Ann. Scuola Norm. Sup. Pisa Cl. Sci. (5)
Vol. IV (2005), 59-78

Summability of semicontinuous supersolutions to a quasilinear parabolic equation

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Abstract. We study the so-called p-superparabolic functions, which are defined as lower semicontinuous supersolutions of a quasilinear parabolic equation. In the linear case, when $p = 2$, we have supercaloric functions and the heat equation. We show that the p-superparabolic functions have a spatial Sobolev gradient and a sharp summability exponent is given.

Mathematics Subject Classification (2000): 35K55.

1. Introduction

The objective of our work is a class of unbounded "supersolutions" of the partial differential equation

$$\frac{\partial u}{\partial t} = \operatorname{div}(|\nabla u|^{p-2}\nabla u), \qquad 1 < p < \infty. \tag{1.1}$$

The functions that we have in mind are pointwise defined as lower semicontinuous functions obeying the comparison principle with respect to the solutions of (1.1). They are called *p-superparabolic functions*. In the linear case $p = 2$ we have the ordinary heat equation and supercaloric functions. In the stationary case supercaloric functions are nothing else but superharmonic functions, well-known in the classical potential theory. The p-superparabolic functions play an important role in the Perron method in a nonlinear potential theory, described in [7]. We seize the opportunity to mention that the p-superparabolic functions are precisely the *viscosity supersolutions* of (1.1), which fact will not be considered in the present work, see [5].

It is important to observe that in their definition (to be given below) the p-superparabolic functions are not required to have any derivatives. The only tie

This research was partially done at Mittag-Leffler Institute during a special year in PDE's in 1999-2000.
Pervenuto alla Redazione il 17 maggio 2004 e in forma definitiva il 19 novembre 2004.

to the differential equation is through the comparison principle. The celebrated Barenblatt solution (see Section 2 below) is the most important explicit example of a p-superparabolic function.

This work is a continuation of our previous paper [6]. There we proved that every locally bounded p-superparabolic function has a spatial gradient in Sobolev's sense and that it is a weak supersolution of the equation in the usual sense with test functions under the integral sign. Our present task is to study unbounded p-superparabolic functions and for them we have the following sharp result.

Teorema 1.1. *Let $p \geq 2$. Suppose that $v = v(x,t)$ is a p-superparabolic function in an open set Ω in $\mathbf{R}^{n+1}$. Then $v \in L^q_{\mathrm{loc}}(\Omega)$ for every q with $0 < q < p-1+p/n$. Moreover, the Sobolev derivative*

$$\nabla v = \left(\frac{\partial v}{\partial x_1}, \dots, \frac{\partial v}{\partial x_n}\right)$$

exists and the local summability

$$\iint_{\Xi} |\nabla v|^q \, dx \, dt < \infty$$

holds for all $\Xi \subset\subset \Omega$, whenever $0 < q < p - 1 + 1/(n+1)$.

This is our main result and the proof is presented in Section 3. A logarithmic estimate in Section 4 complements the theorem. The Barenblatt solution shows that these critical summability exponents for a p-superparabolic function and its gradient are optimal. A direct calculation reveals that the Barenblatt solution does not attain these exponents. There is a difference compared to the stationary case. The corresponding critical exponents related to the elliptic equation

$$\operatorname{div}(|\nabla u|^{p-2} \nabla u) = 0$$

are larger, see [10]. The "fundamental solution" for the stationary case is

$$v(x) = \begin{cases} C|x|^{(p-n)/(p-1)}, & 1 < p < n, \\ -C \log |x|, & p = n, \end{cases}$$

in $\mathbb{R}^n$, but the Barenblatt solution is far more intricate.

In the case $p = 2$ the proof of Theorem 1.1 can be extracted from the linear representation formulas in [15] and [16]. Then all superparabolic functions can be represented in terms of the heat kernel. For $p > 2$ the principle of superposition is not available. Instead we approximate the given p-superparabolic function v by $v_j = \min(v, j)$, $j = 1, 2, \dots$ A priori estimates for the approximants are derived through variational inequalities and these estimates are passed over to the limit.

The proof consists of three steps depending on the exponents. First, an iteration based on a test function used by Kilpeläinen and Malý in the elliptic case, see [8], implies that v is locally summable to any exponent q with $0 < q < p-2$. Second, the passage over $p-2$ requires an iteration taking the influence of the time variable into account. This procedure reaches all exponents q with $0 < q < p-1$. Third, a more sophisticated arrangement of the estimates is needed to bound the quantity involving integrals over time slices. Finally, Sobolev's inequality yields the correct critical exponents for the function and its Sobolev derivative. In other words, there are two different iteration methods involved in the proof.

In the elliptic case the proof can also be based on Moser's iteration technique, see [10]. For parabolic equations Moser's method has been studied in [12], [13] and [14]. See also [9] and [11]. Moser's technique applies also in the parabolic case, if we already know that v is locally integrable to a power $q > p-2$. However, we have not been able to settle the passage over $p-2$ in the parabolic case by using merely Moser's approach and hence we present an alternative proof of the passage.

Our argument is based on a general principle and it applies to other equations as well. It can be extended to include equations like

$$\frac{\partial u}{\partial t} = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(\left|\sum_{k,m=1}^{n} a_{km}(x)\frac{\partial u}{\partial x_k}\frac{\partial u}{\partial x_m}\right|^{(p-2)/2} a_{ij}(x)\frac{\partial u}{\partial x_j}\right),$$

where the matrix (a_{ij}) with bounded measurable coefficients satisfies the standard condition

$$\sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \geq |\xi|^2$$

for all $\xi = (\xi_1, \xi_2, \dots, \xi_n)$ in $\mathbb{R}^n$. The exponents q are the same as for the p-superparabolic functions. For the equation

$$\frac{\partial(|u|^{m-1}u)}{\partial t} = \operatorname{div}(|\nabla u|^{p-2}\nabla u), \qquad 1 \leq m < \infty,$$

our method produces the critical local summability exponents $p-1+mp/n$ for u and $p-1+m/(m+n)$ for ∇u and they are sharp. We have tried to keep our exposition as short as possible, omitting such obvious generalizations. We have also deliberately decided to exclude the case $p < 2$. On the other hand, we think that some features might be interesting even for the ordinary heat equation, to which everything reduces when $p = 2$.

We refer to the books [3] and [17] for background infromation on the p-parabolic equation, which is also known as the evolutionary p-Laplacian and as the non-Newtonian filtration equation. See also the current account [4].

2. Preliminaries

We begin with some notation. In what follows, Q will always stand for a parallelepiped

$$Q = (a_1, b_1) \times (a_2, b_2) \times \cdots \times (a_n, b_n), \quad a_i < b_i, \quad i = 1, 2, \ldots, n,$$

in $\mathbb{R}^n$ and the abbreviations

$$Q_T = Q \times (0, T), \qquad Q_{t_1,t_2} = Q \times (t_1, t_2),$$

where $T > 0$ and $t_1 < t_2$, are used for the space-time boxes in $\mathbb{R}^{n+1}$. The parabolic boundary of Q_T is

$$\Gamma_T = (\overline{Q} \times \{0\}) \cup (\partial Q \times [0, T]).$$

Observe that the interior of the top $\overline{Q} \times \{T\}$ is not included. Similarly, Γ_{t_1,t_2} is the parabolic boundary of Q_{t_1,t_2}. The parabolic boundary of a space-time cylinder $D_{t_1,t_2} = D \times (t_1, t_2)$, where $D \subset \mathbb{R}^n$, has a similar definition. The n-dimensional volume of the parallelepiped Q is denoted by $|Q|$ and the $(n+1)$-dimensional volume of the space-time box Q_T is denoted by $|Q_T|$.

Let $1 < p < \infty$. In order to describe the appropriate function spaces, we recall that $W^{1,p}(Q)$ denotes the Sobolev space of functions $u \in L^p(Q)$, whose first distributional partial derivatives belong to $L^p(Q)$ with the norm

$$\|u\|_{W^{1,p}(Q)} = \|u\|_{L^p(Q)} + \|\nabla u\|_{L^p(Q)}.$$

The Sobolev space with zero boundary values, denoted by $W_0^{1,p}(Q)$, is the completion of $C_0^\infty(Q)$ in the norm $\|u\|_{W^{1,p}(Q)}$. We denote by $L^p(t_1, t_2; W^{1,p}(Q))$ the space of functions such that for almost every t, $t_1 \le t \le t_2$, the function $x \to u(x,t)$ belongs to $W^{1,p}(Q)$ and

$$\int_{t_1}^{t_2} \int_Q \left(|u(x,t)|^p + |\nabla u(x,t)|^p\right) dx\, dt < \infty.$$

Notice that the time derivative u_t is deliberately avoided. The definition of the space $L^p(t_1, t_2; W_0^{1,p}(Q))$ is analogous.

The Sobolev inequality is valid in the following form, see for instance Proposition 3.1 on page 7 of [3].

Lemma 2.1. *Suppose that $u \in L^p(0, T; W_0^{1,p}(Q))$. Then there is $c = c(n, p)$ such that*

$$\int_0^T \int_Q |u|^{p(1+2/n)}\, dx\, dt \le c \int_0^T \int_Q |\nabla u|^p\, dx\, dt \left(\operatorname*{ess\,sup}_{0<t<T} \int_Q |u|^2\, dx \right)^{p/n}. \quad (2.1)$$

To be on the safe side we give the definition of the (super)solutions, interpreted in the weak sense. The reader should carefully distinguish between the supersolutions and the p-superparabolic functions, which are defined later.

Definition 2.2. Let Ω be an open set in $\mathbf{R}^{n+1}$ and suppose that $u \in L^p(t_1, t_2; W^{1,p}(Q))$ whenever $\overline{Q_{t_1,t_2}} \subset \Omega$. Then u is called a *solution* of (1.1) if

$$\int_{t_1}^{t_2} \int_Q \left(|\nabla u|^{p-2} \nabla u \cdot \nabla \varphi - u \frac{\partial \varphi}{\partial t} \right) dx\, dt = 0 \tag{2.2}$$

whenever $\overline{Q_{t_1,t_2}} \subset \Omega$ and $\varphi \in C_0^\infty(Q_{t_1,t_2})$. If, in addition, u is continuous, then u is called *p-parabolic*. Further, we say that u is a *supersolution* of (1.1) if the integral (2.2) is non-negative for all $\varphi \in C_0^\infty(\Omega)$ with $\varphi \geq 0$. If this integral is non-positive instead, we say that u is a *subsolution*.

By parabolic regularity theory the solutions are Hölder continuous after a possible redefinition on a set of measure zero, see [3] or [17]. In general the time derivative u_t does not exist in Sobolev's sense. In most cases one can easily overcome this default by using an equivalent definition in terms of Steklov averages, as on pages 18 and 25 of [3] and in Chapter 2 of [17]. Alternatively, one can proceed using convolutions with smooth mollifiers as on pages 199–121 of [1].

Remark 2.3. If the test function φ is required to vanish only on the lateral boundary $\partial Q \times [t_1, t_2]$, then the boundary terms

$$\int_Q u(x, t_1)\varphi(x, t_1)\, dx = \lim_{\sigma \to 0} \frac{1}{\sigma} \int_{t_1}^{t_1+\sigma} \int_Q u(x, t)\varphi(x, t)\, dx\, dt$$

and

$$\int_Q u(x, t_2)\varphi(x, t_2)\, dx = \lim_{\sigma \to 0} \frac{1}{\sigma} \int_{t_2-\sigma}^{t_2} \int_Q u(x, t)\varphi(x, t)\, dx\, dt$$

have to be included. In the case of a supersolution the condition becomes

$$\begin{aligned} &\int_{t_1}^{t_2} \int_Q \left(|\nabla u|^{p-2} \nabla u \cdot \nabla \varphi - u \frac{\partial \varphi}{\partial t} \right) dx\, dt \\ &\qquad + \int_Q u(x, t_2)\varphi(x, t_2)\, dx - \int_Q u(x, t_1)\varphi(x, t_1)\, dx \geq 0 \end{aligned} \tag{2.3}$$

for almost all $t_1 < t_2$ with $\overline{Q_{t_1,t_2}} \subset \Omega$. There is an abuse of the notation $u(x, t_1)$ and $u(x, t_2)$ in the integrals, which directly evaluated, without the limit procedure with σ, may have another value. In the presence of discontinuities one has to pay due attention to the interpretation given above.

The supersolutions of the p-parabolic equation do not form a good closed class of functions. For example, consider the *Barenblatt solution* $\mathcal{B}_p \colon \mathbf{R}^{n+1} \to [0, \infty)$,

$$\mathcal{B}_p(x, t) = \begin{cases} t^{-n/\lambda} \left(C - \dfrac{p-2}{p} \lambda^{1/(1-p)} \left(\dfrac{|x|}{t^{1/\lambda}} \right)^{p/(p-1)} \right)_+^{(p-1)/(p-2)}, & t > 0, \\ 0, & t \leq 0, \end{cases} \tag{2.4}$$

where $\lambda = n(p-2)+p,\ p>2$, and the constant C is usually chosen so that

$$\int_{\mathbf{R}^n} \mathcal{B}_p(x,t)\,dx = 1$$

for every $t>0$. It formally satisfies the equation

$$\frac{\partial \mathcal{B}_p}{\partial t} - \operatorname{div}(|\nabla \mathcal{B}_p|^{p-2}\nabla \mathcal{B}_p) = \delta,$$

where the right-hand side is Dirac's delta at the origin. In the case $p=2$ we have the heat kernel

$$W(x,t) = \begin{cases} \dfrac{1}{(4\pi t)^{n/2}} e^{-|x|^2/4t}, & t>0 \\ 0, & t \le 0. \end{cases}$$

In contrast with the heat kernel, which is strictly positive, the Barenblatt solution has a bounded support at a given instance $t>0$. Hence the disturbancies propagate with finite speed when $p>2$. -The Barenblatt solution describes the propagation of the heat after the explosion of a hydrogen bomb in the atmosphere. This function was discovered in [2]. The Barenblatt solution is not a supersolution in an open set that contains the origin. It is the a priori summability of $\nabla \mathcal{B}_p$ that fails. Indeed,

$$\int_{-1}^{1}\int_{Q} |\nabla \mathcal{B}_p(x,t)|^p\,dx\,dt = \infty,$$

where $Q = [-1,1]^n \subset \mathbf{R}^n$. However, the Barenblatt solution is a p-superparabolic function according to the following definition.

Definition 2.4. A function $v\colon \Omega \to (-\infty,\infty]$ is called *p-superparabolic* if

(1) v is lower semicontinuous,
(2) v is finite in a dense subset of Ω,
(3) v satisfies the following comparison principle on each subdomain $D_{t_1,t_2} = D\times(t_1,t_2)$ with $D_{t_1,t_2} \subset\subset \Omega$: if h is p-parabolic in D_{t_1,t_2} and continuous in $\overline{D_{t_1,t_2}}$ and if $h \le v$ on the parabolic boundary of D_{t_1,t_2}, then $h \le v$ in D_{t_1,t_2}.

It follows immediately from the definition that, if u and v are p-superparabolic functions, so are their pointwise minimum $\min(u,v)$ and $u+\alpha$, $\alpha\in\mathbf{R}$. Observe that $u+v$ and αu are not superparabolic in general. This is well in accordance with the corresponding properties of supersolutions. The following modification of a p-superparabolic function is useful. If v is a non-negative p-superparabolic function in Ω, then also

$$w(x,t) = \begin{cases} v(x,t), & t>t_0, \\ 0, & t\le t_0, \end{cases} \tag{2.5}$$

is p-superparabolic in Ω.

Notice that *a p-superparabolic function is defined at every point in its domain.* The semicontinuity is an essential assumption. By contrast, no differentiability is presupposed in the definition. The only tie to the differential equation is through the comparison principle. To illuminate this we mention that every function of the form $v(x,t) = g(t)$, where $g = g(t)$ is a non-decreasing lower semicontinuous step function, is p-superparabolic. The interpretation of v_t requires caution.

We recall the following theorem stating that bounded p-superparabolic functions are supersolutions. This is based on the fact that a p-superparabolic function can be approximated from below with solutions of obstacle problems, see [6].

Theorem 2.5. *Let $p \geq 2$. Suppose that v is a p-superparabolic function in Ω and locally bounded above. Then the Sobolev gradient ∇v exists and $\nabla v \in L^p_{\mathrm{loc}}(\Omega)$. Moreover, the function v is a supersolution of equation* (1.1) *in Ω.*

Thus the variational inequality (2.3) is at our disposal for bounded functions.

3. Summability of supersolutions

A locally bounded p-superparabolic function v possesses a certain degree of summability. In particular, v and ∇v belong to $L^p_{\mathrm{loc}}(\Omega)$. In this section we drop the assumption on boundedness and study the question for an arbitrary p-superparabolic function.

If v is p-superparabolic, so are the functions

$$v_j = v_j(x,t) = \min(v(x,t), j), \quad j = 1, 2, \ldots,$$

and, because they are locally bounded, Theorem 2.5 above applies. Our method is to derive estimates for the functions v_j and then pass to the limit

$$v = \lim_{j \to \infty} v_j.$$

If $\overline{Q_T} \subset \Omega$, we have $v_j \in L^p(0,T; W^{1,p}(Q))$, but to begin with we assume that $v_j \in L^p(0,T; W_0^{1,p}(Q))$ and that

$$\lim_{\sigma \to 0} \frac{1}{\sigma} \int_0^\sigma \int_Q v_j(x,t)\, dx\, dt = 0.$$

If this holds, we simply write that $v_j(x,0) = 0$ in Q. Here we use the same interpretation of the boundary values as in (2.3). At the end our construction will reduce the proof to this situation.

Lemma 3.1. *Let $p > 2$ and let $\Omega \subset \mathbf{R}^{n+1}$ be a domain with $\overline{Q_T} \subset \Omega$. Suppose that $v \geq 0$ is a function in Ω such that $v_j = \min(v, j)$ is a supersolution of* (1.1) *in Ω, $v_j \in L^p(0, T; W_0^{1,p}(Q))$ and $v_j(x, 0) = 0$ in Q for every $j = 1, 2, \ldots$ Then*

$$\int_0^T \int_Q |\nabla v_j|^p \, dx \, dt \leq j^2 K, \quad j = 1, 2, \ldots, \tag{3.1}$$

where

$$K = \int_0^T \int_Q |\nabla v_1|^p \, dx \, dt + |Q|. \tag{3.2}$$

Proof. First we select an instant τ, $0 < \tau \leq T$, and fix the index j. Define

$$\varphi_k = (v_k - v_{k-1}) - (v_{k+1} - v_k), \qquad k = 1, 2, \ldots, j,$$

where $v_k = \min(v, k)$. Notice carefully that $\varphi_k \geq 0$ and that $\varphi_k \in L^p(0, T; W_0^{1,p}(Q))$. Thus φ_k will do as a test function in (2.2) for the supersolution v_j. Again we encounter the difficulty with the forbidden time derivative. Let us postpone this difficulty and proceed formally in order to keep the ideas more transparent. Since v_j is a supersolution in Q_τ we have

$$\int_0^\tau \int_Q |\nabla v_j|^{p-2} \nabla v_j \cdot \nabla \varphi_k \, dx \, dt + \int_0^\tau \int_Q \varphi_k \frac{\partial v_j}{\partial t} \, dx \, dt \geq 0.$$

After some rearrangements we arrive at

$$\begin{aligned}
&\int_0^\tau \int_Q |\nabla v_j|^{p-2} \nabla v_j \cdot \nabla (v_{k+1} - v_k) \, dx \, dt + \int_0^\tau \int_Q (v_{k+1} - v_k) \frac{\partial v_j}{\partial t} \, dx \, dt \\
&\quad \leq \int_0^\tau \int_Q |\nabla v_j|^{p-2} \nabla v_j \cdot \nabla (v_k - v_{k-1}) \, dx \, dt + \int_0^\tau \int_Q (v_k - v_{k-1}) \frac{\partial v_j}{\partial t} \, dx \, dt
\end{aligned}$$

or, abbreviated in an obvious way,

$$a_{k+1}(\tau) \leq a_k(\tau).$$

It follows that

$$\sum_{k=1}^{j} a_k(\tau) \leq j a_1(\tau). \tag{3.3}$$

The notation hides the fact that $a_1(\tau)$ depends on the chosen index j. The left-hand side of (3.3) is

$$\begin{aligned}
\sum_{k=1}^{j} a_k(\tau) &= \int_0^\tau \int_Q |\nabla v_j|^p \, dx \, dt + \int_0^\tau \int_Q v_j \frac{\partial v_j}{\partial t} \, dx \\
&= \int_0^\tau \int_Q |\nabla v_j|^p \, dx \, dt + \frac{1}{2} \int_Q v_j^2(x, \tau) \, dx.
\end{aligned}$$

Here we used the assumption that $v_j(x,0) = 0$ in Q. We estimate

$$a_1(\tau) = \int_0^\tau \int_Q |\nabla v_1|^p \, dx \, dt + \int_0^\tau \int_Q v_1 \frac{\partial v_j}{\partial t} \, dx \, dt$$

on the right-hand side of (3.3) using

$$\begin{aligned}\int_0^\tau \int_Q v_1 \frac{\partial v_j}{\partial t} \, dx \, dt &= \int_0^\tau \int_Q v_1 \frac{\partial v_1}{\partial t} \, dx \, dt + \int_0^\tau \int_Q v_1 \frac{\partial}{\partial t}(v_j - v_1) \, dx \, dt \\ &= \frac{1}{2} \int_Q v_1^2(x,\tau) \, dx + \int_Q \big(v_j(x,\tau) - v_1(x,\tau)\big) \, dx \\ &\le \int_Q v_j(x,\tau) \, dx.\end{aligned}$$

This implies that

$$\int_0^\tau \int_Q |\nabla v_j|^p \, dx \, dt \le j \int_0^\tau \int_Q |\nabla v_1|^p \, dx \, dt + j \int_Q v_j(x,\tau) \, dx, \tag{3.4}$$

from which we conclude that

$$\int_0^T \int_Q |\nabla v_j|^p \, dx \, dt \le j^2 \left(\int_0^T \int_Q |\nabla v_1|^p \, dx \, dt + |Q| \right).$$

This proves the estimates under the hypothesis that the time derivative v_t is available at the intermediate steps of the proofs. To justify this calculation we use the convolution

$$(f * \rho_\sigma)(x,t) = \int_{-\infty}^{\infty} f(x, t-s) \rho_\sigma(s) \, ds,$$

where ρ_σ is the Friedrichs mollifier

$$\rho_\sigma(s) = \begin{cases} \dfrac{c}{\sigma} e^{-\sigma^2/(\sigma^2 - s^2)}, & |s| < \sigma, \\ 0, & |s| \ge \sigma. \end{cases}$$

First we slightly enlarge Q_T by replacing it with $Q_{-\delta,T}$ where $\delta > 0$ is chosen so that $\overline{Q_{-\delta,T}} \subset \Omega$. Then we define $v_j(x,t) = 0$, when $t \le 0$. The extended v_j is p-superparabolic in $Q_{-\delta,T}$ because $v_j \ge 0$, see (2.5). We restrict the mollification parameter $\sigma > 0$ so that $\sigma < \delta/2$. When $-\delta/2 < \tau < T - \delta/2$, we have

$$\int_{-\delta/2}^\tau \int_Q \left(\left((|\nabla v_j|^{p-2} \nabla v_j) * \rho_\sigma \right) \cdot \nabla \varphi + \varphi \frac{\partial}{\partial t}(v_j * \rho_\sigma) \right) dx \, dt \ge 0$$

for all test functions $\varphi \geq 0$ vanishing on the lateral boundary. Replace v_k in the proof of Lemma 3.1 by

$$\widetilde{v}_k = \min(v_j * \rho_\sigma, k)$$

and choose

$$\varphi_k = (\widetilde{v}_k - \widetilde{v}_{k-1}) - (\widetilde{v}_{k+1} - \widetilde{v}_k).$$

Since the convolution with respect to the time does not affect the zero boundary values on the lateral boundary and since $\sigma < \delta/2$, we conclude that $\widetilde{v}_k$ vanishes on the parabolic boundary of $Q_{-\delta/2,T-\delta/2}$. (Observe that the functions $v_k * \rho_\sigma$ instead of $(v * \rho_\sigma)_k$ do not work well in this proof.) The same calculation as before yields

$$\widetilde{a}_{k+1}(\tau) \leq \widetilde{a}_k(\tau) \qquad \text{and} \qquad \sum_{k=1}^{j} \widetilde{a}_k(\tau) \leq j\widetilde{a}_1(\tau),$$

where

$$\begin{aligned}\widetilde{a}_k(\tau) = &\int_{-\delta/2}^{\tau}\int_Q \Big((|\nabla v_j|^{p-2}\nabla v_j) * \rho_\sigma\Big) \cdot \nabla(\widetilde{v}_k - \widetilde{v}_{k-1})\, dx\, dt \\ &+ \int_{-\delta/2}^{\tau}\int_Q (\widetilde{v}_k - \widetilde{v}_{k-1})\frac{\partial}{\partial t}(v_j * \rho_\sigma)\, dx\, dt.\end{aligned}$$

Summing up, we obtain

$$\begin{aligned}\sum_{k=1}^{j} \widetilde{a}_k(\tau) = &\int_{-\delta/2}^{\tau}\int_Q \Big((|\nabla v_j|^{p-2}\nabla v_j) * \rho_\sigma\Big) \cdot \nabla\widetilde{v}_j\, dx\, dt \\ &+ \int_{-\delta/2}^{\tau}\int_Q \widetilde{v}_j\frac{\partial}{\partial t}(v_j * \rho_\sigma)\, dx\, dt,\end{aligned}$$

where the last integral can be written as

$$\frac{1}{2}\int_Q (v_j * \rho_\sigma)^2(x,\tau)\, dx.$$

For $\widetilde{a}_1(\tau)$ we again get an estimate free of time derivatives. Therefore we can safely first let $\sigma \to 0$ and then $\delta \to 0$. This leads to (3.4), from which the lemma follows. □

The following lemma holds for rather general functions. The case $\gamma = 2$ is our starting point in view of (3.1) above.

Lemma 3.2. *Let $p > 2$ and suppose that $v \geq 0$ is a function on Q_T such that $v_j \in L^p(0, T; W_0^{1,p}(Q))$ for every $j = 1, 2, \ldots$, where $v_j = \min(v, j)$. If there are $K > 0$ and $0 < \gamma < p$, independent of j, such that*

$$\int_0^T \int_Q |\nabla v_j|^p \, dx \, dt \leq j^\gamma K, \quad j = 1, 2, \ldots, \tag{3.5}$$

then for every q with $0 < q < p - \gamma$ there is $c = c(n, p, q, \gamma)$ such that

$$\int_0^T \int_Q v^q \, dx \, dt \leq |Q_T| + cK. \tag{3.6}$$

In particular, $v \in L^q(Q_T)$ for every q with $0 < q < p - \gamma$.

Proof. Let $\kappa = 1 + 2/n$ and define the sets

$$E_j = \{(x, t) \in Q_T : j \leq v(x, t) < 2j\}, \quad j = 1, 2, \ldots$$

Lemma 2.1 (Sobolev's inequality) implies that

$$\begin{aligned} j^{\kappa p} |E_j| &\leq \iint_{E_j} v_{2j}^{\kappa p} \, dx \, dt \leq \int_0^T \int_Q v_{2j}^{\kappa p} \, dx \, dt \\ &\leq c \int_0^T \int_Q |\nabla v_{2j}|^p \, dx \, dt \left(\operatorname*{ess\,sup}_{0<t<T} \int_Q v_{2j}^2 \, dx \right)^{p/n} \\ &\leq cK j^{\gamma + 2p/n}, \quad j = 1, 2, \ldots, \end{aligned}$$

where $c = c(n, p)$. It follows that

$$|E_j| \leq cK j^{\gamma - p}, \quad j = 1, 2, \ldots$$

From this we conclude that the sum in

$$\int_0^T \int_Q v^q \, dx \, dt \leq |Q_T| + \sum_{j=1}^\infty \int_{E_{2^{j-1}}} v^q \, dx \, dt$$

can be majorized by

$$\sum_{j=1}^\infty \int_{E_{2^{j-1}}} v^q \, dx \, dt \leq \sum_{j=1}^\infty 2^{jq} |E_{2^{j-1}}| \leq cK \sum_{j=1}^\infty 2^{j(q+\gamma-p)}.$$

The series converges if $0 < q < p - \gamma$. □

Next we show that the function v is locally summable.

Theorem 3.3. *Let $p > 2$. Suppose that $\overline{Q_T} \subset \Omega$ and that $v \geq 0$ is a function such that $v_j = \min(v, j)$ is a supersolution of* (1.1) *in Ω, $v_j \in L^p(0, T; W_0^{1,p}(Q))$ and $v_j(x, 0) = 0$ in Q for every $j = 1, 2, \dots$ Then $v \in L^1(Q_{t_1})$, when $0 < t_1 < T$, for every q with $0 < q < p - 1$.*

Proof. From Lemma 3.1 and Lemma 3.2 we conclude that for every q with $0 < q < p - 2$, there is $c = c(n, p, q)$ such that

$$\int_0^T \int_Q v^q \, dx \, dt \leq c \int_0^T \int_Q \left(1 + |\nabla v_1|^p\right) dx \, dt + c|Q| < \infty. \tag{3.7}$$

Thus $v \in L^q(Q_T)$ for every q with $0 < q < p - 2$. Since $p > 2$ there is $\varepsilon > 0$ such that

$$\int_0^T \int_Q v^\varepsilon \, dx \, dt < \infty.$$

We may assume that $0 < \varepsilon \leq 1$. Estimate (3.4) implies that

$$\int_0^{t_1} \int_Q |\nabla v_j|^p \, dx \, dt \leq j \int_0^T \int_Q |\nabla v_1|^p \, dx \, dt + j \int_Q v_j(x, \tau) \, dx$$

when $t_1 \leq \tau \leq T$. We integrate the inequality with respect to τ over $[t_1, T]$ and obtain

$$\begin{aligned}
(T - t_1) & \int_0^{t_1} \int_Q |\nabla v_j|^p \, dx \, dt \\
& \leq j(T - t_1) \int_0^T \int_Q |\nabla v_1|^p \, dx \, dt + j \int_{t_1}^T \int_Q v_j(x, t) \, dx \, dt \\
& \leq j(T - t_1) \int_0^T \int_Q |\nabla v_1|^p \, dx \, dt + j^{2-\varepsilon} \int_{t_1}^T \int_Q v^\varepsilon(x, t) \, dx \, dt.
\end{aligned}$$

Therefore

$$\begin{aligned}
& \int_0^{t_1} \int_Q |\nabla v_j|^p \, dx \, dt \\
& \qquad \leq j \int_0^T \int_Q |\nabla v_1|^p \, dx \, dt + \frac{j^{2-\varepsilon}}{T - t_1} \int_{t_1}^T \int_Q v^\varepsilon(x, t) \, dx \, dt
\end{aligned} \tag{3.8}$$

and consequently

$$\int_0^{t_1} \int_Q |\nabla v_j|^p \, dx \, dt \leq j^{2-\varepsilon} K, \quad j = 1, 2, \dots,$$

where

$$K = \int_0^T \int_Q |\nabla v_1|^p \, dx \, dt + \frac{1}{T - t_1} \int_0^T \int_Q v^\varepsilon(x, t) \, dx \, dt < \infty.$$

Lemma 3.2 implies that for every q with $0 < q < p-2+\varepsilon$ there is $c = c(n,p,q,\varepsilon)$ such that

$$\int_0^{t_1}\int_Q v^q\,dx\,dt \le |Q_T| + cK. \tag{3.9}$$

Hence we have shown that if $v \in L^\varepsilon(Q_T)$ then (3.9) holds and, in particular, $v \in L^q(Q_{t_1})$ for every q with $0 < q < p-2+\varepsilon$. The crucial passage over $p-2$ has now been accomplished.

Next we iterate this procedure. If $p-2+\varepsilon > 1$, then we know that $v \in L^1(Q_{t_1})$ and we may choose $\varepsilon = 1$ to begin with. (Observe that if $p \ge 3$, then this is always possible.) In this case the claim follows from (3.9) even after the first step of iteration.

If $p-2+\varepsilon \le 1$, then (3.9) holds for every q with $0 < q < 2(p-2)+\varepsilon$. At the kth step we have $0 < q < k(p-2)+\varepsilon$. We continue this until $k(p-2)+\varepsilon > 1$. Observe that, at each step of iteration, we have to choose a slightly smaller t_1. However, this happens only finitely many times and does not cause any trouble. The claim follows. □

In order to effectively utilize Sobolev's inequality (Lemma 2.1), we need a better estimate for the integral

$$\int_Q v_j^2(x,t)\,dx$$

than the trivial $j^2|Q|$. We return to equation (2.3). Let $0 < t_1 < T$ and $t_1 \le \tau \le T$. Choosing $u = v_j$ and $\varphi = v_j$ in (2.3) we have the basic estimate

$$\frac{1}{2}\int_Q v_j^2(x,t)\,dx \le \int_0^\tau\int_Q |\nabla v_j|^p\,dx\,dt + \frac{1}{2}\int_Q v_j^2(x,\tau)\,dx,$$

where $0 < t < t_1$. See Section 2.1 in [17] for a proof in terms of Steklov averages. Together with (3.4) this implies that

$$\begin{aligned}
\operatorname*{ess\,sup}_{0<t<t_1}\int_Q v_j^2(x,t)\,dx &\le 2\int_0^\tau\int_Q |\nabla v_j|^p\,dx\,dt + \int_Q v_j^2(x,\tau)\,dx \\
&\le 2j\int_0^\tau\int_Q |\nabla v_1|^p\,dx\,dt + 3j\int_Q v_j(x,\tau)\,dx \\
&\le 2j\int_0^T\int_Q |\nabla v_1|^p\,dx\,dt + 3j\int_Q v_j(x,\tau)\,dx.
\end{aligned}$$

An integration with respect to τ over the interval $[t_1, T]$ yields

$$\begin{aligned}
&\operatorname*{ess\,sup}_{0<t<t_1}\int_Q v_j^2(x,t)\,dx \\
&\qquad \le 2j\int_0^T\int_Q |\nabla v_1|^p\,dx\,dt + \frac{3j}{T-t_1}\int_{t_1}^T\int_Q v_j(x,t)\,dx\,dt
\end{aligned}$$

after a division by $T - t_1$. We choose $\varepsilon = 1$ in (3.8) and we obtain

$$\begin{aligned}\int_0^{t_1}\int_Q |\nabla v_j|^p\,dx\,dt + \operatorname*{ess\,sup}_{0<t<t_1}\int_Q v_j^2(x,t)\,dx \\ \leq 3j\int_0^T\int_Q |\nabla v_1|^p\,dx\,dt + \frac{4j}{T-t_1}\int_{t_1}^T\int_Q v(x,t)\,dx\,dt,\end{aligned}\tag{3.10}$$

where, of course, v is p-superparabolic. At least with a slightly smaller T than the original one, the right-hand side of (3.10) is a finite number, in fact of order $O(j)$, by Theorem 2.5 and Theorem 3.3.

Keeping the fundamental estimate (3.10) in mind, we formulate the lemma below. It reaches the correct exponent.

Lemma 3.4. *Let $v \geq 0$ be a function on Q_T and suppose that $v_j \in L^p(0,T; W_0^{1,p}(Q))$, where $v_j = \min(v, j)$. If there is a constant $K > 0$, independent of j, such that*

$$\int_0^T\int_Q |\nabla v_j|^p\,dx\,dt + \operatorname*{ess\,sup}_{0<t<T}\int_Q v_j^2\,dx \leq jK, \quad j = 1, 2, \dots, \tag{3.11}$$

then $v \in L^q(Q_T)$ for every q with $0 < q < p - 1 + p/n$. Moreover, the function v has the Sobolev gradient ∇v and $\nabla v \in L^q(Q_T)$ for every q with $0 < q < p - 1 + 1/(n+1)$.

Proof. Let $\kappa = 1 + 2/n$ and define the sets

$$E_j = \{(x,t) \in Q_T : j \leq v(x,t) < 2j\}, \quad j = 1, 2, \dots$$

Sobolev's inequality in Lemma 2.1 implies that

$$\begin{aligned} j^{\kappa p}|E_j| &\leq \iint_{E_j} v_{2j}^{\kappa p}\,dx\,dt \leq \int_0^T\int_Q v_{2j}^{\kappa p}\,dx\,dt \\ &\leq c\int_0^T\int_Q |\nabla v_{2j}|^p\,dx\,dt\left(\operatorname*{ess\,sup}_{0<t<T}\int_Q v_{2j}^2\,dx\right)^{p/n} \\ &\leq cK^{1+p/n}j^{1+p/n}, \quad j = 1, 2, \dots, \end{aligned}$$

where $c = c(n,p)$. It follows that

$$|E_j| \leq cK^{1+p/n}j^{1-p-p/n}, \quad j = 1, 2, \dots$$

From this we conclude that the last term in

$$\int_0^T\int_Q v^q\,dx\,dt \leq T|Q| + \sum_{j=1}^{\infty}\int_{E_{2^{j-1}}} v^q\,dx\,dt$$

can be majorized by

$$\sum_{j=1}^{\infty}\int_{E_{2^{j-1}}} v^q\,dx\,dt \le \sum_{j=1}^{\infty} 2^{jq}|E_{2^{j-1}}| \le cK^{1+p/n}\sum_{j=1}^{\infty} 2^{j(q+1-p-p/n)}.$$

The series converges if $0 < q < p-1+p/n$.

To estimate the summability of the gradient, let $k \in \mathbf{N}$. Then

$$\begin{aligned}
\int_0^T\int_Q |\nabla v_k|^q\,dx\,dt &\le \int_{E_0}|\nabla v|^q\,dx\,dt + \sum_{j=1}^{\infty}\int_{E_{2^{j-1}}}|\nabla v_k|^q\,dx\,dt\\
&\le \sum_{j=1}^{\infty}\left(\int_{E_{2^{j-1}}}|\nabla v_k|^p\,dx\,dt\right)^{q/p}|E_{2^{j-1}}|^{1-q/p}\\
&\le c\sum_{j=1}^{\infty}2^{(j-1)(1-q/p)(1-p-p/n)}\left(\int_{E_{2^{j-1}}}|\nabla v_{2^{j-1}}|^p\,dx\,dt\right)^{q/p},
\end{aligned}$$

where $c = c(n, p, q, K)$. Here we also used the fact that $|\nabla v_k| \le |\nabla v_{2^{j-1}}|$ a.e. in $E_{2^{j-1}}$. We use the assumption to get the majorant

$$c\sum_{j=1}^{\infty}2^{(j-1)(1-p-p/n+q+q/n)},$$

which converges if $0 < q < p-1+1/(n+1)$. This is the correct critical exponent. Since $p > 2$ we can at least find an allowed $q > 1$. This implies that (∇v_k) is a bounded sequence in $L^q(Q_T)$ and hence it has a weakly converging subsequence, denoted again by (∇v_k). From this it follows that ∇v exists and that (∇v_k) converges weakly to ∇v in $L^q(Q_T)$. Consequently

$$\int_0^T\int_Q|\nabla v|^q\,dx\,dt \le \liminf_{k\to\infty}\int_0^T\int_Q|\nabla v_k|^q\,dx\,dt < \infty.$$

This concludes the proof. □

For p-superparabolic functions the results of this section are summarized in the following corollary. The estimates needed in this result are the main ingredients in the proof of Theorem 1.1. It still suffers from the restriction on the boundary values.

Corollary 3.5. *Let $p > 2$. Suppose that $\overline{Q_T} \subset \Omega$ and $0 < t_1 < T$. If $v \ge 0$ is a p-superparabolic function in Ω such that $v_j(x,0) = 0$ in Q and $v_j \in L^p(0,T;W_0^{1,p}(Q))$, $j = 1,2,\dots$, then $v \in L^q(Q_{t_1})$ for every q with $0 < q < p-1+p/n$. Moreover, the function v has the Sobolev gradient ∇v and $\nabla v \in L^q(Q_{t_1})$ for every q with $0 < q < p-1+1/(n+1)$.*

Now we are ready to give a proof for our main result.

Proof of Theorem 1.1. In order to conclude the proof of Theorem 1.2 we have to modify the p-superparabolic function v near the parabolic boundary of Q_T so that Corollary 3.5. applies. Therefore we assume that $\overline{Q_T} \subset \Omega$. Let $Q' \subset\subset Q$ and select t_1 and t_2 so that $0 < t_1 < t_2 < T$. Then $\overline{Q'_{t_1,t_2}} \subset Q_T$. By adding a constant to v we may assume that $v \geq 0$ in Q_T. Furthermore, we can redefine v so that $v(x,t) = 0$, when $t \leq t_1$. The obtained function v is also p-superparabolic in Q_T, see (2.5). We aim at proving the summability in Q'_{t_1,t_2}.

Roughly speaking, we want to redefine v in $Q_T \setminus Q'_{t_1,T}$ in the following way:

$$w = \begin{cases} v & \text{in } \overline{Q'_{t_1,T}}, \\ h & \text{in } Q_T \setminus \overline{Q'_{t_1,T}}, \end{cases} \tag{3.12}$$

where h is the p-parabolic function in $Q_T \setminus \overline{Q'_{t_1,T}}$ with zero boundary values on the parabolic boundary of Q_T and $h = v$ on the parabolic boundary of $Q'_{t_1,T}$. Notice that h and v both are zero when $t \leq t_1$. We will show that w is p-superparabolic.

Let us first construct h. The lower semicontinuity implies that there is a sequence of functions $\psi_k \in C^\infty(\Omega)$, $k = 1, 2, \ldots$, such that

$$0 \leq \psi_1 \leq \psi_2 \leq \ldots \qquad \text{and} \qquad \lim_{k\to\infty} \psi_k = v$$

at every point of Ω. We assume, as we may, that $\psi_k = 0$ in $Q \times [0, t_1]$. Let h_k denote the unique p-parabolic function in $(Q \setminus \overline{Q'}) \times (0, T)$ with the following boundary values

$$h_k = \begin{cases} \psi_k & \text{in } \partial Q' \times [0,T], \\ 0 & \text{in } \partial Q \times [0,T], \\ 0 & \text{in } (Q \setminus Q') \times \{0\}. \end{cases}$$

We can extend h_k continuously to the boundary so that $h_k \in C((\overline{Q} \setminus Q') \times [0,T])$. Actually, $h_k(x,t) = 0$ when $t \leq t_1$. We have

$$h_1 \leq h_2 \leq \ldots \qquad \text{and} \qquad h_k \leq v \qquad \text{in} \qquad Q_T \setminus \overline{Q'_{t_1,T}}.$$

By Harnack's convergence theorem (see Remark 3.2 in [7]), a consequence of the intrinsic Harnack estimate on pages 157 and 184 in [3], the function

$$h = \lim_{k\to\infty} h_k$$

is p-parabolic in $Q_T \setminus \overline{Q'_{t_1,T}}$ and clearly $h \leq v$. Thus $w \leq h$.

It remains to verify the comparison principle for w. Let $D_{a,b} = D \times (a,b)$ be a subdomain of Q_T and suppose that $H \in C(\overline{D_{a,b}})$ is p-parabolic and $w \geq H$ on the parabolic boundary of $D_{a,b}$. Since $v \geq w$, the comparison principle valid for v yields $v \geq H$ in $D_{a,b}$. In particular, $H(x,t) \leq 0$ when $t \leq t_1$ (if $a < t_1$). If $D \subset Q'$ we are done. If not, then a comparison has to be performed in (each component of) $(D \setminus Q') \times (a, b)$. We have that $h \geq H$ on the parabolic boundary of this set. The points on $\partial Q' \times (a, b)$ require some care. Let (x_0, t_0) be a point on $\partial Q' \times (a, b)$. From the construction of h we can deduce that, given $\varepsilon > 0$, there is an index k such that

$$H(x_0, t_0) < h_k(x_0, t_0) + \varepsilon.$$

This implies that

$$H(x_0, t_0) \leq \liminf_{(x,t)\to(x_0,t_0)} h(x, t).$$

Thus $h \geq H$ by the comparison principle. This concludes the proof of the inequality $w \geq H$ in $D_{a,b}$.

Therefore the function w defined by (3.12) is p-superparabolic in Q_T, according to Definition 2.4. In particular, w is continuous in $Q_T \setminus \overline{Q'_{t_1,T}}$ and has zero boundary values on the parabolic boundary of Q_T.

Next we apply (3.10) for w in Q_{t_1,t_3}, where $t_2 < t_3 < T$. We obtain

$$\int_{t_1}^{t_2} \int_Q |\nabla w_j|^p \, dx \, dt + \operatorname*{ess\,sup}_{t_1<t<t_2} \int_Q w_j^2(x, t) \, dx \leq jK,$$

where the quantity

$$K = 3 \int_{t_1}^{t_3} \int_Q |\nabla w_1|^p \, dx \, dt + \frac{4}{t_3 - t_2} \int_{t_2}^{t_3} \int_Q w(x, t) \, dx \, dt \tag{3.13}$$

has to be estimated. Recall that $w_1 = \min(w, 1)$. When $Q' \subset\subset Q'' \subset\subset Q$, the integral

$$\int_{t_1}^{t_3} \int_{Q''} |\nabla w_1|^p \, dx \, dt + \frac{4}{t_3 - t_2} \int_{t_2}^{t_3} \int_{Q''} w(x, t) \, dx \, dt,$$

is finite because of Theorem 2.5 and Theorem 3.3. Then by the remark of the proof of Lemma 2.16 in [6] (in particular, see (2.21)), there is $c = c(p)$ such that

$$\int_{t_1}^{t_3} \int_{Q\setminus Q'} |\nabla h|^p \zeta^p \, dx \, dt \leq c \int_{t_1}^{t_3} \int_{Q\setminus Q'} h^p |\nabla \zeta|^p \, dx \, dt,$$

where $\zeta = \zeta(x) > 0$ is a smooth cutoff function depending only on the spatial variable x and vanishing on the lateral boundary $\partial Q' \times [t_1, t_3]$. Observe, that there is no requirement for ζ on $\partial Q \times [t_1, t_3]$. Since $|\nabla w_1| \leq |\nabla h|$, it follows that the integral

$$\int_{t_1}^{t_3} \int_{Q\setminus Q''} |\nabla w_1|^p \, dx \, dt$$

is finite. Since w is continuous in $\overline{Q \setminus Q''} \times [t_1, t_3]$ it is bounded in that set. From this we conclude that K in (3.13) is a finite number.

We can use Corollay 3.16 to conclude that $w \in L^q(Q_{t_3})$. A fortiori $v \in L^q(Q'_{t_1,t_2})$. The same concerns the summability of the gradient. This concludes our proof of Theorem 1.1. □

4. A logarithmic Caccioppoli type estimate

This section is devoted to an elementay local summability estimate of $\nabla \log u$ for supersolutions. *The a priori bound is independent of u, if $u \geq 1$.* It also holds for p-superparabolic functions and complements our main result.

Lemma 4.1. *Let $p > 2$. Suppose that $u \geq 0$ is a supersolution of* (1.1) *in Ω. If $\overline{Q_T} \subset \Omega$ and if $\zeta \in C^\infty(Q_T)$, $\zeta \geq 0$ and $\zeta = 0$ on Γ_T, then there exists a constant $c = c(p)$ such that*

$$\begin{aligned}\int_0^T \int_Q |\nabla \log u|^p \zeta^p \, dx\, dt \\ \leq c \int_0^T \int_Q u^{2-p} \left| \frac{\partial(\zeta^p)}{\partial t} \right| dx\, dt + c \int_0^T \int_Q |\nabla \zeta|^p \, dx\, dt.\end{aligned} \tag{4.1}$$

Proof. We may assume that $u(x,t) \geq \alpha > 0$ by first proving the lemma for the supersolution $u(x,t) + \alpha$ and then letting $\alpha \to 0$. Formally, the test function $\varphi = \zeta^p u^{1-p}$ in (2.3) will lead to the result. In order to avoid the forbidden time derivative u_t, we use the convolution

$$u_\sigma(x,t) = \int_{\mathbf{R}} u(x, t-\tau) \rho_\sigma(\tau)\, d\tau$$

with respect to the time variable. Here $\rho_\sigma = \rho_\sigma(t)$ is the standard mollifier with support in $[-\sigma, \sigma]$. Let $\varphi = \zeta^p u_\sigma^{1-p}$ in the averaged equation

$$\begin{aligned}\int_0^T \int_Q \Big((|\nabla u|^{p-2} \nabla u) * \rho_\sigma \cdot \nabla \varphi - u_\sigma \frac{\partial \varphi}{\partial t} \Big) dx\, dt \\ + \int_Q u_\sigma(x,T) \varphi(x,T)\, dx \geq \int_Q u_\sigma(x,0)\varphi(x,0)\, dx \geq 0,\end{aligned}$$

where $\sigma > 0$ is small. The term with $\dfrac{\partial \varphi}{\partial t}$ becomes

$$\begin{aligned}&- \int_0^T \int_Q u_\sigma \frac{\partial}{\partial t}(\zeta^p u_\sigma^{1-p})\, dx\, dt \\ &\quad = - \int_Q \zeta(x,T)^p u_\sigma(x,T)^{2-p}\, dx + \frac{1}{2-p} \int_0^T \int_Q \zeta^p \frac{\partial}{\partial t}(u_\sigma^{2-p})\, dx\, dt\end{aligned}$$

and so

$$\begin{aligned}-\int_0^T \int_Q u_\sigma \frac{\partial \varphi}{\partial t}\, dx\, dt + \int_Q u_\sigma(x,T)\varphi(x,T)\, dx &= \frac{1}{2-p} \int_0^T \int_Q \zeta^p \frac{\partial}{\partial t}(u_\sigma^{2-p})\, dx\, dt \\ = -\frac{1}{p-2} \int_Q \zeta(x,T)^p u_\sigma(x,T)^{2-p}\, dx &+ \frac{1}{p-2} \int_0^T \int_Q u_\sigma^{2-p} \frac{\partial(\zeta^p)}{\partial t}\, dx\, dt.\end{aligned}$$

We obtain

$$\frac{1}{p-2}\int_0^T\int_Q u_\sigma^{2-p}\frac{\partial(\zeta^p)}{\partial t}\,dx\,dt+\int_0^T\int_Q(|\nabla u|^{p-2}\nabla u)*\rho_\sigma\cdot\nabla(\zeta^p u_\sigma^{1-p})\,dx\,dt$$
$$\geq\frac{1}{p-2}\int_Q\zeta(x,T)^p u_\sigma(x,T)^{2-p}\,dx\geq 0.$$

This estimate is free of problematic time derivatives and hence we may safely let $\sigma\to 0$.

We are left with

$$\frac{1}{p-2}\int_0^T\int_Q u^{2-p}\frac{\partial(\zeta^p)}{\partial t}\,dx\,dt+\int_0^T\int_Q|\nabla u|^{p-2}\nabla u\cdot\nabla(\zeta^p u^{1-p})\,dx\,dt\geq 0$$

where

$$\nabla(\zeta^p u^{1-p})=p\zeta^{p-1}u^{1-p}\nabla\zeta-(p-1)\zeta^p u^{-p}\nabla u.$$

Thus

$$(p-1)\int_0^T\int_Q u^{-p}|\nabla u|^p\zeta^p\,dx\,dt\leq\frac{1}{p-2}\int_0^T\int_Q u^{2-p}\left|\frac{\partial(\zeta^p)}{\partial t}\right|\,dx\,dt$$
$$+\,p\int_0^T\int_Q\zeta^{p-1}|\nabla\log u|^{p-2}\nabla\log u\cdot\nabla\zeta\,dx\,dt.$$

We use Young's inequality

$$p\zeta^{p-1}|\nabla\log u|^{p-1}|\nabla\zeta|\leq(p-1)\beta^q\zeta^p|\nabla\log u|^p+\beta^{-p}|\nabla\zeta|^p$$

with $q=p/(p-1)$ and $\beta>0$ small enough ($\beta^q=1-1/p$ will do) to estimate the last integral. The integral of $\beta^q\zeta^p|\nabla\log u|^p$ is absorbed by the left-hand side. This is the so-called Peter-Paul principle. This proves (4.1) for $p>2$. □

Remark 4.2. In the case $p=2$ the estimate reads

$$\int_0^T\int_Q|\nabla\log u|^2\zeta^2\,dx\,dt\leq 2\int_Q|\log u(x,T)|\zeta(x,T)^2\,dx$$
$$+\,2\int_0^T\int_Q|\log u|\left|\frac{\partial(\zeta^2)}{\partial t}\right|\,dx\,dt+4\int_0^T\int_Q|\nabla\zeta|^2\,dx\,dt.$$

We have the following result for unbounded p-superparabolic functions.

Theorem 4.3. *Let v be p-superparabolic in Ω and suppose that $v\geq 1$. Then the Sobolev derivative $\nabla\log v$ exists and*

$$\int_0^T\int_Q|\nabla\log v|^p\zeta^p\,dx\,dt\leq c\int_0^T\int_Q v^{2-p}\left|\frac{\partial(\zeta^p)}{\partial t}\right|\,dx\,dt+\int_0^T\int_Q|\nabla\zeta|^p\,dx\,dt$$

whenever $\overline{Q_T}\subset\Omega$ and $\zeta\geq 0$ is a test function vanishing on Γ_T.

Proof. We know from Theorem 1.1 that $\log v\in L^p(Q_T)$. Now $v_k=\min(v,k)$ is a supersolution for every $k=1,2,\dots$ (Theorem 2.5) and so estimate (4.1) holds for every v_k. Letting $k\to\infty$ we get the desired estimate. It follows that $\log v\in L^p(0,T;W^{1,p}(Q))$. □

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