$.

If t is a vertex of $G(\mathcal{C})$ corresponding to an irreducible component V of $\tilde{\mathcal{C}}$, then $x \in C$ maps to t if and only if the reduction $\pi(x)$ is a regular point of $\tilde{\mathcal{C}}$ contained in V. The set of these reductions is an irreducible open subset of $\tilde{\mathcal{C}}$ contained in V and may be covered by finitely many $\tilde{\mathcal{U}}$ as above. We conclude that $\|1\|$ is constant on $p^{-1}(t)$.

If t is on a loop γ, then there is a unique irreducible component V through the corresponding double point. As above, we conclude that $\|1(x)\|$ is constant on $p^{-1}(\gamma)$.

It remains to consider an interior point t of an edge (which is not a loop). Then $p^{-1}(t) \subset \pi^{-1}(\tilde{x})$ for a double point $\tilde{x}$ of $\tilde{\mathcal{C}}$. We can identify this formal fibre with an open annulus with coordinate ζ. For every unit γ on a formal open neighbourhood of $\pi^{-1}(\tilde{x})$, there is $m \in \mathbb{Z}$ and $\alpha \in K^\times$ with $|\gamma(x)| = |\alpha\zeta(x)^m|$ for all $x \in \pi^{-1}(\tilde{x})$ ([BGR], Lemma 9.7.1/1). By [BL1], Proposition 2.3(ii), this identity extends to a formal open neighbourhood $\mathcal{U}$ of $\pi^{-1}(\tilde{x})$. By (7), $f(t) := \|1 \circ p^{-1}(t)\|$ is well-defined and linear on the whole edge with slope and constant term in G. Continuity is also clear and so the proof of the first claim is complete.

The arguments may be reversed to get the second claim. □

REMARK 7.5. By $\| \ \| \leftrightarrow \sum f(V)\, V$, the $\mathcal{C}$-admissible metrics on O_C may be identified with the G-vector space of one dimensional cycles on $\tilde{\mathcal{C}}$ with coefficients in G.

A blowing up of $\mathcal{C}$ in a double point leads to a refinement of the intersection graph $G(\mathcal{C})$ corresponding to a subdivision of the open annulus in two annuli. Hence we can construct admissible metrics with $-\log\|1(x)\|$ arbitrarily close to any continuous function on $G(\mathcal{C})$. If we blow up in a regular point $\tilde{x} = \pi(x)$ of $\tilde{\mathcal{C}}$, then we have to add a new vertex and a new edge.

For a semistable model $\mathcal{C}$ over an discrete valuation ring, the intersection graph is the same as in [CR], [Zh1] and the results are similar. By using the rigid analytic structure of the formal fibre, our construction is absolute working for any complete valuation of height 1.

PROPOSITION 7.6. *Let $\mathfrak{X}$ be a reduced formal analytic variety with generic fibre X. Let s be an invertible meromorphic section of a line bundle L on X and let $\| \ \|$ be a $\mathfrak{X}$-admissible metric on L. For every irreducible component W of $\tilde{\mathfrak{X}}$, there is a number $\|s(W)\| \in \exp(G)$ with $\|s(x)\| = \|s(W)\|$ for all $x \in X$ with reduction $\pi(x)$ neither in $\pi|\operatorname{div}(s)|$ nor in any other irreducible component of $\tilde{\mathfrak{X}}$ different from W.*

PROOF. The claim is local on $\mathfrak{X}$, so we may assume that the metric is given by a product of G-powers of formal metrics on $L = O_X$. By linearity, we may assume that the metric is formal. The points in question form a formal open neighbourhood $\mathcal{U}$ of $\mathfrak{X}$. We may assume $\mathcal{U} = \mathfrak{X}$ formal affinoid with irreducible special fibre. Thus $s \in \mathcal{O}_X(X)^\times$ and multiplicativity of the maximum norm ([BGR], Proposition 6.2.3/5) yields $|s(x)| = |s|_{\max}$ constant on X. □

7.7. The *order* of s in W is $\operatorname{ord}(s, W) := -\log\|s(W)\|$. As in [Gu3], 3.10 and 4.1, we get a *Weil divisor* $\operatorname{cyc}(\widehat{\operatorname{div}}_{\mathfrak{X}}(s))$ and a *proper intersection product* $\widehat{\operatorname{div}}_{\mathfrak{X}}(s).Z \in Z(\mathfrak{X}, G)$ for every horizontal cycle Z on $\mathfrak{X}$ intersecting $\operatorname{div}(s)$ properly in X. By projection formula, we extend the definition to admissible formal schemes generalizing the proper intersection product of Cartier divisors with horizontal cycles from 4.3. The projection formula 4.4(b) and the flat pull-back rule 4.4(d) still hold for horizontal cycles. By local linearity, this follows from the formal case.

7.8. Let $\mathfrak{X}$ be a reduced formal analytic variety over K with generic fibre X, let L be a line bundle on X with $\mathfrak{X}$-admissible metric $\| \ \|$ and let s be an invertible meromorphic section of L. A remarkable fact is that for a vertical cycle V of codimension 0 in $\tilde{\mathfrak{X}}$, we can define an *intersection product* $\widehat{\operatorname{div}}_{\mathfrak{X}}(s).V$ well-defined as a vertical cycle with coefficients in G. There is a K°-model $\mathcal{L}$ of L on $\mathfrak{X}$ such that $\| \ \|/\| \ \|_{\mathcal{L}}$ is a $\mathfrak{X}$-admissible metric on O_X given by (7) with respect to a formal open covering $\{\mathcal{U}\}_{\mathcal{U}\in I}$ of $\mathfrak{X}$ trivializing L on the generic fibre. Let f be the rational function corresponding to s in the trivialization. By linearity, we may assume that V is an irreducible component of $\tilde{\mathfrak{X}}$. Then we define the cycle $\widehat{\operatorname{div}}_{\mathfrak{X}}(s).V$ on $\tilde{\mathcal{U}} \cap V$ by

$$\operatorname{div}((f/f(V))^{\sim}) + \sum_j \lambda_j \operatorname{div}((\gamma_j/\gamma_j(V))^{\sim}). \tag{8}$$

LEMMA 7.9. *These cycles fit to a cycle on V depending only on $\| \ \|$.*

PROOF. The rational functions $(f/f(V))^{\sim}$ on $\tilde{\mathcal{U}} \cap V$ form a Cartier divisor on V with corresponding line bundle isomorphic to $\mathcal{L}|_V$. So we may assume $L = O_X$ and $s = 1$. It is enough to show that the order of (8) in an irreducible closed subset W of codimension 1 in V depends only on the metric and the formal fibre over W. Using noetherian normalization with respect to W and base change, we reduce to the case of an irreducible and reduced formal analytic curve $\mathfrak{X}$ over a complete (stable) field Q (cf. [Gu3], Lemma 5.6 and proof of Theorem 5.9).

By passing to formal open affinoid subspaces, we may assume that there is an analytic function g on X with maximum norm ≤ 1 such that W is an isolated zero of the reduction $\tilde{g}$ in $\tilde{\mathfrak{X}}$. By [BL1], Lemma 2.4, we know that for $r \in |K^{\times}|$, $r < 1$ sufficiently close to 1, the periphery $\{x \in X \mid \pi(x) = W, |g(x)| \geq r\}$ of the formal fibre over W decomposes into n connected components $G_1, \ldots, G_n$. Here, π still denotes reduction. These components correspond to the points $\tilde{y}_1, \ldots, \tilde{y}_n$ in the normalization of $\tilde{\mathfrak{X}}$ lying over W. Moreover, G_k is isomorphic to the semi-open annulus $\{\zeta \in \mathbb{B}^1 \mid r^{1/\operatorname{ord}_{\tilde{y}_k}(g)} \leq |\zeta| < 1\}$. Let $\tilde{y}_k$ be contained in the normalization V' of V and let $\gamma \in \mathcal{O}(\mathcal{U}^{\mathrm{an}})^{\times}$. By [BL1], Lemma 2.5, we have $|\gamma(x)/\gamma(V)| = |\zeta(x)|^{\operatorname{ord}(\gamma/\gamma(V), \tilde{y}_k)}$ for all $x \in G_k$. From (7), we get

$$\|1(x)\| = |\gamma_1(V)|^{\lambda_1} \cdots |\gamma_r(V)|^{\lambda_r} |\zeta(x)|^{\sum_j \lambda_j \operatorname{ord}(\gamma_j/\gamma_j(V), \tilde{y}_k)}$$

for all $x \in G_k$ and therefore $\sum_j \lambda_j \operatorname{ord}(\gamma_j/\gamma_j(V), \tilde{y}_k)$ depends only on the metric. By projection formula applied to the normalization morphism over V and to the divisor of $(\gamma_j/\gamma_j(V))^{\sim}$ on V, we have $\operatorname{ord}(\gamma_j/\gamma_j(V), W) = \sum_k \operatorname{ord}(\gamma_j/\gamma_j(V), \tilde{y}_k)$ where k ranges over all $\tilde{y}_k$ contained in V'. We conclude that the order of (8) in W depends only on $\| \ \|$ and not on the choice of $\mathcal{U}$. □

THEOREM 7.10. *Let $\mathfrak{X}$ be a reduced formal analytic variety over K with generic fibre X. Let L, L' be line bundles on X with $\mathfrak{X}$-admissible metrics $\| \ \|, \| \ \|'$ and invertible meromorphic sections s, s' such that* $\operatorname{div}(s)$ *and* $\operatorname{div}(s')$ *intersect properly in X. Then*

$$\widehat{\operatorname{div}}_{\mathfrak{X}}(s).\operatorname{cyc}(\widehat{\operatorname{div}}_{\mathfrak{X}}(s')) = \widehat{\operatorname{div}}_{\mathfrak{X}}(s').\operatorname{cyc}(\widehat{\operatorname{div}}_{\mathfrak{X}}(s)) \in Z(\mathfrak{X}, G).$$

PROOF. The statement is proved for formal metrics in [Gu3], Theorem 5.9. The claim is local and then an admissible metric is a product of G-powers of formal metrics. □

REMARK 7.11. Let C be a smooth projective curve over K. There is a semistable K°-model $\mathcal{C}$ of C such that $\tilde{\mathcal{C}}$ has smooth irreducible components. If necessary, we may choose $\mathcal{C}$ larger than any given K°-model of C. This follows from the semistable reduction theorem ([BL1], Theorem 7.1) and its proof. In fact, the semistable model was obtained by successive refinements of a given formal analytic structure and a further refinement as in the introductory remark to Lemma 2.5 of [BL1] leads to smooth irreducible components of $\tilde{\mathcal{C}}$. By dimensionality, 7.7 and 7.8, $\widehat{\operatorname{div}}_{\mathcal{C}}(s).\mathfrak{Z}$ is defined for all $\mathfrak{Z} \in Z(\mathcal{C}, G)$ with $\mathfrak{Z}^{\mathrm{an}}$ intersecting $\operatorname{div}(s)$ properly in C.

THEOREM 7.12. *Let $\mathcal{C}$ be as in Remark 7.11 and let L be a line bundle on C with $\mathcal{C}$-admissible metric $\| \ \|$. For every invertible meromorphic section s of L, there is a unique $\hat{c}_1(s)_{\mathcal{C}} \in CH^1_{|\operatorname{div}(s)| \cup \tilde{\mathcal{C}}}(\mathcal{C}, G)$ such that for $\mathfrak{Z} \in Z(\mathcal{C}, G)$, we have*:

(a) *If $\operatorname{div}(s)$ intersects $\mathfrak{Z}$ properly in the generic fibre, then*

$$\hat{c}_1(s)_{\mathcal{C}} \cap \mathfrak{Z} = \widehat{\operatorname{div}}_{\mathcal{C}}(s).\mathfrak{Z} \in CH_*^{(|\operatorname{div}(s)| \cup \tilde{\mathcal{C}}) \cap |\mathfrak{Z}|}(\mathcal{C}, G).$$

(b) *If s' is another invertible meromorphic section of L, then*

$$\hat{c}_1(s')_{\mathcal{C}} \cap \mathfrak{Z} - \hat{c}_1(s)_{\mathcal{C}} \cap \mathfrak{Z} = \operatorname{div}_{\mathcal{C}}(s'/s).\mathfrak{Z} \in CH_*^{(|\operatorname{div}(s')| \cup |\operatorname{div}(s)| \cup \tilde{\mathcal{C}}) \cap |\mathfrak{Z}|}(\mathcal{C}, G).$$

PROOF. First, we prove uniqueness. Let $\bar{\psi} : \mathfrak{X}' \to \mathcal{C}$ be a proper morphism of admissible formal schemes with generic fibre ψ. Let s be an invertible meromorphic section of L such that $\psi^* \operatorname{div}(s)$ is a well-defined Cartier divisor on the generic fibre X' intersecting a horizontal cycle Z' properly in X'. Then we claim that

$$\hat{c}_1(s)_{\mathcal{C}} \cap_{\bar{\psi}} Z' = \widehat{\operatorname{div}}_{\mathfrak{X}'}(s \circ \psi).Z' \in \tilde{Z}(\mathfrak{X}', G) \tag{9}$$

is necessary. Note that the right hand side is the intersection product from 7.7 with respect to the $\mathfrak{X}'$-admissible metric $\psi^* \| \ \|$ of $\psi^* L$. To prove (9), we may assume Z' prime and $Z' = X'$. Let $\rho : X' \to Y := \psi X'$ be the map induced by ψ. By [Ha], Proposition III.9.7, it is flat. To prove (9), we may replace $\mathcal{C}$ and $\mathfrak{X}'$ by sufficiently large K°-models for C and X' (use projection formula and (C1)). By Proposition 5.2, we may assume that ρ extends to a flat morphism $\bar{\rho} : \mathfrak{X}' \to \bar{Y}$, where $\bar{Y}$ is the closure of Y in $\mathcal{C}$. Note that $\bar{\rho}$ is induced by $\bar{\psi}$. By the flat pull-back rule, $\bar{\rho}^* Y = Z'$ and (a), we deduce

$$\widehat{\operatorname{div}}_{\mathfrak{X}'}(s \circ \psi).Z' = \bar{\rho}^*(\widehat{\operatorname{div}}_{\mathcal{C}}(s).Y) = \bar{\rho}^*(\hat{c}_1(s)_{\mathcal{C}} \cap Y) = \hat{c}_1(s)_{\mathcal{C}} \cap_{\bar{\psi}} Z'$$

in $\tilde{Z}(\mathfrak{X}', G)$. This proves (9). Note that we already get uniqueness of $\hat{c}_1(s)$ on horizontal cycles because we may use (b) to reduce to the proper intersection case.

Now let V be a vertical prime cycle on $\mathfrak{X}'$. Let $\rho : V \to W := \tilde{\psi} V$ be the morphism induced by the reduction $\tilde{\psi}$ of $\bar{\psi}$. Then ρ is again flat. By axiom (C4), $\hat{c}_1(s)_{\mathcal{C}}$ operates on vertical cycles by $\tilde{c} \in CH^1_{\tilde{\mathcal{C}}}(\tilde{\mathcal{C}}, G)$. Using (C2) for $\tilde{c}$ and $\rho^* W = V$, we get uniqueness from

$$\hat{c}_1(s)_{\mathcal{C}} \cap_{\bar{\psi}} V = \rho^*(\tilde{c} \cap W) = \rho^*(\hat{c}_1(s)_{\mathcal{C}} \cap W) = \rho^*(\widehat{\operatorname{div}}_{\mathcal{C}}(s).W) \in CH_*(V, G). \tag{10}$$

We come to the proof of existence which is clear for formal metrics (Example 6.3), so we may assume $L = O_C$ and $s = 1$. First, we define the Chow cohomology class $\tilde{c}$ from axiom (C4) inducing the action on vertical cycles. Let $\{\mathcal{U}\}_{\mathcal{U} \in I}$ be a formal open covering of $\mathcal{C}$ such that $\| \ \|$ is given by (7). For an irreducible component W, we may view $(\gamma_j/\gamma_j(W))^\sim$ as a rational function on W with support $S(\mathcal{U}, j, W)$ contained in the double points (cf. proof of Proposition 7.4). For $S := \bigcup_{\mathcal{U},j,W} S(\mathcal{U}, j, W)$, we define $\tilde{c} \in CH^1_S(\tilde{\mathcal{C}})$ by the following procedure: let V be an integral scheme and let $\psi : V \to \tilde{\mathcal{C}}$ be a proper morphism. Inspired by (10), we consider the flat $\rho : V \to W := \psi V$ induced by ψ and we set

$$\tilde{c} \cap_\psi V := \rho^*(\widehat{\mathrm{div}}_{\mathcal{C}}(1).W) \in CH_{\dim V-1}(\psi^{-1}S, G). \tag{11}$$

Note that it is a cycle if W is an irreducible component of $\tilde{\mathcal{C}}$ and 0 otherwise. By passing to prime components, this leads to an action $\tilde{c} \cap_\psi \cdot$ on all cycles of a proper scheme over $\tilde{\mathcal{C}}$.

We claim that $\tilde{c} \in CH^1_S(\tilde{\mathcal{C}}, G)$. Axioms (C1) and (C2) follow easily from projection formula and flat pull-back rule for cycles. Let $\psi : V' \to \tilde{\mathcal{C}}$ be a morphism over $\tilde{K}$, let V be a cycle on V' and let D' be a pseudo-divisor on V'. For (C3), we have to check

$$D'.(\tilde{c} \cap_\psi V) = \tilde{c} \cap_\psi (D'.V) \in CH_{\dim V-2}(V \cap \psi^{-1}S \cap |D'|, G). \tag{12}$$

We may assume V prime, $V = V'$ and that D' is a Cartier divisor. Let $W := \psi V$, then we may assume that W is an irreducible component of $\tilde{\mathcal{C}}$ otherwise both sides in (12) are zero. Let U be the complement of all non-proper intersection components of $\psi^{-1}(S) \cap |D'|$ in V. On the open subset U of V, (12) is an identity of cycles. It may be checked locally over $\mathcal{U}$ where $\tilde{c}$ is the refined intersection product with $\sum_j \lambda_j \operatorname{div}((\gamma_j/\gamma_j(W))^\sim)$. Each Cartier divisor commutes with D' proving (12) on U. So let Y be a non-proper intersection component of $\psi^{-1}(S) \cap |D'|$. Then $\psi(Y)$ is a point, hence there is an $\mathcal{U} \in I$ such that $Y \subset \psi^{-1}(\mathcal{U})$. The same argument as above proves that (12) holds in a neighbourhood of Y. This proves (C3) for cycles. We conclude that $\tilde{c}$ passes to rational equivalence and hence $\tilde{c} \in CH^1_S(\tilde{\mathcal{C}}, G)$.

Now we define the operations $\hat{c}_1(1)_{\mathcal{C}}$. Let X' be a proper scheme over K, let $\psi : \mathfrak{X}' \to \mathcal{C}$ be a morphism of K°-models. For α' vertical and Z' horizontal on $\mathfrak{X}'$, let

$$\hat{c}_1(1)_{\mathcal{C}} \cap_\psi \alpha' := \tilde{c} \cap_{\tilde{\psi}} \alpha', \qquad \hat{c}_1(1)_{\mathcal{C}} \cap_\psi Z' := \widehat{\mathrm{div}}_{\mathfrak{X}}(1).Z'$$

using the $\mathfrak{X}'$-admissible metric $\psi^*\| \ \|$ on $O_{X'}$. Axioms (C1)-(C3) are clear on vertical cycles using the corresponding axioms for $\tilde{c}$. For a horizontal cycle, axioms (C1) and (C2) follow from projection formula and flat pull-back rule. To prove (C3) for Z', we first reduce to $Z' = X'$ prime. Replacing sections if necessary, we reduce to the proper intersection case and (C3) follows from Theorem 7.10. Finally, (C4) is by definition. By (C3), the actions $\hat{c}_1(1)_{\mathcal{C}}$ factor through rational equivalence and hence $\hat{c}_1(1)_{\mathcal{C}} \in \widehat{CH}^1_{\tilde{\mathcal{C}}}(\mathcal{C}, G)$. Clearly, (a) and (b) are true. $\square$

REMARK 7.13. Let C be a smooth projective curve over K and let L be a line bundle on C with an admissible metric $\| \ \|$. There is a semistable

K°-model $\mathcal{C}$ of C such that $\tilde{\mathcal{C}}$ has smooth irreducible components and such that $\| \ \|$ is $\mathcal{C}$-admissible. For an invertible meromorphic section s of L, let $\hat{c}_1(s) \in \widehat{CH}^1_{|\operatorname{div}(s)|}(C, G)$ be induced from $\hat{c}_1(s)_{\mathcal{C}}$ (Theorem 7.12). By (9) and (10), it does not depend on the choice of $\mathcal{C}$. Note that if $\| \ \|$ is a formal metric, then $\hat{c}_1(s)$ agrees with $\hat{c}_1(\operatorname{div}(s))$ from Example 6.13.

COROLLARY 7.14. *Let C, C' be smooth projective curves over K with invertible meromorphic sections s, s' of admissibly metrized line bundles L and L', respectively. If X is a proper scheme over K with morphisms $\varphi : X \to C$ and $\varphi' : X \to C'$, then*

$$\varphi^* \hat{c}_1(s) \cup \varphi'^* \hat{c}_1(s') = \varphi'^* \hat{c}_1(s') \cup \varphi^* \hat{c}_1(s) \in \widehat{CH}^2_{|\operatorname{div}(s)| \cap |\operatorname{div}(s')|}(X, G).$$

PROOF. The argument is completely similar to the proof of axiom (C3) in Theorem 7.12. We can always reduce to a local question where the divisors are G-linear combinations of formal Cartier divisors. We omit the details. □

DEFINITION 7.15. Let X be a rigid analytic variety proper over K. For a K°-model $\pi : \mathfrak{X} \to \operatorname{Spf} K^\circ$ and $\alpha_{\mathfrak{X}} \in \tilde{Z}_{-1}(\mathfrak{X}, G)$, let us denote by $\int_{\mathfrak{X}} \alpha_{\mathfrak{X}}$ the number in G corresponding to $\pi_*(\alpha_{\mathfrak{X}}) \in \tilde{Z}_{-1}(\operatorname{Spf} K^\circ, G) \cong G$ (Example 6.8). If $\alpha \in \hat{Z}_{-1}(X, G)$, then $\int_{\mathfrak{X}} \alpha_{\mathfrak{X}}$ is independent of the choice of $\mathfrak{X} \in M_{X_{\mathrm{red}}}$ and is denoted by $\int_X \alpha$.

If $c_{\mathfrak{X}} \in CH^{\dim(X)+1}_{\mathrm{fin}}(\mathfrak{X}, G)$ induces $\hat{c} \in \widehat{CH}^{\dim(X)+1}_{\mathrm{fin}}(X, G)$ for algebraic X, then

$$\int_X \hat{c} := \int_{\mathfrak{X}} c_{\mathfrak{X}} := \int_X \hat{c} \cap \operatorname{cyc}(X) = \int_{\mathfrak{X}} c_{\mathfrak{X}} \cap \operatorname{cyc}(X).$$

REMARK 7.16. Let $\mathcal{C}$ be a semistable K°-model of a smooth projective curve C over K. By Remark 7.5, $\alpha \in Z_1(\tilde{\mathcal{C}}, G)$ induces a unique $\mathcal{C}$-admissible metric $\| \ \|_\alpha$ on O_C with $\alpha = \operatorname{cyc}(\widehat{\operatorname{div}}^{\alpha}_{\mathcal{C}}(1))$. For $\alpha, \beta \in Z_1(\tilde{\mathcal{C}}, G)$, we get a pairing

$$\langle \alpha \mid \beta \rangle := \int_C \widehat{\operatorname{div}}^{\alpha}_{\mathcal{C}}(1).\beta \in G.$$

Let L be a line bundle C with invertible meromorphic section s and $\mathcal{C}$-admissible metric $\| \ \|$. Let ver be the vertical part of $\operatorname{cyc}(\widehat{\operatorname{div}}_{\mathcal{C}}(1))$. By Remark 7.13, the $\mathcal{C}$-admissible metrics $\| \ \|_{\mathrm{ver}}$ and $\| \ \|_{\mathrm{hor}} := \| \ \| / \| \ \|_{\mathrm{ver}}$ induce

$$\hat{c}_1^{\mathrm{ver}} := \hat{c}_1^{\mathrm{ver}}(1) \in \widehat{CH}^1_{\mathrm{fin}}(C, G), \qquad \hat{c}_1^{\mathrm{hor}}(s) = \hat{c}_1(s) - \hat{c}_1^{\mathrm{ver}} \in \widehat{CH}^1_{|\operatorname{div}(s)|}(C, G)$$

called the *vertical and horizontal part* of $\hat{c}_1(s)$. For a morphism $\psi : X' \to C$ such that $\psi^* \operatorname{div}(s)$ is a well-defined Cartier divisor on X', it is clear that $\hat{c}_1^{\mathrm{hor}}(s) \cap \operatorname{cyc}(X') = \operatorname{cyc}(\operatorname{div}(s \circ \psi))$.

The following is a local version of the *Hodge index theorem* for arithmetic surfaces proved by Faltings ([Fa1], Theorem 4), and Hriljac ([Hr], Theorem 3.4). The arguments are similar.

THEOREM 7.17. *The above pairing is a symmetric negative semidefinite bilinear form on the G-vector space $Z_1(\tilde{\mathcal{C}}, G)$. If C is irreducible, then the kernel of the pairing is $G \cdot \operatorname{cyc}(\tilde{\mathcal{C}})$.*

PROOF. Clearly, the pairing is bilinear and symmetry follows from Theorem 7.10. Let $\alpha = \sum_V m_V V$ and $\beta = \sum_W n_W W$ with V and W ranging over all irreducible components of $\tilde{\mathcal{C}}$. Because a multiple of $\operatorname{cyc}(\tilde{\mathcal{C}}) = \sum_V V$ is rationally equivalent to 0, we get

$$\langle \alpha \mid \beta \rangle = -\frac{1}{2} \sum_{V,W} (m_V - n_W)^2 \langle V \mid W \rangle. \tag{13}$$

Let $V \neq W$. We compute the multiplicity of the cycle $\widehat{\operatorname{div}}_{\mathcal{C}}^{V}(1).W$ in $P \in W$. If P is regular, then the admissible metric $\| \ \|_V$ is constant on the formal fibre over P and so the multiplicity is zero. If P is a double point, then the formal fibre is isomorphic to an open annulus $\{\zeta \in K \mid r < |\zeta| < 1\}$ of height $r \in |K^\times|, r < 1$. Let W' be the other irreducible component of $\tilde{\mathcal{C}}$ passing through P, it may happen $W' = W$. If $W' \neq V$, then Proposition 7.4 shows that the metric $\| \ \|_V$ is constant 1 on the formal fibre and again the multiplicity is 0. So we may assume that $W' = V$, i.e. $P \in V \cap W$. Allowing the usual change of coordinates $\zeta \leftrightarrow r/\zeta$ if necessary, the metric $\| \ \|_V$ is given on the formal fibre over P by $\|1(x)\|_V = |\zeta(x)|^{-1/\log r}$. We conclude that the multiplicity is $-1/\log r > 0$ (cf. proof of Lemma 7.9). Hence we have proved that $\langle V \mid W \rangle \geq 0$ with equality if and only if $V \cap W = \emptyset$. Using (13), we conclude that the bilinear form $\langle \ | \ \rangle$ is negative semidefinite. If C is irreducible, then $\tilde{\mathcal{C}}$ is connected. Thus for $V \neq W$, there is a chain $V_0 = V, V_1, \dots, V_r = W$ of irreducible components of $\tilde{\mathcal{C}}$ such that $V_{j-1} \cap V_j \neq \emptyset$. By (13), we conclude that α is in the kernel of $\langle \ | \ \rangle$ if and only if $\alpha \in G \cdot \sum_V V$. □

COROLLARY 7.18. *Let C be a smooth projective curve over K and let $L \in \operatorname{Pic}^0(C)$. Then there is an admissible metric $\| \ \|$ on L such that for any invertible meromorphic section s of L and any $\alpha \in \widehat{CH}_0^{\mathrm{fin}}(C, G)$, we have $\int_C \hat{c}_1(s) \cap \alpha = 0$. If C is irreducible, then the metric is uniquely determined up to multiples in* $\exp(G)$ *and is called a canonical metric.*

PROOF. We may assume C irreducible. Note that the operation of $\hat{c}_1(s)$ on $\widehat{CH}_*^{\mathrm{fin}}(C, G)$ is independent of s. Let $\mathcal{C}$ be a semistable K°-model of C. Since we may compute the intersection number $\int_C \hat{c}_1(s) \cap \alpha$ on $\mathcal{C}$, we have to look for a $\mathcal{C}$-admissible metric $\| \ \|$ on L with $\int_C \widehat{\operatorname{div}}_{\mathcal{C}}(s).\alpha = 0$ for all $\alpha \in Z_1(\tilde{\mathcal{C}}, G)$. Let $\| \ \|_{\mathcal{L}}$ be any formal metric on L with K°-model $\mathcal{L}$ which we may assume to live on $\mathcal{C}$, then the corresponding action of $\widehat{\operatorname{div}}_{\mathcal{C}}^{\mathcal{L}}(s)$ on $Z_1(\tilde{\mathcal{C}}, G)$ is an G-linear form Φ. Using commutativity with divisors of constants and $L \in \operatorname{Pic}^0(C)$, it is clear that Φ vanishes on $\operatorname{cyc}(\tilde{\mathcal{C}})$. By the local Hodge index theorem, there is $\beta \in Z_1(\tilde{\mathcal{C}}, G)$ with $\Phi(\alpha) = \langle \beta \mid \alpha \rangle$ for all $\alpha \in Z_1(\tilde{\mathcal{C}}, G)$. We conclude that the $\mathcal{C}$-admissible metric $\| \ \|_{\mathcal{L}} / \| \ \|_\beta$ satisfies the claim. By using a sufficiently large K°-model, uniqueness also follows from the local Hodge index theorem. □

The *Néron pairing* $\langle\ ,\ \rangle_{\text{Nér}}$ on a smooth projective curve C over K (cf. [La], Theorem 11.3.6) is closely related to the canonical metric $\|\ \|$ of $L \in \operatorname{Pic}^0(C)$ from Corollary 7.18:

COROLLARY 7.19. *For any invertible meromorphic section s of L and any divisor Z of degree* 0 *with* $|\operatorname{div}(s)| \cap |Z| = \emptyset$, *we have* $-\log \|s(Z)\| = \langle \operatorname{div}(s), Z\rangle_{\text{Nér}}$.

PROOF. This follows from the characteristic properties of the Néron pairing. Symmetry follows from Corollary 7.14 and boundedness is clear because admissible metrics are bounded. □

EXAMPLE 7.20. Let $q \in K^\times$, $|q| < 1$, and let $C = \mathbb{G}_m/q^{\mathbb{Z}}$ be Tate's elliptic curve (cf. [BGR], Example 9.3.4/4 and 9.7.3). For $a \in K^\times$, $|q| < |a| < 1$, let $\mathcal{C}$ be the formal analytic variety given by the formal open affinoid covering $\mathcal{U}_1 := \{\zeta \in \mathbb{G}_m \mid |a| \leq |\zeta| \leq 1\}$ and $\mathcal{U}_2 := \{\zeta \in \mathbb{G}_m \mid |q| \leq |\zeta| \leq |a|\}$ of C. Then $\mathcal{C}$ is a semistable K°-model of C with intersection graph equal to a circle of circumference $-\log|q|$ divided up in two arcs of length $-\log|a|$ and $\log|a| - \log|q|$, respectively. The canonical metric on $O([a]-[1])$ is given by

$$\|s_{[a]-[1]}(\zeta)\| = \begin{cases} r \cdot \left|\zeta^{\lambda_2}\dfrac{\zeta-a}{\zeta-1}\right| & \text{if } \zeta \in \mathcal{U}_1, \\ r \cdot \left|\zeta^{\lambda_1}\dfrac{\zeta-a}{\zeta-q}\right| & \text{if } \zeta \in \mathcal{U}_2 \end{cases}$$

for any $r \in \exp(G)$, where $\lambda_1 = 1 - \frac{\log|a|}{\log|q|}$, $\lambda_2 = -\frac{\log|a|}{\log|q|}$. This is an easy exercise. It shows that admissible metrics are indispensable for dealing with canonical metrics.

8. – Cohomological models of line bundles

Let K and G be as in Section 7. All spaces denoted by greek letters $X, X', \ldots$ are assumed to be proper schemes over K endowed with their rigid analytic structures.

We define admissible first Arakelov-Chern classes for line bundles in Arakelov-Chow cohomology. Since the cup product is not known to be commutative, we have to work inside the centralisator. The Arakelov-Chern classes generalize the divisoral operations of formally metrized line bundles on any proper scheme over K and also of admissibly metrized line bundles on curves considered in the previous section. They give rise to associated metrics on the line bundles. The main result is that for every line bundle L algebraically equivalent to 0, there is an admissible first Arakelov-Chern class which is orthogonal to vertical cycles. The associated metric is given by the Néron symbol. Using a correspondence of L to a curve, this is deduced from the corresponding result on curves proved in the Section 7.

Definition 8.1. Let $\hat{B}^*(X,G)$ be the set of all $\hat{c} \in \widehat{CH}^*_{\mathrm{fin}}(X,G)$ such that for any projective smooth curve C', any admissible metric $\| \ \|$ on $O_{C'}$ and any morphisms $\psi : X' \to X, \phi : X' \to C'$, we have

$$\psi^*\hat{c} \cup \phi^*\hat{c}_1(1) = \phi^*\hat{c}_1(1) \cup \psi^*\hat{c} \in \widehat{CH}^*_{\mathrm{fin}}(X',G).$$

Definition 8.2. Let $\hat{C}^*_{\mathrm{fin}}(X,G)$ be the set of $\hat{c} \in \widehat{CH}^*_{\mathrm{fin}}(X,G)$ satisfying

$$\hat{c}' \cup \psi^*\hat{c} = \psi^*\hat{c} \cup \hat{c}' \in \widehat{CH}^*_{\mathrm{fin}}(X',G)$$

for all morphisms $\psi : X' \to X$ and all $\hat{c}' \in \hat{B}^*(X',G)$.

Remark 8.3. If $\| \ \|$ is a formal metric on O_X, then (C3) implies $\hat{c}_1(\mathrm{div}(1)) \in \hat{C}^1_{\mathrm{fin}}(X,G)$. If $\| \ \|$ is an admissible metric on O_C for a projective smooth curve C, then Corollary 7.14 proves $\hat{c}_1(1) \in \hat{B}^1(C,G)$. Clearly, the above groups are closed under pull-back and cup-product. We conclude that $\hat{C}^*_{\mathrm{fin}}(X,G) \subset \hat{B}^*(X,G)$, hence $\hat{C}^*_{\mathrm{fin}}(X,G)$ is commutative with respect to cup-product. Clearly, $\hat{C}^*_{\mathrm{fin}}(C,G)$ contains $\hat{c}_1(1)$ for an admissible metric $\| \ \|$ on O_C.

Definition 8.4. Let L be a line bundle on X. An *admissible first Arakelov-Chern class* $\hat{c}_1(L)$ for L is a family of elements $\hat{c}_1(D) \in \widehat{CH}^1_{|D|}(X,G)$ for every pseudo-divisor $D = (L, |D|, s_D)$ on X. It is required that all these operations are given by one formal metric $\| \ \|_{\mathcal{L}}$ of L and one $\hat{c}_1^G \in \hat{C}^1_{\mathrm{fin}}(X,G)$ through $\hat{c}_1(D) = \hat{c}_1^{\mathcal{L}}(D) + \hat{c}_1^G$ where $\hat{c}_1^{\mathcal{L}}(D)$ is from Example 6.13 using $\| \ \|_{\mathcal{L}}$. If s is an invertible meromorphic section of L, then we write $\hat{c}_1(s) := \hat{c}_1(\mathrm{div}(s))$.

Remark 8.5. Note that a formal metric $\| \ \|_{\mathcal{L}}$ and a class $\hat{c}_1^G \in \hat{C}^1_{\mathrm{fin}}(X,G)$ induce always an admissible first Arakelov-Chern class. We may extend the constructions of sum and pull-back. The assumption $\hat{c}_1^G \in \hat{C}^1_{\mathrm{fin}}(X,G)$ shows that admissible first Arakelov-Chern classes commute with respect to the cup product. By Remark 7.13, every admissibly metrized line bundle on a smooth projective curve induces an admissible first Arakelov-Chern class.

8.6. For an admissible first Arakelov-Chern class $\hat{c}_1(L)$, the *associated metric* on L is

$$\|s(x)\|_{\hat{c}_1(L)} := \int_{\{x\}} \hat{c}_1(s(x)) \in G$$

for $x \in X, s(x) \in L_x$. If $\hat{c}_1(L)$ is induced by a formal metric (resp. an admissible metric in case of a smooth projective curve), then the associated metric is equal to the original metric.

By Proposition 5.2, there is a K°model $\mathcal{L}$ on $\mathfrak{X} \in M_X$ such that $\hat{c}_1(L)$ is induced by $\| \ \|_{\mathcal{L}}$ and the corresponding $\hat{c}_1^G$ is induced by $\hat{c}^G_{1,\mathfrak{X}} \in \widehat{CH}^1_{\mathrm{fin}}(\mathfrak{X},G)$. We say that $\hat{c}_1(L)$ *lives on* $\mathfrak{X}$.

PROPOSITION 8.7. *Let $\hat{c}_1(L)$ be an admissible first Arakelov-Chern class on a reduced X. Then $\hat{c}_1(L)$ lives on a formal analytic model $\mathfrak{X}$ of X. We consider an invertible meromorphic section s of L and an irreducible component V of $\tilde{\mathfrak{X}}$. For any $x \in X$ with reduction $\pi(x) \in V \cap \tilde{\mathfrak{X}}_{\mathrm{reg}}$ and with $\pi(x) \notin \pi|\operatorname{div}(s)|$, the multiplicity of $\hat{c}_1(s) \cap \operatorname{cyc}(X)$ in V is equal to $-\log \|s(x)\|_{\hat{c}_1(L)}$. This number is denoted by $-\log\|s(V)\|$.*

PROOF. The result holds for the first Arakelov-Chern class of a formally metrized line bundle (Proposition 7.6), hence we may assume $L = O_X$, $s = 1$ and $\hat{c}_1(s) = \hat{c}_1^G$. By Proposition 5.2 and passing to the associated formal analytic variety, it is clear that such a $\mathfrak{X}$ exists.

The proof of the multiplicity claim is by induction on the dimension of X. If X is 0-dimensional, the claim is by definition of $\| \ \|_{\hat{c}_1(L)}$. So we may assume $\dim(X) > 0$. We choose $x \in X$ with $\tilde{x} := \pi(x) \in V \cap \tilde{\mathfrak{X}}_{\mathrm{reg}}$. There is a regular function $\tilde{a}$ in an affine irreducible neighbourhood $\tilde{\mathcal{U}}$ of $\tilde{x}$ with $\tilde{a}(\tilde{x}) = 0$ and $d\tilde{a}(\tilde{x}) \neq 0$. Thus $\operatorname{div}(\tilde{a})$ is smooth in a neighbourhood of $\tilde{x}$. We may assume that there is a prime divisor W in V such that $\operatorname{div}(\tilde{a}) = \tilde{\mathcal{U}} \cap W$ is smooth. Let $\mathcal{U} = \operatorname{Spf}\mathcal{A} = \pi^{-1}\tilde{\mathcal{U}}$ be the corresponding formal affinoid open subspace of $\mathfrak{X}$. We choose a lift $a \in \mathcal{A}^\circ$ of $\tilde{a}$. Since the algebraic functions are dense in $\mathcal{A}$, we may assume that a is induced by a rational function on X also denoted by a. We may assume that $a(x) = 0$.

We consider the closed formal subscheme $\operatorname{Spf}\mathcal{A}^\circ/\langle a\rangle$ of $\operatorname{Spf}\mathcal{A}^\circ$. It is obvious that $\mathcal{A}^\circ/\langle a\rangle$ has no K°-torsion. Since the reduction $\tilde{\mathcal{A}}/\langle\tilde{a}\rangle$ is integral, we conclude that $\operatorname{Spf}\mathcal{A}^\circ/\langle a\rangle$ is an admissible formal scheme associated to the integral formal analytic variety $\operatorname{Spf}\mathcal{A}/\langle a\rangle$ (4.1 and [BGR], Proposition 6.2.3/5). In particular, the horizontal $\operatorname{div}(a)$ is prime on $\mathcal{U}$. Let Y be the irreducible component of $\operatorname{div}(a)$ passing through $\mathcal{U}$. Note that W is the intersection of $\mathcal{U}$ with the special fibre of $\bar{Y}$. Applying induction to $\bar{Y}$, we conclude that the multiplicity of $\hat{c}^G_{1,\mathfrak{X}} \cap Y$ in W is equal to $-\log\|1(x)\|_{\hat{c}_1(L)}$. Axiom (C3) shows

$$\text{(14)}\quad \operatorname{div}_{\mathfrak{X}}(a).(\hat{c}^G_{1,\mathfrak{X}} \cap \operatorname{cyc}(X)) = \hat{c}^G_{1,\mathfrak{X}} \cap (\operatorname{div}_{\mathfrak{X}}(a).\operatorname{cyc}(X)) \in CH^{|\operatorname{div}_{\mathfrak{X}}(a)|\cap\tilde{\mathfrak{X}}}_{\dim(X)-2}(\mathfrak{X}, G)\,.$$

Note that W is an irreducible component of the support $|\operatorname{div}_{\mathfrak{X}}(a)| \cap \operatorname{cyc}(\tilde{\mathfrak{X}})$ and W has relative dimension $\dim(X) - 2$. The multiplicity of the left hand side of (14) in W is equal to the multiplicity of $\hat{c}^G_{1,\mathfrak{X}} \cap \operatorname{cyc}(X)$ in V (by construction of a and working on $\mathcal{U}$). The multiplicity of the right hand side of (14) in W is $-\log\|1(x)\|_{\hat{c}_1(L)}$. □

PROPOSITION 8.8. *Let Z be a t-dimensional cycle on a smooth proper variety X over K algebraically equivalent to 0. Then there is $\alpha^{\mathrm{fin}} \in \widehat{CH}_t^{\mathrm{fin}}(X, G)$ such that $\int_X \hat{c} \cap (Z + \alpha^{\mathrm{fin}}) = 0$ for all $\hat{c} \in \hat{C}^{t+1}_{\mathrm{fin}}(X, G)$.*

PROOF. By [Fu], Example 10.3.2, we may assume that there is a smooth projective curve C over K, a correspondence Γ on $X \times C$ and $t_0, t_1 \in C$ such that $Z = p_{1*}(p_2^* c_1(O([t_1] - [t_0])) \cap \Gamma) \in CH_d(X)$ where p_i are the projections of $X \times C$ onto the factors. Now we choose a canonical metric $\| \ \|$ on $O(t_1 - t_0)$

from Corollary 7.18. We may assume $p_{1*}(p_2^*\hat{c}_1([t_1]-[t_0])\cap\Gamma) = Z+\alpha^{\mathrm{fin}}$ for a suitable $\alpha^{\mathrm{fin}} \in \widehat{CH}_t^{\mathrm{fin}}(X,G)$. We get

$$\int_X \hat{c}\cap(Z+\alpha^{\mathrm{fin}}) = \int_{X\times C} p_1^*\hat{c}\cap p_2^*\hat{c}_1([t_1]-[t_0])\cap\Gamma = \int_C \hat{c}_1([t_1]-[t_0])\cap p_{2*}(p_1^*\hat{c}\cap\Gamma)\,.$$

Because $p_{2*}(p_1^*\hat{c}\cap\Gamma)$ is vertical, this equals 0. $\square$

THEOREM 8.9. *Let X be a smooth proper variety over K and let $L\in \mathrm{Pic}^0(X)$. Then there is an admissible first Arakelov-Chern class $\hat{c}_1(L)$ for L such that $\int_X \hat{c}_1(L)\cap\alpha = 0$ for all $\alpha\in\widehat{CH}_0^{\mathrm{fin}}(X,G)$. This determines the associated metric $\|\ \|_{\hat{c}_1(L)}$ up to multiples in e^G.*

PROOF. We may assume that X is irreducible. Since L is algebraically equivalent to 0, there is an irreducible smooth projective curve C over K with points $t_0, t_1\in C$ and $M\in\mathrm{Pic}(X\times C)$ with $M_{t_0}=O_X$ and $M_{t_1}=L$. We endow the line bundle $O([t_1]-[t_0])$ on C with a canonical admissible metric $\|\ \|$ from Corollary 7.18. By Remark 7.11, there is a semistable K°-model $\mathcal{C}$ of C such that $\|\ \|$ is $\mathcal{C}$-admissible and such that $\tilde{\mathcal{C}}$ has smooth irreducible components. By Proposition 5.2, there is a K°-model $\mathcal{M}$ of M living on the K°-model $\mathfrak{Y}$ of $X\times C$ such that the projections p_i of $X\times C$ to the factors extend to morphisms $\bar{p}_1:\mathfrak{Y}\to\mathfrak{X}$ and $\bar{p}_2:\mathfrak{Y}\to\mathcal{C}$ of K°-models. The formal metric $\|\ \|_{\mathcal{M}}$ induces a formal metric $\|\ \|_t$ on M_t for any $t\in C$. Replacing $\mathfrak{X}$ and $\mathfrak{Y}$ by larger K°-models, we may assume that $\|\ \|_{t_0}$ is the trivial metric on $M_{t_0}=O_X$, $\|\ \|_{t_1}=\|\ \|_{\mathcal{L}_1}$ for a K°-model $\mathcal{L}_1$ of $L=M_{t_1}$ on $\mathfrak{X}$ and $\bar{p}_1$ is flat (Proposition 5.2).

We choose any invertible meromorphic section s_M of M. Note that we may assume that $\mathrm{div}(s_M)$ intersects any finite number of cycles properly in $X\times C$ ([Gu2], Lemma 3.6). So we may assume that s_M restricts to a well-defined invertible meromorphic section s_t of M_t for $t=0,1$ and that $s_{t_0}=1$. We use vertical and horizontal parts for $\hat{c}_1([t_1]-[t_0])$ from Remark 7.16 componentwise. We claim that there is $\hat{c}_1^G\in\hat{C}^1_{\mathrm{fin}}(X,G)$ induced on horizontal cycles by

$$\bar{p}_{1*}(\hat{c}_1^{\mathcal{M}}(s_M)_{\mathfrak{Y}}\cup\bar{p}_2^*\hat{c}_{1,\mathcal{C}}^{\mathrm{ver}}) \tag{15}$$

and on vertical cycles by

$$\bar{p}_{1*}(\hat{c}_1^{\mathcal{M}}(s_M)_{\mathfrak{Y}}\cup\bar{p}_2^*\hat{c}_1([t_1]-[t_0])_{\mathcal{C}}) - \hat{c}_1^{\mathcal{L}}(s_{t_1})_{\mathfrak{X}}\,. \tag{16}$$

This defines operations $\hat{c}_1^G$ on $\hat{Z}(X',G)$ for every proper morphism $\psi: X'\to X$ which do not depend on the choice of s_M. Note that $\hat{c}_1^G\cap_\psi\alpha'$ is always vertical for $\alpha'\in\hat{Z}(X',G)$. Let $c^G_{1,\mathfrak{X}}\cap_{\bar{\psi}}$ be the operation on $\tilde{Z}(\mathfrak{X}',G)$ given by (15) and (16) where $\bar{\psi}:\mathfrak{X}'\to\mathfrak{X}$ is a proper morphism of admissible formal schemes. We will prove that $c^G_{1,\mathfrak{X}}$ satisfies the axioms (C1)-(C4) on the cycle level. Hence the operations pass to rational equivalence and give a well-defined

$c_{1,\mathfrak{X}}^G \in CH^1_{\mathrm{fin}}(\mathfrak{X}, G)$ inducing $\hat{c}_1^G \in \widehat{CH}^1_{\mathrm{fin}}(X, G)$. Finally, we will prove that $\hat{c}_1^G \in \hat{C}^1_{\mathrm{fin}}(X, G)$.

Axioms (C1) and (C2) may be checked on horizontal and vertical cycles separately, and axiom (C4) concerns only vertical cycles. So they follow from the corresponding axioms for (15) or (16). For a pseudo-divisor D' on $\mathfrak{X}'$, the same argument proves (C3) for vertical $\alpha' \in \tilde{Z}(\mathfrak{X}, G)$. So we may assume α' horizontal, $\alpha' = X'$ prime and that D' is a Cartier divisor. The problem is now that $c_{1,\mathfrak{X}}^G$ acts on vertical and horizontal cycles simultaneously. Note that (15) and (16) agree on horizontal cycles if we allow the larger support $S := p_1(|\operatorname{div}(s_M)| \cap (X \times \{t_0, t_1\})) \cup \tilde{\mathfrak{X}}$. Using (C3) for (16), we get

$$c_{1,\mathfrak{X}}^G \cap_{\bar{\psi}} (D'.X') = D'.(c_{1,\mathfrak{X}}^G \cap_{\bar{\psi}} X') \in CH^{\psi^{-1}S \cap |D'|}_{\dim(X)-2}(\mathfrak{X}', G)\,. \tag{17}$$

We may choose s_M such that all occuring intersections of $\operatorname{div}(s_M)$ in the above supports are proper in the generic fibre. By dimensionality, (17) holds also for $S = \tilde{\mathfrak{X}}$ proving (C3) for $c_{1,\mathfrak{X}}^G$.

Next, we prove that $\hat{c}_1^G \in \hat{C}_{\mathrm{fin}}(X, G)$. Let $\psi : X' \to X$ be a morphism and let $\hat{c}' \in \hat{B}^*(X', G)$. We have to prove that $\psi^*\hat{c}_1^G$ and $\hat{c}'$ commute. It is enough to check this on a cycle α' of a sufficiently large K°-model $\mathfrak{X}'$ of X'. So we may assume that ψ extends to a morphism $\bar{\psi} : \mathfrak{X}' \to \mathfrak{X}$ and that $\hat{c}'$ is induced by $c'_{\mathfrak{X}'} \in CH^*_{\mathrm{fin}}(\mathfrak{X}', G)$. We have to check the identity

$$c_{1,\mathfrak{X}}^G \cap_{\bar{\psi}} (c'_{\mathfrak{X}'} \cap \alpha') = c'_{\mathfrak{X}'} \cap (c_{1,\mathfrak{X}}^G \cap_{\bar{\psi}} \alpha') \in CH_*^{\mathrm{fin}}(\mathfrak{X}', G)\,. \tag{18}$$

It is clear that $\hat{c}'$ commutes with any first Arakelov-Chern class for a formally metrized line bundle and with any pull-back of a first Arakelov-Chern class of an admissibly metrized line bundle on a projective smooth curve. If α' is vertical, then working on the fibre square $\mathfrak{Y} \times_{\mathfrak{X}} \mathfrak{X}'$ and using projection formula as well as fibre square rule, we deduce (18) from (16). If α' is horizontal, then as in the proof of axiom (C3) above, we can prove that $c_{1,\mathfrak{X}}^G$ in (18) may be replaced by (16) and the same argument as in the vertical case proves (18).

We define a first Arakelov-Chern class $\hat{c}_1(L)$ for L by the formal metric $\|\ \|_{t_1}$ and by $\hat{c}_1^G \in \hat{C}^1_{\mathrm{fin}}(X, G)$. Let $\alpha \in \widehat{CH}_0^{\mathrm{fin}}(X, G)$. By (16), we get

$$\begin{aligned}\int_X \hat{c}_1(L) \cap \alpha &= \int_{\mathfrak{Y}} \bar{p}_2^*\hat{c}_1([t_1] - [t_0])_C \cap \hat{c}_1^{\mathcal{M}}(s_M)_{\mathfrak{Y}} \cap \bar{p}_1^*\alpha_{\mathfrak{X}} \\ &= \int_C \hat{c}_1([t_1] - [t_0])_C \cap \bar{p}_{2*}(\hat{c}_1^{\mathcal{M}}(s_M)_{\mathfrak{Y}} \cap \bar{p}_1^*\alpha_{\mathfrak{X}})\,.\end{aligned}$$

Because $\bar{p}_{2*}(\hat{c}_1^{\mathcal{M}}(s_M)_{\mathfrak{Y}} \cap \bar{p}_1^*\alpha_{\mathfrak{X}})$ is vertical, this is zero proving existence.

To prove uniqueness of the associated metric, we may assume $L = O_X$. Let us fix $x_0 \in X$ and let x be any point of X. Then $x - x_0$ is algebraically equivalent to 0. By Proposition 8.8, there is $\alpha^{\mathrm{fin}} \in \widehat{CH}_0^{\mathrm{fin}}(X, G)$ such that

$$0 = \int_X \hat{c}_1(1) \cap (x - x_0 + \alpha^{\mathrm{fin}}) = \log \|1(x_0)\|_{\hat{c}_1(O_X)} - \log \|1(x)\|_{\hat{c}_1(O_X)}\,.$$

Hence $\|\ \|_{\hat{c}_1(O_X)}$ is equal to $\|1(x_0)\|_{\hat{c}_1(O_X)}$ times the trivial metric on O_X. □

Corollary 8.10. *Let X be a smooth proper scheme over K and let $L \in \mathrm{Pic}^0(X)$ with an admissible first Arakelov-Chern class $\hat{c}_1(L)$ of type as in Theorem* 8.9. *For every invertible meromorphic section s of L and every* 0*-dimensional cycle Z of degree* 0 *with $|\operatorname{div}(s)| \cap |Z| = \emptyset$, we have $-\log \|s(Z)\|_{\hat{c}_1(L)} = \langle \operatorname{div}(s), Z\rangle_{\text{Nér}}$ and the* canonical metric $\| \ \|_{\hat{c}_1(L)}$ *is bounded.*

Proof. For the Néron symbol $\langle \ , \ \rangle_{\text{Nér}}$, we refer to [Ne], Théorème 3. On curves, the claim follows from Corollary 7.9. We use the notation of the proof of Theorem 8.9. We may choose s_M with $|\operatorname{div}(s_M)| \cap (|Z| \times \{t_0, t_1\}) = \emptyset$. Replacing again (15) by (16), we get

$$\begin{aligned} -\log \|s_{t_1}(Z)\|_{\hat{c}_1(L)} &= \int_{\mathfrak{X}} \bar{p}_{1*}(\hat{c}_1^{\mathcal{M}}(s_M)_{\mathfrak{Y}} \cup \bar{p}_2^* \hat{c}_1([t_1]-[t_0])_{\mathcal{C}}) \cap Z \\ &= \int_{\mathcal{C}} \hat{c}_1([t_1]-[t_0])_{\mathcal{C}} \cap \bar{p}_{1*}(\hat{c}_1^{\mathcal{M}}(s_M)_{\mathfrak{Y}} \cap \bar{p}_1^* Z) \, . \end{aligned}$$

Using the case of curves and the correspondence $\Gamma = \operatorname{div}(s_M)$, we get

$$-\log \|s_{t_1}(Z)\|_{\hat{c}_1(L)} = \langle t_1 - t_0, \Gamma(Z)\rangle_{\text{Nér}} = \langle \operatorname{div}(s_{t_1}), Z\rangle_{\text{Nér}} \, .$$

The last step was by the reciprocity law ([Ne], Théorème 4) and $s_{t_0} = 1$. By the axiom of the Néron symbol for principal divisors, we get the desired identity also for s. Using the boundedness axiom, it is clear that the metric is bounded. □

9. – Local heights over non-archimedean fields

We use the same assumptions and notation as in Section 8.

9.1. A metric associated to an admissible first Arakelov-Chern class is called a *cohomological metric*. Every formal metric is a cohomological metric. On a projective smooth curve, every admissible metric is a cohomological metric. A *cohomological pseudo-divisor* is a pseudo-divisor with a cohomological metric (defined similarly as in 2.5).

Definition 9.2. Let $\hat{D}_0, \ldots, \hat{D}_t$ be cohomological pseudo-divisors with metrics associated to $\hat{c}_1(O(D_0)), \ldots, \hat{c}_1(O(D_t))$ and let Z be a cycle on X of dimension t. We say that the *local height* $\lambda(Z)$ of Z with respect to $\hat{D}_0, \ldots, \hat{D}_t$ is *well-defined* if $|D_0| \cap \cdots \cap |D_t| \cap |Z| = \emptyset$. Then

$$\lambda(Z) := \lambda_{\hat{D}_0, \ldots, \hat{D}_t}(Z) := \int_Z \hat{c}_1(D_0) \cup \cdots \cup \hat{c}_1(D_t) \, .$$

Remark 9.3. It follows from Proposition 8.7 that the dependence of the local height with respect to $\hat{c}_1(O(D_0)), \ldots, \hat{c}_1(O(D_t))$ is determined by the associated metrics. The local height has similar properties as in Section 3. Obviously, it is multilinear and symmetric in $\hat{D}_0, \ldots, \hat{D}_t$, and linear in Z. By 6.12, the analogue of *functoriality* 3.6 holds.

PROPOSITION 9.4. *Let X be smooth and let $\hat{D}_0, \ldots, \hat{D}_t$ be cohomological pseudo-divisors such that the local height $\lambda(Z)$ of the t-dimensional cycle Z is well-defined. We assume that $O(D_0) \in \mathrm{Pic}^0(X)$ is endowed with the canonical metric $\| \ \|$ (cf. Corollary 8.10). Let Y be a cycle representing $D_1 \ldots D_t.Z \in CH_0(|D_1| \cap \ldots \cap |D_t| \cap |Z|)$. Then $\lambda(Z) = -\log \|s_{D_0}(Y)\|$ holds.*

PROOF. Using the same notation as in the proof of Proposition 3.7, we get

$$\log \|s_0(Y')\| - \log \|s_0(Y)\| = \int_{\mathrm{div}(\mathbf{f})} \hat{c}_1(s_0) = \int_X \hat{c}_1(s_0) \cap \widehat{\mathrm{div}}(\mathbf{f}) = 0$$

proving independence of the representative Y. The formula follows from

$$\lambda(Z) = \int_Z \hat{c}_1(D_0) \cap Y = -\log \|s_0(Y)\|$$

because the horizontal part of $\hat{c}_1(D_1) \cup \cdots \cup \hat{c}_1(D_t)$ is Y and the verical part plays no role. □

REMARK 9.5. There is also an analogue of the *induction formula*. We use the same assumptions and notation as in Proposition 3.5. There is a K°-model $\mathfrak{X}$ of X such that $\hat{c}_1(O(D_j))$ lives on $\mathfrak{X}$ for all $j = 0, \ldots, t$. Let $\tilde{c}_1(O(D_j)) \in CH^1(\tilde{\mathfrak{X}}, G)$ inducing the action of $\hat{c}_1(D_j)$ on vertical cycles of $\mathfrak{X}$. By Proposition 8.7, we get

$$\lambda_{\hat{D}_0,\ldots,\hat{D}_t}(Z) = \lambda_{\hat{D}_0,\ldots,\hat{D}_{t-1}}(Y) - \int_{\mathfrak{Z}} \sum_V \log \|s_{t,Z}(V)\| \cdot \tilde{c}_1(O(D_0)) \cap \cdots \cap \tilde{c}_1(O(D_{t-1})) \cap V$$

where V ranges over the irreducible components of $\tilde{\mathfrak{Z}}$ for $\mathfrak{Z} := (\bar{Z})^{\text{f-an}}$.

Let $\lambda'(Z)$ be the local height obtained by exchanging the metric $\| \ \|$ on $O(D_0)$ by another cohomological metric $\| \ \|'$. For $\rho := \log(\|s_{D_0}\|'/\|s_{D_0}\|)$, symmetry and induction formula imply

$$\lambda(Z) - \lambda'(Z) = \int_Z \sum_V \rho(V)\, \tilde{c}_1(O(D_1)) \cap \cdots \cap \tilde{c}_1(O(D_t)) \cap V\,. \tag{19}$$

9.6. Let L be a line bundle on X. An admissible first Arakelov-Chern class $\hat{c}_1(L)$ is called *semipositive* if it satisfies the following property: There is a K°-model $\mathfrak{X}$ of X such that $\hat{c}_1(L)$ lives on $\mathfrak{X}$ with $\tilde{c}_1(L) \in CH^1(\tilde{\mathfrak{X}}, G)$ inducing the action on vertical cycles. Then we assume that $\deg(\tilde{c}_1(L) \cap V) \geq 0$ for all 1-dimensional prime cycles V on $\tilde{\mathfrak{X}}$. Note that the condition is independent of the choice of $\mathfrak{X}$ (use [Kl], Lemma I.4.1).

A metric on L is called *cohomologically semipositive* if it is bounded and associated to a semipositive admissible first Arakelov-Chern class for L.

These notions are closed under tensor product and pull-back. The formal metric of a K°-model $\mathcal{L}$ is cohomologically semipositive if $\mathcal{L}$ is generated by global sections.

LEMMA 9.7. *Let $\hat{c}_1(L_1), \ldots, \hat{c}_1(L_t)$ be semipositive admissible first Arakelov-Chern classes for line bundles $L_1, \ldots, L_t$ on X. Then they live on a K°-model $\mathfrak{X}$ of X such that the operations on vertical cycles are induced by $\tilde{c}_1(L_j) \in CH^1(\tilde{\mathfrak{X}}, G)$. For effective t-dimensional cycles Z on X and V on $\tilde{\mathfrak{X}}$, we have* $\deg_{L_1,\ldots,L_t}(Z) \geq 0$ *and* $\deg(\tilde{c}_1(L_1) \cap \tilde{c}_1(L_2) \cap \cdots \cap \tilde{c}_1(L_t) \cap \mathrm{cyc}(V)) \geq 0$.

PROOF. The existence of the K°-model follows from Proposition 5.2. By the alteration theorem of de Jong ([dJ], Theorem 4.1), there is a smooth scheme V' and a proper morphism $\psi : V' \to V$ which is the composition of a birational morphism with a finite surjective morphism. By Poincaré duality ([Fu], Corollary 17.4), the Chow cohomology classes $\psi^*\tilde{c}_1(L_1), \ldots, \psi^*\tilde{c}_1(L_t)$ are the first Chern classes of numerically positive line bundles. By projection formula and a result of Kleiman ([Kl], Corollary II.2.2), we deduce the second inequality.

To prove the first inequality, let $[v]$ be the special fibre of $\mathrm{Spf}K^\circ$, we may view it as an G-power of a formal Cartier divisor. If π is the morphism of structure, then the above implies

$$\deg_{L_1,\ldots,L_t}(X)[v] = \pi_*(\hat{c}_1(L_1) \cap \cdots \cap \hat{c}_1(L_t) \cap \pi^*[v] \cap X) \geq 0\,. \qquad \square$$

10. – Canonical local heights

Let K be an algebraically closed field with a complete absolute value $|\ |_v$. First, we extend the result of local heights of subvarieties obtained in Sections 3 and 9 allowing certain uniform limits of metrics. In the archimedean case, we always pass to the underlying reduced complex analytic space to apply the previous results and in the non-archimedean case, we fix a subfield G as in the previous sections. Then we study canonical local heights in dynamic situations.

DEFINITION 10.1. Let L be a line bundle on a proper scheme X over K. For two bounded metrics $\|\ \|, \|\ \|'$ on L, let $\left(\|\ \|'/\|\ \|\right)(x) := \|s(x)\|'/\|s(x)\|$ where $x \in X$ and $s(x) \in L_x \setminus \{0\}$. This is a bounded function on X and we define the *distance*

$$d(\|\ \|, \|\ \|') := \max_{x\in X} |\log(\|\ \|'/\|\ \|)(x)|\,.$$

DEFINITION 10.2. Let $\mathfrak{g}_X^+$ be the set of isometry classes of line bundles with semipositive curvature forms (resp. cohomologically semipositive metrics). Let $\hat{\mathfrak{g}}_X^+$ be the set of isometry classes of metrized line bundles $(L, \|\ \|)$ on X satisfying the following property: For all $n \in \mathbb{N}$, there is a proper surjective morphism $\varphi_n : X_n \to X$ and a metric $\|\ \|_n$ on $\varphi_n^* L$ with $(\varphi_n^* L, \|\ \|_n) \in \mathfrak{g}_{X_n}^+$ such that $\lim_{n\to\infty} d_{X_n}(\varphi_n^*\|\ \|, \|\ \|_n) = 0$. Let $\mathfrak{g}_X := \mathfrak{g}_X^+ - \mathfrak{g}_X^+$.

Let $\hat{\mathfrak{g}}_X$ be the set of isometry classes of metrized line bundles $\overline{L}$ on X with a proper surjective morphism $\varphi : X' \to X$ and $\overline{M}, \overline{N} \in \hat{\mathfrak{g}}_{X'}^+$ such that $\varphi^*\overline{L} = \overline{M} \otimes \overline{N}^{-1}$. A $\hat{\mathfrak{g}}_X$*-pseudo-divisor* is a metrized pseudo-divisor $\hat{D}$ with $\overline{O}(D) \in \hat{\mathfrak{g}}_X$ and similarly, we proceed for $\mathfrak{g}_X, \mathfrak{g}_X^+$ or $\hat{\mathfrak{g}}_X^+$.

REMARK 10.3. Note that we may always assume that the morphisms in Definition 10.2 are from projective varieties (Chow lemma) with disjoint irreducible components and that the morphism is generically finite. The latter follows by intersection with generic hyperplanes until the right dimension is obtained. Thus the definitions of $\hat{\mathfrak{g}}_X^+$ and $\hat{\mathfrak{g}}_X$ agree with those in [Gu2], Section 1.

It is clear that $\mathfrak{g}_X^+, \hat{\mathfrak{g}}_X^+$ are semigroups and $\hat{\mathfrak{g}}_X$ is a group with respect to tensor product. If $\overline{L}$ is a metrized line bundle with $\overline{L}^{\otimes n} \in \hat{\mathfrak{g}}_X^+$ for some $n \in \mathbb{N} \setminus \{0\}$, then $\overline{L} \in \hat{\mathfrak{g}}_X^+$. Every metric of a line bundle in $\hat{\mathfrak{g}}_X$ is bounded. The semigroup $\hat{\mathfrak{g}}_X^+$ is closed under uniform convergence.

Let $\varphi : X' \to X$ be a proper morphism with a metrized line bundle $\overline{L}$ on X. If $\overline{L} \in \hat{\mathfrak{g}}_X^+$ (resp. $\hat{\mathfrak{g}}_X$), then $\varphi^*\overline{L} \in \hat{\mathfrak{g}}_{X'}^+$ (resp. $\hat{\mathfrak{g}}_{X'}$). The converse holds for φ surjective ([Gu2, Proposition 1.18).

PROPOSITION 10.4. *Hermitian and formally metrized line bundles are both in* $\hat{\mathfrak{g}}_X$*, hence every line bundle has a metric in* $\hat{\mathfrak{g}}_X$*.*

PROOF. The archimedean case is in [Gu2], Example 1.20. The proof for a formal metric is similar. By Chow's lemma and Remark 10.3, we may assume that X is a projective variety over K. There is a K°-model $\mathcal{L}$ of our line bundle such that the formal metric is $\| \; \|_{\mathcal{L}}$. By Lemma 10.5 below, we may assume that $\mathcal{L}$ lives on a projective K°-model $\mathfrak{X}$ of X. Therefore, $\mathcal{L}$ is the difference of two very ample line bundles giving $\| \; \|_{\mathcal{L}}$ as the quotient of $\mathfrak{g}_X^+$-metrics. Finally, existence follows by a partition of unity argument and Proposition 5.2. □

PROPOSITION 10.5. *Let X be a projective scheme over K with K°-model $\mathfrak{X}$. Then there is a projective K°-model $\mathfrak{X}_1$ of $\mathfrak{X}$ with $\mathfrak{X}_1 \geq \mathfrak{X}$.*

PROOF. We fix a closed embedding $X \subset \mathbb{P}^N$. There is a projective flat K°-model $\mathfrak{X}_0$ of X associated to a flat projective subscheme $\mathfrak{X}_0^{\mathrm{alg}}$ of $\mathbb{P}^N_{K^\circ}$. By [BL2], Section 4, there is a K°-model $\mathfrak{X}_1$ of X with $\mathfrak{X}_1 \geq \mathfrak{X}_0, \mathfrak{X}_1 \geq \mathfrak{X}$. Moreover, $\mathfrak{X}_1$ is obtained as a formal blowing up of $\mathfrak{X}_0$ in an open coherent ideal $\mathcal{J}$. Using the GAGA-principle for projective schemes ([Ul], Theorem 6.8), $\mathcal{J}$ is algebraic. Note that the blowing up $\mathfrak{X}_1^{\mathrm{alg}}$ of $\mathfrak{X}_0^{\mathrm{alg}}$ in $\mathcal{J}_{\mathrm{alg}}$ is projective, since $\mathfrak{X}_0^{\mathrm{alg}}$ is quasicompact and $\mathcal{J}_{\mathrm{alg}}$ is of finite type (use [EGA II], Proposition 8.1.7, Proposition 3.4.1, Théorème 5.5.3, Corollaire 5.3.3). It follows from the local description of admissible formal blowing ups ([BL2], Lemma 2.2) that $\mathfrak{X}_1$ is the formal completion of $\mathfrak{X}_1^{\mathrm{alg}}$ along the special fibre. □

THEOREM 10.6. *For a proper scheme X over K with $\hat{\mathfrak{g}}_X$-pseudo-divisors $\hat{D}_0, \ldots, \hat{D}_t$, there is a unique local height $\lambda(Z) = \lambda_{\hat{D}_0,\ldots,\hat{D}_t}(Z)$, well-defined on*

t-dimensional cycles Z of X with $|D_0| \cap \cdots \cap |D_t| \cap |Z| = \emptyset$, satisfying the following properties (*assuming all terms well-defined*):

(a) *$\lambda(Z)$ is multilinear and symmetric in the variables $\hat{D}_0, \ldots, \hat{D}_t$, and linear in Z.*
(b) *Let $\varphi : X' \to X$ be a proper morphism, then we have the functoriality*

$$\lambda_{\varphi^*\hat{D}_0,\ldots,\varphi^*\hat{D}_t}(Z') = \lambda_{\hat{D}_0,\ldots,\hat{D}_t}(\varphi_* Z') .$$

(c) *If $\hat{D}_0 = \widehat{\mathrm{div}}(f)$ for an invertible meromorphic function f on X and if Y is a representative of $D_1 \ldots D_t.Z \in CH_0(|D_1| \cap \cdots \cap |D_t| \cap |Z|)$, then $\lambda(Z) =$* $\log|f(Y)|_v$.
(d) *Let $\overline{O}(D_1), \ldots, \overline{O}(D_t) \in \hat{\mathfrak{g}}_X^+$. Replacing the metric $\| \ \|$ on $O(D_0)$ by a $\hat{\mathfrak{g}}_X$-metric $\| \ \|'$, we get the local height $\lambda'(Z)$. For Z effective, there are finite upper and lower bounds $c_\pm$ of* $\log(\| \ \|'/\| \ \|)$ *restricted to Z, and we have*

$$c_- \cdot \deg_{O(D_1),\ldots,O(D_t)}(Z) \leq \lambda(Z) - \lambda'(Z) \leq c_+ \cdot \deg_{O(D_1),\ldots,O(D_t)}(Z) .$$

(e) *If $\hat{D}_0, \ldots, \hat{D}_t$ are $\mathfrak{g}_X^+$-pseudo-divisors, then $\lambda(Z)$ is the local heights of* 3.3 *or* 9.2.

Proof. We have seen in Sections 3 and 9 that (a)-(d) hold for $\mathfrak{g}_X$-pseudo-divisors. For (d), use Proposition 3.8 and that $c_1(\overline{O}(D_1)) \wedge \cdots \wedge c_1(\overline{O}(D_t))$ is a positive measure on Z with volume $\deg_{O(D_1),\ldots,O(D_t)}(Z)$ in the archimedean case, resp. (19) and the positivity results of Lemma 9.7 in the non-archimedean case. By [Gu2], 1.19, the results extend uniquely to $\hat{\mathfrak{g}}_X$. □

Proposition 10.7. *Let Z be an effective t-dimensional cycle on X such that the local height $\lambda(Z)$ is well-defined with respect to the $\hat{\mathfrak{g}}_X^+$-pseudo-divisors $\hat{D}_0, \ldots, \hat{D}_t$. On $Z \setminus |D_j|$, we assume that $\|s_{D_j}(x)\| \leq C_j \in \mathbb{R}$. Then $\lambda(Z) \geq -\sum_{j=0}^t \log C_j \cdot \deg_{O(D_0),\ldots,O(D_{j-1}),O(D_{j+1}),\ldots,O(D_t)}(Z)$. In particular, if $C_j = 1$ for $j = 0, \ldots, t$, then the local height is non-negative.*

Proof. By continuity and functoriality, we may assume that $\hat{D}_0, \ldots, \hat{D}_t$ are $\mathfrak{g}_X^+$-pseudo-divisors. We proceed by induction on t. We may assume Z prime. For $t = 0$, the local height is $-\log\|s_0(Z)\| \geq -\log C_j$. For $t > 0$, symmetry allows us to assume $Z \not\subset |D_t|$. We claim that the cycle $D_t.Z$ is effective. By functoriality and de Jong's alteration theorem ([dJ], Theorem 4.1), we may assume that D_t is a divisor with normal crossings on the smooth variety $X = Z$. Because $\|s_{D_t}\|$ is bounded on $X \setminus D_t$, we conclude that D_t is effective. As in the proof of (d) above, induction formula shows $\lambda(Z) \geq \lambda(D_t.Z) - \log C_t \cdot \deg_{O(D_0),\ldots,O(D_{t-1})}(Z)$. Induction for the local height $\lambda(D_t.Z)$ with respect to $\hat{D}_0, \ldots, \hat{D}_{t-1}$ proves the claim. □

10.8. We give a slight generalization of a result of Zhang ([Zh2], Theorem 2.2) constructing canonical metrics relative to dynamics. Let X be a proper scheme over K and let $\psi : X \to X$ be a morphism. Let L be a line bundle on X with an isomorphism $\theta : \psi^* L^{\otimes n} \xrightarrow{\sim} L^{\otimes m}$ for some $n, m \in \mathbb{Z}$, $|m| > |n|$. Any other isomorphism has the form $\alpha\theta$ for locally constant α.

THEOREM 10.9. *There is a unique bounded metric $\| \; \|_\theta$ on L with $\| \; \|_\theta^{\otimes m} \circ \theta = \psi^* \| \; \|_\theta^{\otimes n}$.*

PROOF. The metric space of bounded metrics on L with the distance d is a complete metric space non canonically isometric to the space of bounded functions on X. It has a contractive endomorphism Φ with contraction factor $|\frac{n}{m}|$ given by

$$\Phi(\| \; \|) := ((\psi^* \| \; \|^{\otimes n}) \circ \theta^{-1})^{\frac{1}{m}} .$$

By Proposition 10.4, there is at least one bounded metric $\| \; \|$ on L. By Banach's fixed point theorem, there is a unique bounded metric on L with $\Phi(\| \; \|) = \| \; \|$. □

REMARK 10.10. Recall from the proof of Banach's fixed point theorem that $\| \; \|_\theta = \lim_{k \to \infty} \| \; \|_k$ for any bounded metric $\| \; \|$ on L and $\| \; \|_k := \Phi^k(\| \; \|)$. It is a Cauchy sequence by

$$d(\| \; \|_k, \| \; \|_l) \leq \left|\frac{n}{m}\right|^k \left(1 + \left|\frac{n}{m}\right| + \cdots + \left|\frac{n}{m}\right|^{l-k-1}\right) d(\| \; \|_0, \| \; \|_1) \tag{20}$$

for $k < l$. If we replace θ by $\alpha\theta$, then uniqueness implies $\| \; \|_{\alpha\theta} = |\alpha|_v^{\frac{1}{n-m}} \| \; \|_\theta$.

EXAMPLE 10.11. Let A be an abelian variety over K and let $\psi = [m]$ be multiplication with $m \in \mathbb{Z}, |m| \geq 2$. The theorem of the cube implies $[m]^* L \cong L^{\otimes m^2}$ if L is even and $[m]^* L \cong L^{\otimes m}$ if L is odd. Any line bundle L on A is isomorphic to the tensor product of an even and an odd line bundle unique up to 2-torsion in $\mathrm{Pic}(X)$. So we get canonical metrics on any line bundle, unique up to multiples in $|K^\times|_v$. In the same sense, they are not depending on the choice of m.

If v is archimedean, then there is a smooth hermitian metric on L with harmonic Chern form. This metric is unique up to multiples. Since $[m]^*$ transforms harmonic forms to harmonic forms, the canonical metrics are the smooth hermitian metrics with harmonic Chern forms.

Now let v be non-archimedean and let L be odd (or equivalently $L \in \mathrm{Pic}^0(A)$). By Corollary 8.10, we get a canonical metric on L unique up to multiples and given by the Néron symbol. From the basic properties of the Néron symbol, we deduce that these canonical metrics satisfy the characteristic property of Theorem 10.9, hence they agree with the canonical metrics above.

REMARK 10.12. Let X be a smooth proper scheme over K and let $L \in \mathrm{Pic}^0(X)$. From the fundamental property of the Picard variety, there is an abelian variety A, $L' \in \mathrm{Pic}^0(A)$ and a morphism $\varphi : X \to A$ with $\varphi^* L' \cong L$. The pull-back of a canonical metric on L' is called a *canonical metric* on L.

If v is archimedean (resp. non-archimedean), then the canonical metrics are the smooth hermitian metrics on L with first Chern form 0 (resp. the canonical metrics of Theorem 8.9). They are closed under pull-back and tensor product. Note that canonical metrics are in $\mathfrak{g}_X^+$.

PROPOSITION 10.13. *Let Z be a t-dimensional cycle on a smooth proper scheme X over K such that the local height $\lambda(Z)$ is well-defined with respect to $\hat{\mathfrak{g}}_X$-pseudo-divisors $\hat{D}_0, \ldots, \hat{D}_t$. Suppose that $O(D_0) \in \mathrm{Pic}^0(X)$ and that its metric $\| \ \|$ is canonical. For a cycle Y representing the refined intersection $D_1 \ldots D_t.Z$, the identity $\lambda(Z) = -\log \|s_{D_0}(Y)\|$ holds.*

PROOF. By functoriality and de Jong's alteration theorem ([dJ], Theorem 4.1), we may assume that $\hat{D}_1, \ldots, \hat{D}_t$ are differences of $\hat{\mathfrak{g}}_X^+$-pseudo-divisors. By multilinearity choosing the subtrahends generic, we reduce to $\hat{D}_1, \ldots, \hat{D}_t$ $\hat{\mathfrak{g}}_X^+$-pseudo-divisors. Theorem 10.6(d) shows that $\lambda(Z)$ is independent of the $\hat{\mathfrak{g}}_X^+$-metrics on $O(D_1), \ldots, O(D_t)$ using $\overline{O}(D_0) \in \mathfrak{g}_X^+$ (Remark 10.12). Passing again to an equidimensional covering, we may assume that $\hat{D}_1, \ldots, \hat{D}_t$ are $\mathfrak{g}_X^+$-pseudo-divisors and the claim follows from Propositions 3.7 and 9.4. □

10.14. Next, we are going to define canonical local heights in the dynamic situation of 10.8. Let $\psi : X \to X$ be a morphism of a proper scheme over K. For $j = 0, \ldots, t$, we fix $m_j, n_j \in \mathbb{Z}, |m_j| > |n_j|$. Let L_j be a line bundle on X with a metric in $\hat{\mathfrak{g}}_X^+$. Such a metric exists if L is generated by global sections, if L is ample or, for X smooth, if $L \in \mathrm{Pic}^0(X)$. Suppose that we have isomorphisms $\theta_j : \psi^* L_j^{\otimes n_j} \xrightarrow{\sim} L_j^{\otimes m_j}$.

Let $D_j = (L_j, |D_j|, s_j)$ be a pseudo-divisor. By Remark 10.10, the canonical metric $\| \ \|_{\theta_j}$ is uniform limit of metrics in $\hat{\mathfrak{g}}_X^+$ giving rise to a $\hat{\mathfrak{g}}_X^+$-pseudo-divisor $\hat{D}_j^{\theta_j}$. By Theorem 10.6, the local height of a t-dimensional cycle Z with respect to $\hat{D}_0^{\theta_0}, \ldots, \hat{D}_t^{\theta_t}$ is well-defined if $|D_0| \cap \cdots \cap |D_t| \cap |Z| = \emptyset$. It is called the *canonical local height* $\hat{\lambda}(Z)$ of Z with respect to $(D_0, \theta_0), \ldots, (D_t, \theta_t)$.

10.15. If X is irreducible and if we replace the isomorphisms θ_j by other isomorphisms θ_j' inducing the canonical local height $\hat{\lambda}'(Z)$, then there are $\alpha_j \in K^\times$ with $\theta_j' = \alpha_j \theta_j$ and Remark 10.10 proves

$$\hat{\lambda}(Z) - \hat{\lambda}'(Z) = \sum_{j=0}^{t} \frac{\log |\alpha_j|_v}{n_j - m_j} \deg_{L_0, \ldots, L_{j-1}, L_{j+1}, \ldots, L_t}(Z).$$

10.16. We assume that $(|D_0| \cup \psi^{-1}|D_0|) \cap \cdots \cap (|D_t| \cup \psi^{-1}|D_t|) \cap |Z| = \emptyset$. For a representative Y_j of the refined intersection $D_0 \ldots D_{j-1}.D_{j+1} \ldots D_t.Z$, we deduce from Theorem 10.6

$$m_0 \cdots m_t \hat{\lambda}(Z) - n_0 \cdots n_t \hat{\lambda}(\psi_* Z) = \sum_{j=0}^{t} \log \left| \frac{\theta_j \circ s_j^{\otimes n_j} \circ \psi}{s_j^{\otimes m_j}}(Y_j) \right|_v .$$

PROPOSITION 10.17. *Let A be an abelian variety over K with a line bundle $\overline{L}$ endowed with a canonical metric as in Example* 10.11. *Then $\overline{L} \in \hat{\mathfrak{g}}_X$.*

PROOF. We may assume that L is even or odd. For L odd, we know that $\overline{L} \in \mathfrak{g}_X^+$ (Remark 10.12). So we may assume that L is even. There is a very ample even line bundle H on A. By considering $L \otimes H^{\otimes n}$ for large powers n, we may assume that L is even and very ample. Thus L has a $\mathfrak{g}_X^+$-metric. By Remark 10.14, we get $\overline{L} \in \hat{\mathfrak{g}}_X^+$. □

10.18. Let $\hat{D}_0, \dots, \hat{D}_t$ be pseudo-divisors on A endowed with canonical metrics. For a t-dimensional cycle Z on A with $|D_0| \cap \cdots \cap |D_t| \cap |Z| = \emptyset$, Theorem 10.6 provides us with a *canonical local height* $\hat{\lambda}(Z)$ of Z. It is multilinear in the variables $\hat{D}_0, \dots, \hat{D}_t$, linear in Z and functorial with respect to homomorphisms of abelian varieties always under the assumption that all local heights are well-defined.

If we replace the canonical metrics $\| \ \|_j$ on L_j by other canonical metrics $\| \ \|_j'$ for $j = 0, \dots, t$, then there is $r_j \in \mathbb{R}$ with $\| \ \|_j' = r_j \| \ \|_j$ and Theorem 10.6(a) and (c) proves

$$\hat{\lambda}(Z) - \hat{\lambda}'(Z) = \sum_{j=0}^{t} \log r_j \deg_{L_0, \dots, L_{j-1}, L_{j+1}, \dots, L_t}(Z) .$$

11. – Global heights

In this section, K denotes an M-field with defect of product formula d. All spaces considered are assumed to be proper schemes over K and are denoted by $X, X', \dots$

DEFINITION 11.1. Let K be a field and let (M, μ) be a positive measure space. For every $\alpha \in K$, let $M \to \mathbb{R}_+, v \mapsto |\alpha|_v$, be a μ-almost everywhere defined map with

(a) $|\alpha + \beta|_v \leq |\alpha|_v + |\beta|_v \quad \mu$-ae
(b) $|\alpha\beta|_v = |\alpha|_v |\beta|_v \quad \mu$-ae
(c) $\log |\gamma|_v \in L^1(M, \mu)$ and $|0|_v = 0 \quad \mu$-ae

for all $\alpha, \beta \in K$ and $\gamma \in K^\times$. Then K is called an *M-field*. For $\alpha \in K^\times$, we call $d_\alpha := \int_M \log |\alpha|_v \, d\mu(v)$ the *defect of product formula*. If $d_\alpha = 0$ for all $\alpha \in K^\times$, then K is said to satisfy the product formula.

EXAMPLE 11.2. Every number field K is an M_K-field with M_K the set of places of K endowed with the discrete $\mu(v) = N_v/[K : \mathbb{Q}]$, where N_v is the local degree of $v \in M_K$. Similarly, every function field K is an M_K-field for M_K the set of prime divisors.

In Nevanlinna theory, one considers the M_R-field of meromorphic functions on $\mathbb{C}$, where $M_R := \{v \in \mathbb{C} \mid |v| \leq R\}$ with the counting measure in the interior and the Lebesque probability measure on the boundary. Only in the interior, we have honest absolute values induced by the order. On the boundary, evaluation induces only almost absolute values. The defect of product formula may be easily computed by Jensen's formula. In all these examples, the algebraic closure has a canonical M-field structure. For details, we refer to [Gu2] or 11.23.

11.3. We recall from [Gu2], Section 2, the concept of boundedness and metrics for M-fields. For simplicity, we assume that almost all $|\ |_v$ are absolute values.

If M is a proper set of absolute values on K, then M-bounded sets and functions are well-known in diophantine geometry (cf. [La] or [Ne]). For generalization to M-fields, one has to replace all M-constants in the bounds by integrable functions on M.

Let $\mathbb{K}_v$ be the completion of the algebraic closure of the completion of K with respect to $|\ |_v$. It is a complete algebraically closed field with respect to the unique extension of $|\ |_v$ to an absolute value ([BGR], Proposition 3.4.1/3).

An *M-metric* $\|\ \|$ on a line bundle L is a family of metrics $\|\ \|_v$ on $L(\mathbb{K}_v)$ over $\mathbb{K}_v$ for almost every $v \in M$. We also assume that the metric $\|\ \|$ is locally M-bounded, i.e. for every open subset U of X and every $s \in L(U)$, the function $\|s\|$ is locally M-bounded on U.

We denote by $\hat{\mathfrak{g}}_X$ the set of isometry classes of line bundles L on X endowed with an M-metric $\|\ \|$ such that $\|\ \|_v$ is in $\hat{\mathfrak{g}}_{X\otimes\mathbb{K}_v}$ for almost all $v \in M$ (using Definition 10.2). Similaly, we proceed for $\mathfrak{g}_X$, $\mathfrak{g}_X^+$ and $\hat{\mathfrak{g}}_X^+$.

Let $\hat{D}_0, \dots, \hat{D}_t$ be $\hat{\mathfrak{g}}_X$-pseudo-divisors. Let Z be a t-dimensional cycle on X. Then the *local height* is said to be *well-defined in* $v \in M$ if $|D_0| \cap \cdots \cap |D_t| \cap |Z| = \emptyset$ and we set

$$\lambda(Z, v) := \lambda_{\widehat{D_0}^v, \dots, \widehat{D_t}^v}(Z)\,.$$

DEFINITION 11.4. A t-dimensional prime cycle Z on X is called *integrable* with respect to $\overline{L}_0, \dots, \overline{L}_t \in \hat{\mathfrak{g}}_X$ if there are invertible meromorphic sections $s_{0,Z}, \dots, s_{t,Z}$ of $L_0|_Z, \dots, L_t|_Z$ such that all partial intersections formed out of $\mathrm{div}(s_{0,Z}), \dots, \mathrm{div}(s_{t,Z})$ are proper in Z and such that the local height with respect to $(\overline{L}_0|_Z, s_{0,Z}), \dots, (\overline{L}_t|_Z, s_{t,Z})$ is well-defined for almost all $v \in M$ and integrable on M. By linearity, we extend this definition to all cycles Z.

For a morphism $\varphi : X' \to X$ and a prime cycle Z' on X', $\varphi_* Z'$ is integrable with respect to $\overline{L}_0, \dots, \overline{L}_t$ if and only if Z' is integrable with respect to $\varphi^*\overline{L}_0, \dots, \varphi^*\overline{L}_t$ (Theorem 10.6(b)).

PROPOSITION 11.5. *Let $\hat{D}_0, \dots, \hat{D}_t$ be $\hat{\mathfrak{g}}_X$-pseudo-divisors and let Z be a t-dimensional cycle integrable with respect to $\overline{O}(D_0), \dots, \overline{O}(D_t)$. If $|D_0| \cap \cdots \cap |D_t| \cap |Z| = \emptyset$, then the local height of Z with respect to $\hat{D}_0, \dots, \hat{D}_t$ is well-defined for almost all $v \in M$ and integrable on M.*

PROOF. In the proper intersection case, this was proved in [Gu2], Corollary 3.8. To reduce the general case to the proper intersection case, we may assume $X = Z$. By de Jong's alteration theorem ([dJ], Theorem 4.1), we reduce to the case of a regular projective variety. By multilinearity and de Jong's alteration theorem again now applied to the divisors, we may assume that $D_0, \dots, D_t$ are all prime divisors and that a list of different representatives intersects properly. Using Theorem 10.6(a),(c) and [Gu2], Lemma 3.6, we may replace repeated divisors by equivalent ones such that $D_0, \dots, D_t$ intersect properly in X. □

DEFINITION 11.6. Let $\hat{D}_0, \dots, \hat{D}_t$ be $\hat{\mathfrak{g}}_X$-pseudo-divisors. The *global height* of a t-dimensional cycle Z on X with respect to $\hat{D}_0, \dots, \hat{D}_t$ is *well-defined* if Z is integrable with respect to $\overline{O}(D_0), \dots, \overline{O}(D_t)$ and if $|D_0| \cap \dots \cap |D_t| \cap |Z| = \emptyset$. Then it is defined by

$$h(Z) := h_{\hat{D}_0,\dots,\hat{D}_t}(Z) := \int_M \lambda_{\hat{D}_0,\dots,\hat{D}_t}(Z, v)\, d\mu(v)\,.$$

11.7. By integration, properties (a)-(d) of Theorem 10.6 hold for global heights. In (c), the defect of product formula gives $h(Z) = d_{f(Y)}$ and in (d), one has to use integrable bounds $c_\pm$ on M leading to the estimates of $h(Z) - h'(Z)$ by $\int c_\pm d\mu$ times the degree. This is possible because X and hence the metrics are M-bounded (cf. [Gu2], Corollary 2.20).

EXAMPLE 11.8. Let Z be a t-dimensional cycle on the multiprojective space $\mathbb{P} = \mathbb{P}^{n_0} \times \dots \times \mathbb{P}^{n_t}$ and let $s_0, \dots, s_t$ be global sections of $O_{\mathbb{P}}(e_0), \dots, O_{\mathbb{P}}(e_t)$ such that $|\operatorname{div}(s_0)| \cap \dots \cap |\operatorname{div}(s_t)| \cap |Z| = \emptyset$. Let $\bar{O}_{\mathbb{P}}(e_0), \dots, \bar{O}_{\mathbb{P}}(e_t)$ be endowed with the standard (resp. Fubini-Study) metrics in the non-archimedean (resp. archimedean) case. Clearly, they are in $\mathfrak{g}_X^+$. Then the corresponding global height $h(Z)$ is well-defined and given in terms of the Chow form F_Z ([Gu2], Lemma 3.4).

REMARK 11.9. Any line bundle on X generated by global sections has a $\mathfrak{g}_X^+$-metric obtained by pull-back of $\bar{O}_{\mathbb{P}^n}(1)$ endowed with standard (resp. Fubini-Study) metrics. More generally, let L be a line bundle on X such that there is $k \in \mathbb{N}, k \geq 1$, with $L^{\otimes k}$ generated by global sections. This includes also all ample line bundles. Then we endow L with the k-th root of a pull-back metric considered above. We denote by $\mathfrak{b}_X^+$ the set of such isometry classes. Using the Segre embedding, it is obvious that $\mathfrak{b}_X^+$ is a submonoid of $\hat{\mathfrak{g}}_X^+$. By Example 11.8 and functoriality, every t-dimensional cycle Z on X is integrable with respect to $\overline{L}_0, \dots, \overline{L}_t \in \mathfrak{b}_X^+$. If $Z = \sum_Y n_Y Y$ is the decomposition into components, then let

$$\delta_{L_0,\dots,L_t}(Z) := \sum_{j=0}^{t} \sum_Y |n_Y| \deg_{L_0,\dots,L_{j-1},L_{j+1},\dots,L_t}(Y)\,.$$

For pseudo-divisors $D_0, \dots, D_t$ such that a positive tensor power of every $O(D_j)$ is generated by global sections, the global height $h(Z)$ with respect to $\hat{D}_0, \dots, \hat{D}_t$ doesn't depend on the $\mathfrak{b}_X^+$-metrics on $O(D_0), \dots, O(D_t)$ up to $O(\delta_{L_0,\dots,L_t})(Z)$. It is well-defined if and only if $|D_0| \cap \dots \cap |D_t| \cap |Z| = \emptyset$. This follows immediately from Theorem 10.6 and generalizes Weil's theorem and the first main theorem of Nevanlinna theory (cf. [Gu2]). For effective Cartier divisors, Proposition 10.7 shows that we may always find a representative h

for the global height with respect to $D_0, \dots, D_t$ such that $h(Z) \geq 0$ for all effective cycles Z on X.

If the product formula is satisfied for the M-field K, then the dependence of the global heights $h(Z)$ on the pseudo-divisors $D_0, \dots, D_t$ is determined by the isomorphism classes $O(D_0), \dots, O(D_t)$. For $L_0, \dots, L_t \in \mathrm{Pic}(X)$ with a positive tensor power generated by global sections, we get a global height $h_{L_0,\dots,L_t}(Z)$, well-defined for all t-dimensional cycles on X and canonical up to $O(\delta_{L_0,\dots,L_t})(Z)$. It is multilinear and symmetric in $L_0, \dots, L_t$ and functorial.

THEOREM 11.10. *If L has an M-metric* (*resp.* $\hat{\mathfrak{g}}_X^+$*-metric*) *in the dynamic situation of* 10.8*, then there is a unique M-metric* (*resp.* $\hat{\mathfrak{g}}_X^+$*-metric*) *satisfying* $\| \ \|_\theta^{\otimes m} \circ \theta = \psi^* \| \ \|_\theta^{\otimes n}$.

PROOF. For almost all $v \in M$, we may assume that $| \ |_v$ is an absolute value (by passing to sufficiently large finitely generated subfields of K, cf. [Gu2], Remark 2.10). By Theorem 10.9, we get a canonical metric $\| \ \|_{\theta,v}$ on L satisfying the required identity. Remark 10.10 shows $\| \ \|_{\theta,v} = \lim_{k\to\infty} \| \ \|_{k,v}$ where $\| \ \|_k := \Phi^k(\| \ \|)$ is an M-metric (resp. $\hat{\mathfrak{g}}_X^+$-metric). By (20) and because $d_v(\| \ \|_{0,v}, \| \ \|_{1,v})$ is bounded by an L^1-function, $\| \ \|_\theta$ is an M- (resp. $\hat{\mathfrak{g}}_X^+$-) metric. □

11.11. Let $\psi : X \to X$ be a morphism and let $m_j, n_j \in \mathbb{N}, m_j > n_j$. For $j = 0, \dots, t$, let L_j be a line bundle on X with a positive tensor power generated by global sections and with isomorphisms $\theta_j : \psi^* L_j^{\otimes n_j} \xrightarrow{\sim} L_j^{\otimes m_j}$. Theorem 11.10 induces a canonical $\overline{L}_j^{\theta_j} \in \hat{\mathfrak{g}}_X^+$.

PROPOSITION 11.12. *Every t-dimensional cycle is integrable with respect to* $\overline{L}_0^{\theta_0}, \dots, \overline{L}_t^{\theta_t}$.

PROOF. We use the above construction of canonical metrics starting with a $\mathfrak{b}_X^+$-metric on every L_j. Because every cycle is integrable with respect to $\mathfrak{b}_X^+$-metrics, the claim follows from the dominated convergence theorem. □

DEFINITION 11.13. Under the assumptions of 11.11, let $\hat{D}_j^{\theta_j} = (\overline{L}_j^{\theta_j}, |D_j|, s_j)$ be a $\hat{\mathfrak{g}}_X^+$-pseudo-divisor. The global height $\hat{h}$ of a t-dimensional cycle Z with respect to $\hat{D}_0^{\theta_0}, \dots, \hat{D}_t^{\theta_t}$ is called the *canonical height* of Z with respect to $(D_0, \theta_0), \dots, (D_t, \theta_t)$. By Proposition 11.12, it is well-defined if and only if $|D_0| \cap \dots \cap |D_t| \cap |Z| = \emptyset$.

THEOREM 11.14. *We assume that the product formula is satisfied. Under the assumptions of* 11.11*, there is a unique real function $\hat{h}$ on the t-dimensional cycles Z satisfying*:

(a) $m_0 \cdots m_t \hat{h}(Z) = n_0 \cdots n_t \hat{h}(\psi_* Z)$.
(b) *For all integrable cycles Z with respect to $\overline{L}_0, \dots, \overline{L}_t \in \hat{\mathfrak{g}}_X^+$, we have*

$$|\hat{h}(Z) - h_{\overline{L}_0,\dots,\overline{L}_t}(Z)| = O(\delta_{L_0,\dots,L_t})(Z)\,.$$

Moreoever, $\hat{h}$ is the canonical height and it is non-negative on effective cycles.

PROOF. Let $\hat{h}$ be the canonical height of 11.13. By integrating 10.15, 10.16, it depends only on $L_0, \ldots, L_t$, thus well-defined for all cycles and satisfies (a). By Theorem 10.6(d), we get (b).

To see $\hat{h}(Z) \geq 0$ for an effective cycle Z, we may assume that K is infinite, otherwise all global heights are zero. We may assume $L_0, \ldots, L_t$ generated by global sections. Then there are global sections s_j of L_j with $|\operatorname{div}(s_0)| \cap \cdots \cap |\operatorname{div}(s_t)| \cap |Z| = \emptyset$. We can construct a $\mathfrak{b}_X^+$-metric $\| \ \|$ on L_j such that $\sup_{x \in X} \|s_j(x)\|_v \leq 1$ for almost all $v \in M$. Then this holds for all $\mathfrak{b}_X^+$-metrics $\| \ \|_k$ from the proof of Proposition 11.12. By Proposition 10.7 and the dominated convergence theorem, we deduce $\hat{h}(Z) \geq 0$.

To prove uniqueness, let $\bar{h}$ be another real function on t-dimensional cycles satisfying (a) and (b). For any t-dimensional cycle Z on X and any $k \in \mathbb{N}$, we get

$$\begin{aligned} |\hat{h}(Z) - \bar{h}(Z)| &= \left(\frac{n_0 \cdots n_t}{m_0 \cdots m_t}\right)^k |\hat{h}(\psi_*^k Z) - \bar{h}(\psi_*^k Z)| \\ &\leq C \left(\frac{n_0 \cdots n_t}{m_0 \cdots m_t}\right)^k \delta_{L_0, \ldots, L_t}(\psi_*^k Z) \\ &= C \sum_{j=0}^{t} \left(\frac{n_j}{m_j}\right)^k \deg_{L_0, \ldots, L_{j-1}, L_{j+1}, \ldots, L_t}(Z) \end{aligned}$$

where the constant C is independent of Z and k. By $k \to \infty$, we get $\hat{h}(Z) = \bar{h}(Z)$. □

EXAMPLE 11.15. Let A be an abelian variety over K. As on any projective variety, every line bundle L has an M-metric because it may be written as the difference of two very ample ones. By the decomposition into even and odd parts and using Theorem 11.10, we get *canonical M-metrics* $\| \ \|$ on L as in Example 10.11. They are unique up to multiplication with the function $v \mapsto |\alpha|_v$ for some $\alpha \in K^\times$. For almost every $v \in M$, the metric $\| \ \|_v$ is the canonical metric of Example 10.11 and hence canonical metrics are in $\hat{\mathfrak{g}}_X$. Canonical M-metrics are closed under tensor product and pull-back with respect to homomorphisms of abelian varieties.

PROPOSITION 11.16. *Let $\overline{L}_0, \ldots, \overline{L}_t$ be line bundles on A endowed with canonical metrics. Then every t-dimensional cycle on A is integrable with respect to $\overline{L}_0, \ldots, \overline{L}_t$.*

PROOF. By multilinearity, we may assume that every L_j is either even and very ample (cf. proof of Proposition 10.17) or odd (i.e. in $\operatorname{Pic}^0(A)$). First, we handle the case where one line bundle is odd, say L_0. Then the local heights do not depend on the choice of the metrics on $L_1, \ldots, L_t$ (Proposition 10.13). Using multilinearity, we may assume that $\overline{L}_1, \ldots, \overline{L}_t \in \mathfrak{b}_A^+$. Clearly, L_0 has a metric $\| \ \|$ lying in $\mathfrak{b}_A^+ - \mathfrak{b}_A^+$. Then the same holds for the sequence $\| \ \|_k := \Phi^k(\| \ \|)$ of metrics on L_0 considered in the proof of Theorem 11.10. Using (20), the convergence of $\| \ \|_k$ to the canonical metric of L_0 is dominated by an integrable function on M. By Remark 11.9, every t-dimensional cycle on A is integrable with respect to $(L_0, \| \ \|_k), \overline{L}_1, \ldots, \overline{L}_t$. Using Theorem 10.6(d) and the dominated convergence theorem, we conclude that the same holds for $\overline{L}_0, \ldots, \overline{L}_t$. By symmetry, this proves the case of at least one odd line bundle.

So we may assume that all line bundles are even. By multilinearity using the argument in Proposition 10.7, we may assume that all line bundles are even and generated by global sections. Then the claim follows by Proposition 11.12. □

DEFINITION 11.17. Let $\hat{D}_0, \ldots, \hat{D}_t$ be pseudo-divisors on an abelian variety A over K endowed with canonical metrics. For a t-dimensional cycle Z on A, the global height with respect to $\hat{D}_0, \ldots, \hat{D}_t$ is called the *Néron-Tate height* of Z. By Proposition 11.16, it is well-defined if and only if $|D_0| \cap \cdots \cap |D_t| \cap |Z| = \emptyset$.

THEOREM 11.18. *Suppose that K satisfies the product formula. For $L_0, \ldots, L_t \in$ Pic(A), there is a real function $\hat{h}_{L_0,\ldots,L_t}$ on all t-dimensional cycles Z of A with*

(a) *$\hat{h}_{L_0,\ldots,L_t}$ is multilinear and symmetric in the variables $L_0, \ldots, L_t$, and linear in Z.*
(b) *If φ is a homomorphism of abelian varieties and Z' is a cycle on A', then*

$$\hat{h}_{\varphi^* L_0,\ldots,\varphi^* L_t}(Z') = \hat{h}_{L_0,\ldots,L_t}(\varphi_* Z') .$$

(c) *If $m \in \mathbb{Z}$, $|m| \geq 2$, if k line bundles of $L_0, \ldots, L_t$ are even and the others are odd, then*

$$m^{k+t+1}\hat{h}(Z) = \hat{h}([m]_* Z) .$$

(d) *For all integrable cycles Z with respect to any $\overline{L}_0, \ldots, \overline{L}_t \in \hat{\mathfrak{g}}_A^+$, we have*

$$|\hat{h}_{L_0,\ldots,L_t}(Z) - h_{\overline{L}_0,\ldots,\overline{L}_t}(Z)| = O(\delta_{L_0,\ldots,L_t})(Z) .$$

(e) *If $L_0, \ldots, L_t$ are even with a positive tensor power generated by global sections and if Z is effective, then $\hat{h}_{L_0,\ldots,L_t}(Z) \geq 0$.*

The function $\hat{h}_{L_0,\ldots,L_t}$ is the Néron-Tate height, uniquely determined by (a), (c) *and* (d).

PROOF. By Theorems 10.6 and 11.14, the Néron-Tate height satisfies (a)-(e). For uniqueness, (a) implies that we may assume L_j even or odd and Theorem 11.14 proves the claim. □

11.19. Let X be a proper smooth scheme over K and let $L \in$ Pic$^0(X)$. We assume that X has a K-rational point which usually may be achieved by base change. Then there is a morphism φ of X to an abelian variety A (e.g. the Albanese variety) over K with $L' \in$ Pic$^0(A)$ such that $L \cong \varphi^* L'$. An M-metric $\| \ \|$ on L is called *canonical* if it is the pull-back of a canonical M-metric on such an L'. For almost all $v \in M$, we get a canonical metric $\| \ \|_v$ in the sense of Remark 10.12. Hence canonical metrics are in $\mathfrak{g}_X^+$. Clearly, canonical metrics are closed under tensor product and pull-back.

If X is irreducible, then we claim that a canonical metric on L is unique up to multiplication by the function $v \mapsto |\alpha|_v$ on M for some $\alpha \in K^\times$. To see it, we use that up to a translation φ factors through the Albanese variety and the pull-back of L' is the fibre of the Poincaré class over L. Then uniqueness follows from Example 11.15.

PROPOSITION 11.20. *Let X be as in 11.19 with $\hat{\mathfrak{g}}_X$-pseudo-divisors $\hat{D}_0, \ldots, \hat{D}_t$. Suppose that $O(D_0) \in \mathrm{Pic}^0(X)$ and that its M-metric is canonical. Then every t-dimensional cycle Z on X is integrable with respect to $\overline{O}(D_0), \ldots, \overline{O}(D_t)$. If $|D_0| \cap \cdots \cap |D_t| \cap |Z| = \emptyset$, then $h_{\hat{D}_0,\ldots,\hat{D}_t}(Z) = h_{\hat{D}_0}(Y)$ for any representative Y of the refined intersection $D_1 \ldots D_t.Z$.*

PROOF. By Proposition 10.13, we have $\lambda_{\hat{D}_0,\ldots,\hat{D}_t}(Z, v) = -\log \|s_{D_0}(Y)\|_v$ for almost all $v \in M$. Using Proposition 11.12 and functoriality, we get the claim. □

REMARK 11.21. Let K be a number field with ring of integers O_K and let $\mathfrak{X}$ be a proper flat scheme over O_K with generic fibre X. Then hermitian line bundles $\overline{\mathcal{L}}_0, \ldots, \overline{\mathcal{L}}_t$ on $\mathfrak{X}$ induce M_K-metrics on the generic fibres. The absolute height of a cycle on X with respect to $\overline{\mathcal{L}}_0, \ldots, \overline{\mathcal{L}}_t$ from arithmetic intersection theory (cf. [BoGS]) is the same as the global height with respect to the corresponding M_K-metrized line bundles using the normalizations of Example 11.2. This follows from Section 3 and [Gu3], Proposition 6.5.

A hermitian line bundle $\overline{\mathcal{L}}$ on $\mathfrak{X}$ is called *nef* if the hermitian metric has semipositive curvature on $X(\mathbb{C})$, if the restrictions of $\mathcal{L}$ to vertical fibres are numerically positive and if $h_{\overline{\mathcal{L}}}(C) \geq 0$ for all 1-dimensional effective cycles on X. Similarly as in 9.6, the height of an effective cycle with respect to nef hermitian line bundles is non-negative. By de Jong's alteration theorem ([dJ], Theorem 8.2), this follows from [Mor], Proposition 2.3.

EXAMPLE 11.22. Let T be an integral normal variety proper and flat over $\mathbb{Z}$ of relative dimension t. We are going to explain Moriwaki's M-field structure ([Mor]) on the function field $K := \mathbb{Q}(T)$ where M_{fin} is the set of prime divisors on T and $M_\infty := T(\mathbb{C})$.

We fix hermitian line bundles $\overline{\mathcal{H}}_1, \ldots, \overline{\mathcal{H}}_t$ assumed to be nef. For $v \in M_{\mathrm{fin}}$, the height $h(v)$ with respect to $\overline{\mathcal{H}}_1, \ldots, \overline{\mathcal{H}}_t$ is non-negative (Remark 11.21) giving rise to the absolute value $|f|_v := e^{-\mathrm{ord}(f,v)h(v)}$ on K. For $v \in M_\infty$, let $|f|_v := |f(v)|$. The positive measure μ is the counting measure on M_{fin} and $c_1(\overline{\mathcal{H}}_1) \wedge \cdots \wedge c_1(\overline{\mathcal{H}}_t)$ on M_∞. Then K is an $M := M_{\mathrm{fin}} \cup M_\infty$-field satisfying the product formula. To see the latter, note that the height of $T_{\mathbb{Q}}$ with respect to $\overline{\mathcal{H}}_1, \ldots, \overline{\mathcal{H}}_t, \overline{\mathcal{O}}_T$ is 0 and then apply the induction formula (3.5 and 9.5) for $Y = \mathrm{div}(f_{\mathbb{Q}})$.

EXAMPLE 11.23. For the M-fields considered in Example 11.2, it is always possible to pass to the algebraic closure. With a similar procedure, we define a canonical M-field structure on the algebraic closure F of $K := \mathbb{Q}(T)$. For a finite subextension L/K, let $T^{(L)}$ be the integral closure of T in L. It is an integral normal variety with a canonical finite morphism $p_L : T^{(L)} \to T$ and function field L ([EGA II], 6.3).

Let $M_{\mathrm{fin}}^{(L)}$ be the set of prime divisors on $T^{(L)}$ and let $M_{\infty}^{(L)} := T^{(L)}(\mathbb{C})$. The height with respect to $p_L^*\overline{\mathcal{H}}_1, \dots, p_L^*\overline{\mathcal{H}}_t$ is denoted by $h^{(L)}$. For $w \in M_{\mathrm{fin}}^{(L)}$ with $v := p_L(w)$, let $N_w := [\hat{L}^w : \hat{K}^v]$ using completions with respect to the discrete valuations $\mathrm{ord}(\cdot, w)$ and $\mathrm{ord}(\cdot, v)$. Because of normality, the classical formula $\sum_w N_w = [L : K]$ holds where w ranges over $p_L^{-1}(v)$. We conclude that the absolute values on L extending $|\ |_v$ are given by $|f|_w := \exp(-\mathrm{ord}(f, w)h^{(L)}(w)/N_w)$, $w \in p_L^{-1}(v)$. If $h(v) \neq 0$, then they are pairwise inequivalent discrete absolute values. For $w \in M_{\infty}^{(L)}$, let $|f|_w := |f(w)|$. The positive measure $\mu^{(L)}$ on $M^{(L)}$ is defined by $\mu^{(L)}(w) = N_w/[L : K]$ on $M_{\mathrm{fin}}^{(L)}$ and by $c_1(p_L^*\overline{\mathcal{H}}_1) \wedge \cdots \wedge c_1(p_L^*\overline{\mathcal{H}}_t)/[L : K]$ on $M_{\infty}^{(L)}$. By Example 11.22, we conclude that L is an $M^{(L)}$-field satisfying the product formula.

Using the complex structure on $M_{\infty}^{(L)}$ and the discrete topology on $M_{\mathrm{fin}}^{(L)}$, we get a locally compact space $M_F := \varprojlim_L M^{(L)}$ given as a closed subset of $\prod_L M^{(L)}$ where L ranges over all finite subextensons of F/K. Let $\bar{p}_L : M_F \to M^{(L)}$ be the projection. For a finite subextension L' of F/K with $L \subset L'$, it follows from projection formula and transformation formula of integrals that $\mu^{(L)}$ is the image measure of $\mu^{(L')}$ with respect to the canonical map $M^{(L')} \to M^{(L)}$. Hence there is a unique regular Borel measure μ_F on M_F such that $\mu^{(L)} = \bar{p}_L(\mu_F)$. For $u \in M_F$ and $f \in F$, let $|f|_u := |f|_w$ where $w := \bar{p}_L(u)$ for a finite subextension L of F/K with $f \in L$. If $v \in M_{\mathrm{fin}}^{(K)}$ and $h(v) \neq 0$, then $\bar{p}_K^{-1}(v)$ gives rise to all absolute values on F extending $|\ |_v$.

By passing to finite subextensions, we conclude that F is an M_F-field satisfying the product formula. The same construction works for every algebraic extension L/K. If it is finite, then it is equal to the $M^{(L)}$-field structure above. If X is a proper scheme over L, then heights of cycles on X are independent of the choice of L and may be computed over $\bar{L}$.

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