

## Stochastic Poisson-Sigma model

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**Abstract.** We produce a stochastic regularization of the Poisson-Sigma model of Cattaneo-Felder, which is an analogue regularization of Klauder's stochastic regularization of the Hamiltonian path integral [23] in field theory. We perform also semi-classical limits.

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### 1. Introduction

Let us consider a manifold  $M$  endowed with a Poisson structure, a bilinear map  $\{.,.\}$  from the space of smooth functions on the manifold into the space of smooth functions on the manifold, anticommutative and satisfying the Jacobi relation. Bayen, Flato, Fronsdal, Lichnerowicz and Sternheimer [6, 7] have introduced the so-called program of deformation quantization. These authors get the following formal series:

$$f * g = \sum i^n h^n P_n(f, g). \quad (1.1)$$

The  $P_n$ 's are differential operators. This series is diverging. The program of deformation quantization was carried out by Kontsevich [24]. We refer to the survey of Dito-Sternheimer about this topic [17].

Cattaneo-Felder [15] have established the link between Kontsevich formula and quantum field theory. Let us suppose that the manifold is  $\mathbb{R}^d$ . They consider the so-called Poisson-Sigma model. Let us recall how it is constructed: we consider the disk  $D$ , 3 points  $\infty, 1, 2$  on the boundary of the disk. They consider the space of forms  $\eta$  on  $D$  and the space of maps  $X$  from  $D$  into  $\mathbb{R}^d$ . Let  $\alpha_{i,j}$  be the components of the Poisson structure on  $\mathbb{R}^d$ . Let  $(t, s) = S$  be the polar coordinates on  $D$ :  $t \in [0, 1], s \in S^1$ . Cattaneo-Felder consider the action:

$$\sum_i \int_D \eta_i(S) \wedge dX_i + \sum_{i,j} \int_D \alpha_{i,j}(X) \eta_i \wedge \eta_j = S(X, \eta) \quad (1.2)$$

where  $X = (X_1, \dots, X_d)$  and where  $\eta_j$  are 1-forms on  $D$ . [15] consider the following formula for the non-perturbative  $*$ -product:

$$f *_h g(x) = \int_{\eta, X, X(\infty)=x} f(X(1))g(X(2)) \exp[iS/h] dD X dD \eta \quad (1.3)$$

where the field  $X$  and the form  $\eta$  are chosen at random according to the formal Lebesgue measure on the configuration space. [15] perform the semi-classical analysis when  $h \rightarrow 0$  and get the asymptotic expansion:

$$f *_h g(x) = \sum (ih)^n P_n(f, g). \quad (1.4)$$

The objects of Cattaneo-Felder are formal (see [20, 21]) and use the heavy apparatus of quantum field theory. Our purpose is to add a stochastic regulator in (1.3) in order to define the functional integral rigorously. We get a stochastic product  $f *_{st, h} g$ .

Let us recall that in (1.3), we have to choose two kinds of objects at random: the field  $X : D \rightarrow \mathbb{R}^d$  and the forms  $\eta$  over  $D$ . So we have to introduce stochastic regulators to define a random field  $X$  and to define random forms  $\eta$ .

In order to define the random field  $X : D \rightarrow \mathbb{R}^d$ , we will follow the procedure of Airault-Malliavin [2]. Airault-Malliavin [2] have defined the Brownian motion over a loop group. Let us recall that infinite dimensional processes over infinite dimensional manifolds have a long history: see works of Kuo [26], Belopol'skaya-Daletskii [8] and Daletskii [16]. Albeverio-Léandre-Röckner [4] have defined the Ornstein-Uhlenbeck process over the free loop space, by using the theory of Dirichlet forms. Brzezniak-Elworthy [12] have given an abstract generalization of the works of Airault-Malliavin.

In this paper, we are concerned with a  $(1 + 1)$ -dimensional theory: this means we consider a diffusion process on the loop space. Various works in this direction were done by Brzezniak-Léandre [13, 14], Léandre [34, 35, 36]. Let us remark that in (1.3), there is the condition  $X(\infty) = x$ . [14, 34, 36] have introduced a convenient Brownian bridge in order to do the conditional expectation by  $X(\infty) = x$ . But there is another procedure to condition functionals: it is the Airault-Malliavin-Sugita procedure [1, 44]. In this work, we will follow this procedure.

In order to define random forms, we will employ the techniques of [37]. This means we will not choose our random forms on  $D$  according to the formal Lebesgue measure on the space of forms, but we will introduce a stochastic Gaussian regulator in order to define the probability measure on the space of forms.

If we do not look at the conditional expectation by  $X(\infty) = x$ , the action  $S$  becomes a stochastic integral, which belongs to all of the Sobolev spaces of the Malliavin Calculus [41]. We consider the measure

$$h \rightarrow E [f(X(1))g(X(2))h(X(\infty)) \exp[iS]] .$$

By Malliavin Calculus, it has a smooth density. Moreover, the magic properties of the Airault-Malliavin equation tell us that the density of the law of  $X(\infty)$  is strictly positive.

We have:

**Theorem A.**

$$f *_{st} g = E[f(X(1))g(X(2)) \exp[iS]|X(\infty) = x] \tag{1.5}$$

defines a continuous bilinear map from  $C_b^\infty(\mathbb{R}^d)$  into  $C_b^\infty(\mathbb{R}^d)$ .

We use the Malliavin Calculus to prove Theorem A.

We perform a semi-classical analysis when  $h \rightarrow 0$ : for that task, we choose a small leading Brownian motion as well as a small stochastic regularization of  $\eta$ . Such considerations were done in [32]. But  $S$  is only a stochastic integral: so, by improving a bit the techniques of [32], we have:

**Theorem B.**

$$f *_{st,h} g = E_h[f(X(1))g(X(2)) \exp[iS/h]|X(\infty) = x] \tag{1.6}$$

has, when  $h \rightarrow 0$  an asymptotic expansion:

$$f *_{st,h} g = \sum h^n Q_n(f, g) \tag{1.7}$$

where the  $Q_n$ 's are differential operators acting on  $f$  and  $g$ .

For that, we use the techniques of asymptotics of Wiener functionals by using the Malliavin Calculus: we refer to the surveys by Léandre [28], Kusuoka [27] and Watanabe [45] for this topic.

The reader interested in the relation existing between analysis over loop space and mathematical physics can consult the survey by Albeverio [3] and the two surveys by Léandre [29, 30].

**2. The model without conditioning**

Let  $\Pi(x)$  be a linear map from  $\mathbb{R}^n$  into  $\mathbb{R}^d$ , which depends smoothly from  $x \in \mathbb{R}^d$ : we suppose that the derivatives of all orders of  $\Pi$  are bounded and that  $(\Pi(x)e_i)_{i=1, \dots, n}$  spans uniformly  $\mathbb{R}^d$  for the canonical basis  $e_1, \dots, e_n$  of  $\mathbb{R}^n$ .

Let  $H = H^{1,2}(S^1; \mathbb{R}^n)$  be the Hilbert space of maps from the circle  $S^1$  into  $\mathbb{R}^n$  such as:

$$\int_0^1 |h(s)|^2 ds + \int_0^1 |d/dsh(s)|^2 ds = \|h\|^2 < \infty. \tag{2.1}$$

We write  $h = (h^1, \dots, h^n)$ . Moreover,

$$h^j(0) = \langle h, e^j \rangle \tag{2.2}$$

where

$$e^j(s) = (0, \dots, 0, \lambda \exp[-s] + \mu \exp[s], 0, \dots, 0) \tag{2.3}$$

for some  $\lambda$  and some  $\mu$  for  $0 \leq s \leq 1$ . Moreover  $e^j(s)$  is smooth on  $]0, 1[$  with half derivatives at all orders at 0 and 1:  $e^j(0) = e^j(1)$  but  $d/dse^j(0) \neq d/dse^j(1)$ .

We have:

$$h^j(s) = \langle h, e^j(\cdot - s) \rangle. \tag{2.4}$$

We consider the Brownian motion with values in  $H$ :

$$B_t(s) = (B_t^1(s), \dots, B_t^n(s)). \tag{2.5}$$

The processes  $B_t^j(\cdot)$  are independent and  $t \rightarrow B_t^j(s)$  is a Brownian motion with values in  $\mathbb{R}$  submitted to the relation:

$$d\langle B_t^j(s), B_t^j(s') \rangle = e(s - s')dt \tag{2.6}$$

where  $e^j(s)$  is the  $j^{th}$  coordinate of  $e(s)$ .

We consider the Airault-Malliavin equation [2, 12]:

$$dx_t(s)(x) = \Pi(x_t(s)(x))d_t B_t(s); x_0(s)(x) = x. \tag{2.7}$$

It is a family of Stratonovitch equations. We have shown that  $s \rightarrow x_t(s)(x)$  is  $1/2 - \epsilon$  Hölder by Gronwall lemma and Kolmogorov lemma [39]: we have an improvement of this result. Namely:

**Proposition 2.1.**  $x \rightarrow (s \rightarrow x_1(s)(x))$  is almost-surely smooth for the Hölder topology.

*Proof.* This comes from the fact that  $s \rightarrow \frac{D^{(r)}}{Dx^{(r)}}x_1(s)(x)$  is almost surely Hölder  $1/2 - \epsilon$  in  $s$  (see [32] for an analogous statement). Namely, the stochastic differential equation of  $\frac{D}{Dx}x_t(s)(x)$  is

$$d\frac{D}{D(x)}x_t(s)(x) = D\Pi(x_t(s)(x))\frac{D}{D(x)}x_t(s)(x)d_t B_t(s) \tag{2.8}$$

and we get by induction the differential equation of  $\frac{D^{(r)}}{Dx^{(r)}}x_t(s)(x)$ . □

Let us write for  $\Delta s$  small:

$$B_t(s + \Delta s) = B_t(s) + \Delta_s B_t(s). \tag{2.9}$$

We have (See [35] Part III):

**Property H(1).** If  $s'$  does not belong to  $]s, s + \Delta s[$ :

$$d\langle \Delta_s B.(s), B.(s') \rangle = dtC(s, s')\Delta s + O(\Delta s^2). \tag{2.10}$$

**Property H(2).** if  $]s, s + \Delta s[\cap]s', s' + \Delta s'[ = \emptyset$ , we have:

$$d\langle \Delta_s B.(s), \Delta_{s'} B.(s') \rangle = dtC(s, s')\Delta s\Delta s' + O(\Delta s + \Delta s')^3. \tag{2.11}$$

Let us consider a sequence of intervals  $]s_i, s_i + \Delta s_i[$ , two intervals being either disjoint or equal. We denote by  $|I|$  the number of intervals and by  $\|I\|$  the number of distinct intervals. Let us consider some points  $r_j$  of the circle which do not belong to the union of the previous open intervals. Let  $\alpha_t(i)$  be some processes, which are  $B.(r_j)$  measurable, previsible, and which are semi-martingales. We suppose that the local characteristic [22] of each  $\alpha_t(i)$  have bounded Sobolev norms in the sense of the Malliavin Calculus [41] for the Gaussian space spanned by the  $B.(r_j)$ . We put iteratively:

$$I^{i+1}(t) = \int_0^t I^i(u)\alpha_u(i)d_u\Delta_{s_i}B_u(s_i) \tag{2.12}$$

and we get an iterated Stratonovitch integral  $I^I(t)$ . Let  $F$  be a measurable functional for the Gaussian space spanned by the  $B.(r_j)$ : we suppose that  $F$  has bounded Sobolev norms in the sense of Malliavin Calculus for the space spanned by the  $B.(r_j)$ . We denote by  $I'$  the set of indices obtained by selecting from  $I$  an interval only one time. The cardinal of  $I'$  is therefore  $\|I\|$ . We have the main lemma:

**Lemma 2.2.**

$$E[FI^I(t)] \leq C \prod_{i \in I'} \Delta s_i \tag{2.13}$$

where  $C$  can be estimated in terms of the Sobolev norms of  $F$  and of the  $\alpha.(i)$ .

*Proof.* We apply the Clark-Ocone formula to  $F$  [41]. We select the Itô term in  $I^I(t)$  and the finite energy term in  $I^I(t)$ . We conclude by applying Itô formula and Properties H(1) and H(2) and property H(3):

**Property H(3).**

$$d_t\langle \Delta_s B.(s), \Delta_s B.(s) \rangle = C(s)\Delta s dt + O(\Delta s^2)dt. \tag{2.14}$$

The statement follows by induction on  $|I|$ . □

**Remark 2.3.** We remark that we have analogue estimates if we consider a product  $\prod_{i \in I} I^i(t)$  of single integrals or if we consider double iterated integrals in the

product. Namely, we can come back to the situation of Lemma 2.2 by using the Itô-Stratonovitch formula. We put:

$$I_t^1(s, \Delta s)(x) = \frac{D}{D(x)} x_t(s)(x) \int_0^t D\Pi(x_u(s)(x)) \frac{D^{-1}}{D(x)} x_u(s)(x) d_u \Delta_s B_u(s) \quad (2.15)$$

and

$$\begin{aligned} I_t^2(s, \Delta s)(x) &= \frac{D}{D(x)} x_t(s)(x) \int_0^t \frac{D^{-1}}{D(x)} x_u(s)(x) \langle D\Pi(x_u(s)(x)), I_u^1(s, \Delta s)(x), d_u \Delta_s B_u(s) \rangle. \end{aligned} \quad (2.16)$$

By using the rule of differentiation of stochastic differential equations along a parameter [10, 25], we have that:

$$x_t(s + \Delta s)(x) = x_t(s)(x) + I_t^1(s, \Delta s)(x) + I_t^2(s, \Delta s)(x) + O(\Delta s^{3/2}). \quad (2.17)$$

The error term is uniform in  $x$  over each compact set of  $\mathbb{R}^d$ .

Let us consider a 1-form on  $[0, 1] \times S^1$ ,  $\eta = \eta_1 ds + \eta_2 dt$ . We put a Gaussian measure on the set of  $\eta$ :  $\eta_1$  and  $\eta_2$  are independent. On the space of  $\eta$  we consider a Gaussian measure whose reproducing Hilbert space is defined as follows: we consider the space of function taking values in  $\mathbb{R}^{2d}$  endowed with the Sobolev norm  $\int_{S^1} \langle (-\frac{d^2}{ds^2} + 1)\eta(s), \eta(s) \rangle ds = \|\eta\|_{H^d}^2$  and the space of forms endowed with the Hilbert norm  $\int_0^1 \|\frac{\partial}{\partial t} \eta_t(\cdot)\|_{H^d}^2 dt$ . The random forms which are obtained in that way are almost surely Hölder. Let us consider  $N = 2^{N_0}$ . We consider the polygonal approximation  $s \rightarrow x_t^N(s)(x)$  of  $s \rightarrow x_t(s)(x)$ . We consider a coordinate  $x_t^{N,j}(s)(x)$  of it. We put:

$$A_t^{N,j}(x) = \int_{S^1} \eta_2^j(s, t) d_s x_t^{N,j}(s)(x). \quad (2.18)$$

We have:

**Proposition 2.4.** *When  $N \rightarrow \infty$ ,  $A_t^{N,j}(x)$  tends in all of the  $L^p$  to a real random variable*

$$\int_{S^1} \eta_2^j(s, t) d_s x_t^j(s)(x). \quad (2.19)$$

*Moreover, the stochastic integral defined in (2.19) depends almost surely smoothly on  $x$  and in all of the  $L^p$ .*

*Proof.* We omit to write the index  $j$ , doing as if the diffusion  $x_t(s)(x)$  was one dimensional. We write:

$$A_t^N(x) = \sum A_i^N = \sum \int_{s_i}^{s_{i+1}} \eta_2(s, t) d_s x_t^N(s)(x). \quad (2.20)$$

Let us decompose  $A_i^N$  as a sum:

$$A_i^N = B_i^N + C_i^N \tag{2.21}$$

where

$$B_i^N = \eta_2(s_i, t) \Delta_{s_i} x_t(s_i)(x) \tag{2.22}$$

and where:

$$C_i^N = \int_{s_i}^{s_{i+1}} (\eta_2(s, t) - \eta_2(s_i, t)) \frac{ds}{s_{i+1} - s_i} \Delta_{s_i} x_t(s_i)(x). \tag{2.23}$$

**First step.** Convergence of  $\sum B_i^N$  in all the  $L^p$ .

We write  $\Delta s_i = 1/N$ . Moreover,

$$\begin{aligned} B_i^N &= B_{i,1}^N + B_{i,2}^N + \text{error} \\ &= \eta_2(s_i, t) I_t^1(s_i, \Delta s_i)(x) + \eta_2(s_i, t) I_t^2(s_i, \Delta s_i)(x) + \text{error}. \end{aligned} \tag{2.24}$$

Let us study first the convergence of  $\sum B_{i,1}^N$  in all the  $L^p$ . Let  $N' = 2^{N_0}$  be an integer larger than  $N$ . We write:

$$D_i^N = B_{i,1}^N - \sum_{[s_{i'}, s_{i'+1}] \subseteq [s_i, s_{i+1}]} B_{i',1}^{N'}. \tag{2.25}$$

In  $B_{i,1}^N$  and in  $I_t^1(s_i, \Delta s_i)$ , we get:

$$d_t \Delta_{s_i} B_t(s_i) = \sum_{[s_{i'}, s_{i'+1}] \subseteq [s_i, s_{i+1}]} d_t \Delta_{s_{i'}} B_t(s_{i'}) \tag{2.26}$$

and we apply Lemma 2.2 in order to get the estimate:

$$E \left[ \prod_{i_j \in I} D_{i_j}^N \right] = o(1) \prod_{i_j \in I'} \Delta s_{i_j} = o(1) C(I). \tag{2.27}$$

But there are at most  $CN^r$  set of multi-indices  $I$  such that  $|I| = p$  and  $\|I\| = r$ . Therefore the result.

Let us study the behaviour of  $\sum B_{i,2}^N$  in (2.24).

In  $I_t^2(s_i, \Delta s_i)$ , we write:

$$d_u \Delta_{s_i} B_u(s_i) d_v \Delta_{s_i} B_v(s_i) = \sum_{[s_j, s_{j+1}], [s_{j'}, s_{j'+1}] \subseteq [s_i, s_{i+1}]} d_u \Delta_{s_j} B_u(s_j) d_v \Delta_{s_{j'}} B_v(s_{j'}). \tag{2.28}$$

We select in the decomposition (2.27) the sum where we have  $s_j = s_{j'}$ , and we get a decomposition of  $B_{i,2}^N$  into  $D_{i,1}^N + D_{i,2}^N$  where in  $D_{i,1}^N = \sum D_{i,j,1}^N$  we consider only the diagonal terms.

We write:

$$\begin{aligned} \sum \eta_2(s_i, t) D_{i,1}^N - \sum \eta_2(s_j, t) B_{j,2}^{N'} \\ = \sum_{[s_j, s_{j+1}] \subseteq [s_i, s_{i+1}]} (\eta_2(s_i, t) - \eta_2(s_j, t)) D_{i,j,1}^N + \\ \sum_{[s_j, s_{j+1}] \subseteq [s_i, s_{i+1}]} \eta_2(s_j, t) (D_{i,j,1}^N - B_{j,2}^{N'}). \end{aligned} \tag{2.29}$$

The first term tends trivially to 0 in all of the  $L^p$ . By applying the remark following Lemma 2.2, the second term tends to zero in all of the  $L^p$  when  $N' \rightarrow \infty$ .

Let us consider  $\sum \eta_2(s_i, t) D_{i,2}^N$ . Let us show that it tends in all the  $L^p$  to zero. Let  $I = \{i_1, \dots, i_{|I|}\}$  with  $\|I\|$  given. According to Lemma 2.2. we have:

$$E \left[ \prod_{i_j \in I} D_{i,2}^N \right] = O(N^{-\|I\|}). \tag{2.30}$$

Hence, we can write  $D_{i,2}^N = \sum_{j \neq j'} D_{j,j',2}^N$ . If we distribute in  $(D_{i,2}^N)^r$ , there are at most  $C(N'/N)^k$  products  $\prod D_{j_l, j_{l'}, 2}^N$  where the cardinal described by  $j_l, j_{l'}$  is  $k$ . But  $k$  is at least equal to 2. Therefore (2.30). We conclude as in (2.27).

**Second step.** Convergence of  $\sum C_i^N$  in all of the  $L^p$ .

We write

$$\sum C_i^N = \sum \alpha^N(s_i) I_t^1(s_i, \Delta s_i)(x) + \text{error} \tag{2.31}$$

where  $\alpha^N(s_i)$  is independent of the system of  $I_t^1(s_i, \Delta s_i)$  and tends to 0 in all of the  $L^p$ . Therefore the sum tends to 0 by the previous considerations in all the  $L^p$ .

In order to show that the stochastic integral defined by (2.19) depends almost surely and in all of the  $L^p$  from  $x$ , we can apply the previous considerations to  $\frac{D^r}{Dx^r} A_t^{N,j}(x)$  and show that it converges in all the  $L^p$  to  $\frac{D^r}{Dx^r} \int_{S^1} \eta_2^j(s, t) dx_t^j(s)(x)$ . The Sobolev imbedding theorem allows to conclude.  $\square$

Let us introduce the stochastic Poisson-Sigma action defined as follows:

$$\begin{aligned} S(x, (\cdot)(x), \eta) = \sum_j \int_{[0,1] \times S^1} \eta^j \wedge dx_t^j(\cdot)(x) \\ + \sum_{i,j} \int_{[0,1] \times S^1} \alpha_{i,j}(x, (\cdot)(x)) \eta^i \wedge \eta^j. \end{aligned} \tag{2.32}$$

Proposition 2.4 allows us to deduce the following theorem:

**Theorem 2.5.** *The random variable  $S(x, \cdot)(x, \eta)$  is real, and is almost surely differentiable in  $x$ . For all  $r$ , all  $p \geq 1$ :*

$$\sup_x E \left[ \left| \frac{D^r}{Dx^r} S(\eta, x, \cdot)(x) \right|^p \right] < \infty. \tag{2.33}$$

This allows us to state the following theorem: let  $C_b^\infty(\mathbb{R}^d)$  be the Fréchet space of smooth functions  $f$  on  $\mathbb{R}^d$  with bounded derivatives at each order endowed with the set of semi-norms:

$$\|f\|_{r, \infty} = \sup_x \left| \frac{D^r}{Dx^r} f(x) \right|. \tag{2.34}$$

**Theorem 2.6.** *The map which sends  $(f, g)$  to:*

$$E[f(x_1(1)(x))g(x_1(2)(x)) \exp[iS(x, \cdot)(x), \eta]] \tag{2.35}$$

*is a continuous bilinear application from  $C_b^\infty(\mathbb{R}^d)$  into  $C_b^\infty(\mathbb{R}^d)$ . 1 and 2 denote in (2.35) two different points of  $S^1$ .*

### 3. A stochastic star product

Let us recall that, if the Malliavin Calculus has a lot of precursors (see the work of Hida, Elworthy, Fomin, Albeverio . . . ), the main novelty of the Malliavin Calculus was to complete the differential operations known at that time on the Wiener space in all of the  $L^p$ . This allowed Malliavin to recover Hörmander’s theorem by probabilistic methods [38]. The first ones who have applied the Malliavin Calculus to other Gaussian spaces than the traditional Wiener space are Nualart and Sanz [42] in order to study the Brownian sheet. Here, we apply the Malliavin Calculus in our situation.

We consider the space  $H(B)$  of maps from  $[0, 1]$  into  $H$ ,  $h_t(\cdot)(B)$ , such that

$$\int_0^1 \left\| \frac{\partial}{\partial t} h_t(\cdot)(B) \right\|_B^2 dt < \infty \tag{3.1}$$

and the space  $H(\eta)$  of maps from  $[0, 1]$  into  $H^d$ ,  $h_t(\cdot)(\eta)$ , such that

$$\int_0^1 \left\| \frac{\partial}{\partial t} h_t(\cdot)(\eta) \right\|_{H^d}^2 dt < \infty. \tag{3.2}$$

$H(B)$  is the Hilbert reproducing space of the Gaussian field  $B(\cdot)$  and  $H(\eta)$  is the Hilbert reproducing space of the Gaussian field  $\eta$ .

If  $F$  is a functional which is  $B_t(\cdot)$  and  $\eta$  measurable, we take its derivative in the direction of  $H(B)$  and  $H(\eta)$ .  $\nabla^r F$  is therefore a random element of  $(H(B) \oplus H(\eta))^{\otimes r}$ . We consider its  $L^p$  norm and we get:

$$\|F\|_{r,p} = E[\|\nabla^r F\|^p]^{1/p} \tag{3.3}$$

which is the collection of Sobolev norms in the sense of the Malliavin Calculus [41].  $F$  is said to be smooth in the Malliavin sense if  $\|F\|_{r,p} < \infty$  for all  $r$  and  $p$ .

**Lemma 3.1.**  $\frac{D^r}{Dx^r} x_t(s)(x)$  and  $\frac{D^{-1}}{Dx} x_t(s)(x)$  are smooth in the sense of Malliavin. Moreover their Sobolev norms are bounded in  $s, t \in [0, 1]$  and  $x$ , and the kernel of their derivatives are  $B_t(s)$ -measurable.

*Proof.* This result is classical [41] if we consider these functionals as  $B_t(s)$ -measurable. But

$$d/dt h_t(s)(B) = d/dt \langle h_t(\cdot)(B), e(\cdot - s) \rangle. \tag{3.4}$$

Therefore the result. □

**Proposition 3.2.**  $\frac{D^r}{Dx^r} A_t^{N,j}(x)$  tends to  $\frac{D^r}{Dx^r} \int_{S^1} \eta_2^j(s, t) d_s x_t^j(s)(x)$  in all the Sobolev spaces and the Sobolev norms of this last stochastic integral are bounded in  $x \in \mathbb{R}^d$ .

*Proof.* If we do not take the derivatives of  $d_u \Delta_s B_u(s)$  and  $d_v \Delta_s B_v(s) d_u \Delta_s B_u(s)$  in (2.15) and in (2.16), the result goes by the same methods as the proof of Proposition 2.4, by applying Lemma 3.1. Let us take the derivatives of  $d_u \Delta_s B_u(s)$  in (2.15) and (2.16). They are given by  $\frac{\partial}{\partial u} \Delta_s h_u(s)(B) = \frac{\partial}{\partial u} \langle h_u(\cdot)(B), e(\cdot - s - \Delta s) - e(\cdot - s) \rangle_H$  and therefore the treatment leads to simpler considerations than in the statement of Proposition 2.4. □

We deduce from Proposition 3.2 that  $\frac{D^r}{Dx^r} S(x(\cdot)(x), \eta)$  is bounded in  $x$  in all the Sobolev spaces. We get, since the stochastic Poisson-Sigma action  $S(x(\cdot)(x), \eta)$  is real, that:

**Proposition 3.3.** Let  $\mu(x)$  be the measure on  $\mathbb{R}^d$  which sends  $h \in C_b(\mathbb{R}^d)$  to:

$$E[f(x_1(1)(x))g(x_1(2)(x))h(x_1(\infty)(x)) \exp[iS(x(\cdot)(x), \eta)]] \tag{3.5}$$

where  $f$  and  $g$  belong to  $C_b^\infty(\mathbb{R}^d)$ .  $\mu(x)$  has a density  $q(x, y)$  with respect to the Lebesgue measure and the uniform norm of  $\frac{D^r}{Dx^r} \frac{D^{r'}}{Dy^{r'}} q(x, y)$  can be estimated in terms of the uniform norms of the derivatives of  $f$  and  $g$ .

*Proof.* This comes from the fact that  $\frac{D^r}{Dx^r} \exp[iS(x(\cdot)(x), \eta)]$  and  $\frac{D^r}{Dx^r} x_1(s)(x)$  have bounded Sobolev norms in the sense of the Malliavin Calculus in  $x$  and from the Malliavin Calculus [41]. □

**Proof of Theorem A.**  $x_1(\infty)(x)$  is given by a diffusion on  $\mathbb{R}^d$ . Its law has a smooth density  $p_1(x, y) > 0$  with bounded derivatives of all orders in  $x$  and  $y$ . By using the Airault-Malliavin-Sugita procedure [1, 44], we get :

$$\frac{\mu(x, x)}{p_1(x, x)} = E[f(x_1(1)(x))g(x_1(2)(x)) \exp[iS(x.(.)(x), \eta)] | x_1(\infty)(x) = x]. \quad (3.6)$$

Then the result follows, since  $p_1(x, x) > c > 0$ . □

### 4. Semi-classical analysis

Following [40] and [18], let us put  $\epsilon = h^{1/2}$ . We replace  $B.(.)$  by  $\epsilon B.(.)$  and  $\eta$  by  $\epsilon\eta$ . We get a random field  $x.(.)(\epsilon)(x)$ .

By using the classical rules of differentiation of  $x_t(s)(\epsilon)(x)$  along the parameter  $\epsilon$  and  $x$  [10, 39, 25] and considerations analog to Lemma 3.1, we get:

**Lemma 4.1.**  $\frac{D^{r'}}{D\epsilon^{r'}} \frac{D^r}{Dx^r} x_t(s)(\epsilon)(x)$  and  $\frac{D^{r'}}{D\epsilon^{r'}} \frac{D^{-1}}{Dx} x_t(s)(\epsilon)(x)$  are smooth in the sense of Malliavin for the total Gaussian space. Moreover, their Sobolev norms are bounded in  $s, t \in [0, 1], \epsilon \in [0, 1]$  and  $x$  in  $\mathbb{R}^d$  and the kernels of their derivatives are  $B.(s)$ -measurable.

We get by adding the new parameter  $\epsilon$ :

**Proposition 4.2.**  $\frac{D^{r'}}{D\epsilon^{r'}} \frac{D^r}{Dx^r} A_t^{N,j}(\epsilon)(x)$  tends to  $\frac{D^{r'}}{D\epsilon^{r'}} \frac{D^r}{Dx^r} \int_{S^1} \epsilon \eta_2^j(s, t) d_s x_t^j(s)(\epsilon)(x)$  in all of the Sobolev spaces of the Malliavin Calculus. The Sobolev norms in the sense of Malliavin Calculus of the last stochastic integral are bounded in  $x \in \mathbb{R}^d$  and  $\epsilon \in [0, 1]$ . Moreover, they are 0 if  $r' = 0$  or  $r' = 1$ .

We get:

**Proposition 4.3.** Let  $\mu_\epsilon(x)$  be the measure on  $\mathbb{R}^d$  which to  $h \in C_b(\mathbb{R}^d)$  assigns:

$$E \left[ f(x_1(1)(\epsilon)(x))g(x_1(2)(\epsilon)(x)) \exp[i/\epsilon^2 S(x.(.)(\epsilon)(x), \epsilon\eta)] h(x_1(\infty)(\epsilon)(x)) \right] \quad (4.1)$$

where  $f$  and  $g$  belong to  $C_b^\infty(\mathbb{R}^d)$ .  $\mu_\epsilon(x)$  has a density  $q_\epsilon(x)$  ( $\epsilon > 0$ ) and when  $\epsilon \rightarrow 0$ :

$$q_\epsilon(x, x) = \epsilon^{-d} \sum_{i=1}^n h^i \tilde{Q}_i(f, g)(x) + O(h^n) \quad (4.2)$$

where  $\tilde{Q}_i$  are differential operators in  $f$  and  $g$ .

*Proof.*  $q_\epsilon(x, x) = \epsilon^{-d} \tilde{q}_\epsilon(x, 0)$  where  $\tilde{q}_\epsilon(x, y)$  is the density of the measure  $\nu_\epsilon$ :

$$E \left[ f(x_1(1)(\epsilon)(x)) g(x_1(2)(\epsilon)(x)) \exp[i/\epsilon^2 S((x, \cdot)(\epsilon)(x), \epsilon \eta)] h \left( \frac{x_1(\infty)(\epsilon)(x) - x}{\epsilon} \right) \right]. \tag{4.3}$$

The result follows by standard arguments of Malliavin Calculus depending on a parameter [28, 27, 45] because, in all the Sobolev spaces of Malliavin Calculus, when  $\epsilon \rightarrow 0$ ,  $x_1(1)(\epsilon)(x) \rightarrow x$ ,  $x_1(2)(\epsilon)(x) \rightarrow x$ ,  $\frac{x_1(\infty)(\epsilon)(x) - x}{\epsilon}$  tends to the nondegenerate Gaussian variable  $\Pi(x)B_1(\infty)$  and

$$\epsilon^{-2} S(x, \cdot)(\epsilon)(x), \epsilon \eta \rightarrow \sum_{i,j} \alpha_{i,j}(x) \int_D \eta_i \wedge \eta_j + \sum_i \int_D \eta_i \wedge dB_i. \tag{4.4}$$

Let us write the measure  $\nu_\epsilon$

$$h \rightarrow E[G_\epsilon h(Z_\epsilon)] \tag{4.5}$$

$G_\epsilon$  depends smoothly on  $\epsilon$  in all of the Sobolev spaces of the Malliavin Calculus as well as  $Z_\epsilon$ . Moreover  $Z_\epsilon$  satisfies uniformly in  $\epsilon$  Malliavin’s nondegeneracy condition:  $\sup_\epsilon E[\langle \nabla Z_\epsilon, \nabla Z_\epsilon \rangle^{-p}] < \infty$  for all positive integers  $p$ .

We have:

$$\frac{d^r}{d\epsilon^r} \nu_\epsilon h = \sum_{|(r')| \leq r} E \left[ \tilde{G}_{(r')}(\epsilon) \frac{\partial^{(r')}}{\partial y^{(r')}} h(Z_\epsilon) \right]. \tag{4.6}$$

But by Malliavin’s condition of nondegeneracy, we can remove the derivative of  $h$  in (4.6) and we get

$$\frac{d^r}{d\epsilon^r} \nu_\epsilon f = E[\overline{G}_r(\epsilon) h(Z_\epsilon)] \tag{4.7}$$

Therefore  $\tilde{q}(x, 0)$  is smooth in  $\epsilon$ .

At  $\epsilon = 0$ , the Malliavin matrix of the Gaussian  $Z_0$  is deterministic and  $\overline{G}_r(0)$  contains only expressions in the Gaussian terms which are of the same parity as  $r$ . If  $r$  is odd

$$E[\overline{G}_r(0) | \Pi(x)B_1(\infty) = 0] = 0 \tag{4.8}$$

because we consider centered Gaussian variables.  $\tilde{q}_\epsilon(x, 0)$  has therefore an asymptotic expansion  $\sum \epsilon^i \overline{Q}_i(f, g)(x)$  but only the even powers of  $\epsilon$  remain in this asymptotic expansion: namely the odd exponent leads to the expectation of odd functionals of some centered Gaussian measures, which are 0. The introduction of derivatives of  $f$  and  $g$  is due to the asymptotic expansion of  $f(x_1(1)(\epsilon)(x))$  and of  $g(x_1(2)(\epsilon)(x))$  when  $\epsilon \rightarrow 0$  because  $x_1(1)(\epsilon)(x)$  and  $x_1(2)(\epsilon)(x)$  go to  $x$  in all the Sobolev spaces of the Malliavin Calculus when  $\epsilon \rightarrow 0$ .  $\square$

On the other hand,  $p_\epsilon(x, x)$  has an asymptotic expansion:

$$p_\epsilon(x, x) = \epsilon^{-d} \sum_{i=1}^n h^i c_i(x) + O(h^n) \quad (4.9)$$

where the coefficients belong to  $C_b^\infty(\mathbb{R}^d)$  and  $c_0(x) > c > 0$ .

**Proof of theorem B.** We get

$$f *_{st,h} g(x) = \frac{\sum h^i \tilde{Q}_i(f, g)(x) + O(h^n)}{\sum_{i=0}^n h^i c_i(x) + O(h^n)}. \quad (4.10)$$

The result holds because  $c_0(x) > c > 0$ .  $\square$

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