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Hartogs theorem for forms: solvability of Cauchy-Riemann operator at critical degree

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Abstract. The Hartogs Theorem for holomorphic functions is generalized in two settings: a CR version (Theorem 1.2) and a corresponding theorem based on it for C^k $\bar\partial$-closed forms at the critical degree, $0 \le k \le \infty$ (Theorem 1.1). Part of Frenkel's lemma in C^k category is also proved.

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1. Introduction

Let P_N denote the unit polydisc in $\mathbb{C}^N$, $N \ge 1$. In $\mathbb{C}^{m+1}$, $m \ge 1$, set

$$\omega = P_m \times \left\{ z_{m+1} \in \mathbb{C} \,\middle|\, \frac{1}{2} < |z_{m+1}| < 1 \right\}, \quad m \ge 1.$$

The classical Hartogs theorem (see [9, p. 55]) states: suppose, for a given holomorphic function f on ω, there is an open set $U \subset P_m$ such that f has a holomorphic extension to $U \times \{z_{m+1} \in \mathbb{C} |\ |z_{m+1}| < 1|\}$, then f can be extended holomorphically to P_{m+1}. This phenomenon in higher fiber dimension is suggested by Frenkel's lemma (see [13, p. 15]).

Let n always be an integer bigger than 1. For $z \in \mathbb{C}^{m+n}$ we write $z = (z', z'')$ with $z' = (z_1, \dots, z_m)$ and $z'' = (z_{m+1}, \dots, z_{m+n})$. Set

$$\Omega = P_m \times \left(P_n \setminus \frac{1}{2} P_n \right)$$

where $\frac{1}{2} P_n = \{z'' \in \mathbb{C}^n |\quad 2z'' \in P_n\}$. The first part of Frenkel's lemma says: the Cauchy-Riemann equation

$$\bar\partial u = f, \quad f \in C^\infty_{(0,q)}(\Omega), \quad 1 \le q \le m+n, \quad \bar\partial f = 0$$

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always has a solution $u \in C^\infty_{(0,q-1)}(\Omega)$ except $q = n-1$. From now on a $(0, n-1)$ form will be called of the "critical degree".

Note that in Hartogs theorem the open set U can be replaced by a subset A of P_m such that A is not contained in a subvariety of codimension one in P_m (see [12, p. 16]). The following is our first theorem.

Theorem 1.1. *With Ω, A as above. Let f be a C^k $\bar\partial$-closed $(0, n-1)$ form on Ω, $0 \le k \le \infty$. For $z' \in P_m$, let $\gamma_{z'} = \Omega \cap (\{z'\} \times \mathbb{C}^n)$ be the fiber over z' in Ω. For every $z' \in A$, suppose the Cauchy-Riemann equation on $\gamma_{z'}$,*

$$\bar\partial_{\gamma_{z'}} v = f,$$

is solvable. Then the Cauchy-Riemann equation on Ω

$$\bar\partial v = f$$

is solvable with $v \in C^k(\Omega)$.

We explain the notation for the (tangential) Cauchy-Riemann operator used in this paper. In general, if there is no ambiguity, it is denoted by $\bar\partial$; if the ground space X is specified, we use $\bar\partial_X$ to denote the Cauchy-Riemann (or tangential Cauchy-Riemann) operator on X. Also in integral representations, we usually use ζ for the dummy variable and z for the resulting variable. In this case, the notation $\bar\partial_\zeta$ (respectively, $\bar\partial_z$) denotes the (tangential) Cauchy-Riemann operator with respect to ζ (respectively, z) variable.

The proof of Theorem 1.1 depends on Theorem 1.2 which is the CR version of Theorem 1.1. Let ρ be a C^k ($k \ge 3$) real valued function in $\mathbb{C}^{m+n}$ which is strictly plurisubharmonic in a neighborhood of $\{\rho \le 0\}$. Let σ be a C^k real valued function in $\mathbb{C}^m$, strictly plurisubharmonic in a neighborhood of $\{\sigma \le 0\}$. We also assume that $\{\sigma < 0\}$ is connected and relatively compact in $\mathbb{C}^m$. Set

$$M = \{\rho = 0\} \cap (\{\sigma < 0\} \times \mathbb{C}^n).$$

Assume that $d\rho$ (respectively, $d\sigma$) does not vanish on $\{\rho = 0\}$ (respectively, $\{\sigma = 0\}$) and $d\rho \wedge d\sigma \ne 0$ on ∂M. Let f be a $(0, q)$ form on M such that $\bar\partial_M f = 0$ in distribution sense. It is proved in [1] that if $f \in L^p_{(0,q)}(M)$ (*i.e.* f is a $(0, q)$ form with coefficients in L^p), $1 \le p \le \infty$, then

$$\bar\partial_M u = f \tag{$*$}$$

is solvable on M with $u \in L^p_{(0,q-1)}(M)$ whenever $1 \le q < n-1$. In the case $q = n-1$, the above equation is solvable with $f \in C^0_{(0,q)}(\bar M)$ (*i.e.* coefficients of f are continuous on $\bar M$,) if and only if f satisfies the moment condition. Our main result is the following:

Theorem 1.2. *Let M be as above and A be a subset of $\{\sigma < 0\}$ not contained in any subvariety of codimension one in $\{\sigma < 0\}$. Let f be a $\bar\partial$-closed $(0, n-1)$ form with $f \in C^0_{(0,n-1)}(\bar M) \cap C^{k'}(M)$, $0 \le k' \le k-3$. The equation $(*)$ is solvable with $u \in C^{k'}_{(0,n-2)}(M)$ if and only if all the holomorphic moments of f on $\Gamma_{z'}$, $z' \in A$ vanish, in other words, for every $z' \in A$*

$$\int_{\Gamma_{z'}} h(z', \zeta'') f(z', \zeta'') d\zeta'' = 0$$

for every function h holomorphic near $\Gamma_{z'}$, where $\Gamma_{z'} = M \cap (\{z'\} \times \mathbb{C}^n)$ is the fiber over z' in M.

Remark 1.3.

(a) When $\Gamma_{z'}$ is strictly pseudoconvex in $\{z'\} \times \mathbb{C}^n$, it is well-known that the solvability of the tangential Cauchy-Riemann equation $\bar\partial_{\Gamma_{z'}} u = f$ with f of top degree is equivalent to the vanishing of all holomorphic moments of f on $\Gamma_{z'}$. So if $\Gamma_{z'}$ is strictly pseudoconvex in $\{z'\} \times \mathbb{C}^n$ for every $z' \in A$, the statement of Theorem 1.2 can be phrased as:

The equation $(*)$ is solvable with $u \in C^0(M)$ if and only if $\bar\partial_{\Gamma_{z'}} v = f$ is solvable for every $z' \in A$.

(b) The CR function version of Hartogs theorem was proved in [2] where the condition for A is stronger but no convexity condition or boundary regularity is required for M.

We recall the representaion formula for $\bar\partial_b$-closed form f on M (cf. [1]):

$$(-1)^q f(z) = \bar\partial \left\{ \int_M f(\zeta) \wedge \Omega_{q-1}(\mathfrak{r}, \mathfrak{r}^*)(\zeta, z) + (-1)^q \int_{\partial M} f(\zeta) \wedge \Omega(\mathfrak{r}, \mathfrak{r}^*, \mathfrak{s})(\zeta, z) \right\}$$
$$+ \int_{\partial M} f(\zeta) \wedge \Omega_q(\mathfrak{r}^*, \mathfrak{s})(\zeta, z), \quad z \in M \tag{2.7}$$

where $\Omega(\mathfrak{r}, \mathfrak{r}^*)$, $\Omega(\mathfrak{r}, \mathfrak{r}^*, \mathfrak{s})$, $\Omega(\mathfrak{r}^*, \mathfrak{s})$ are defined in Section 2. It is clear from this representation that the obstruction to the solvability of $(*)$ is the last integral which is null when $q < n-1$ by type consideration. We therefore in Section 2 define for $f \in C^0_{(0,n-1)}(\bar M)$, $\bar\partial_M f = 0$ in distribution sense, the following transform:

$$Tf(z) = (-1)^{n-1} \int_{\partial M} f(\zeta) \wedge \Omega(\mathfrak{r}^*, \mathfrak{s})(\zeta, z), \tag{2.8}$$

which may be taken as the global moment of f (cf. ((3.1′))). Section 3 consists of the properties of the operator T needed in this paper. For f in the domain of T

we show that Tf is defined in a set containing M and is $\bar{\partial}$-closed there, so (2.7) becomes a "jump formula" for f (see Remark 3.3(a) for more details). Next, we show that T can be defined locally over the base set $\{\sigma < 0\}$. Finally, the operator T is proved to be just defined fiberwise over every point $z' \in \{\sigma < 0\}$. This enables us to define T on M with arbitrary base set B in $\mathbb{C}^m$. In section 4 we define the moment operator $\mathcal{M}_h$ for Tf (or f) with respect to a holomorphic function h. It turns out that $\mathcal{M}_h(f)$ is a holomorphic function in the base set. Using this property we prove a more general Theorem 4.4 which implies Theorem 1.2 immediately. Section 5 contains the proof of Theorem 1.1. The interesting thing here is a procedure which improves the method in [11] to produce a C^k solution for $0 \le k \le \infty$. The results of this paper hold for $(p, n-1)$ forms, $0 \le p \le n$. For simplicity, we only deal with the case $p = 0$.

Finally, closely related to the topics in this paper, there is another Hartogs theorem (see [9, p. 56, 63]) whose higher dimensional analogue is written in a forthcoming paper.

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2. Preliminaries

We write $\zeta \in \mathbb{C}^{m+n}$ as (ζ', ζ'') where $\zeta' \in \mathbb{C}^m$ and $\zeta'' \in \mathbb{C}^n$. Similarly, for differential forms we write $df = (d'f, d''f)$, and $\partial f = (\partial' f, \partial'' f)$, where $d'\cdot$, $d''\cdot$ denote respectively the differentials with repect to the first $2m$ variables and those with respect to the last $2n$ variables; likewise for $\partial'\cdot$ $(\bar{\partial}'\cdot)$ and $\partial''\cdot$ $(\bar{\partial}''\cdot)$. Also we use $d\zeta = d\zeta_1 \wedge \ldots \wedge d\zeta_{m+n}$, $d\zeta' = d\zeta_1 \wedge \ldots \wedge d\zeta_m$ and $d\zeta'' = d\zeta_{m+1} \wedge \ldots \wedge d\zeta_{m+n}$; similarly for $d\bar{\zeta}$, $d\bar{\zeta}'$, $d\bar{\zeta}''$, etc..

The following notations and exterior calculus developed by Harvey and Polking [5] will be used in construting kernels needed in this paper:

Let $E^1, \ldots, E^\alpha$ (which are called sections) be a collection of N-tuples of C^2 functions in $(\zeta, z) \in \mathbb{C}^N \times \mathbb{C}^N$. Following Harvey-Polking [5] we use

$$\Omega(E^1, \ldots, E^\alpha) = \frac{\langle E^1, \vec{d\zeta}\rangle}{\langle E^1, \zeta - z\rangle} \wedge \cdots \wedge \frac{\langle E^\alpha, \vec{d\zeta}\rangle}{\langle E^\alpha, \zeta - z\rangle} \wedge \sum_{\lambda_1 + \ldots + \lambda_\alpha = N-\alpha} \left(\frac{\langle \bar{\partial}_{\zeta,z} E^1, \vec{d\zeta}\rangle}{\langle E^1, \zeta - z\rangle}\right)^{\lambda_1} \wedge \cdots \wedge \left(\frac{\langle \bar{\partial}_{\zeta,z} E^\alpha, \vec{d\zeta}\rangle}{\langle E^\alpha, \zeta - z\rangle}\right)^{\lambda_\alpha} \tag{2.1}$$

where $\langle x, y\rangle = \sum x_i y_i$ for vectors x, y in $\mathbb{C}^N$ and $\vec{d\zeta}$ here is understood to be the N-vector $(d\zeta_1, \ldots, d\zeta_N)$. Then Ω is C^1 away from the singular set $Z = \bigcup_1^\alpha \{(\zeta, z) | \langle E^j, \zeta - z\rangle = 0\}$. We can rewrite Ω as $\Omega(E^1, \ldots, E^\alpha) = \sum_0^{N-1} \Omega_q(E^1, \ldots, E^\alpha)$, where Ω_q is the sum of components of Ω which are of degree q in $d\bar{z}_j,\ j = 1, \ldots, N$. Outside the singular set Z we have the following identity:

$$\bar{\partial}_{\zeta,z}\Omega(E^1, \ldots, E^\alpha) = \sum_{j=1}^{\alpha}(-1)^j \Omega(E^1, \ldots, \widehat{E^j}, \ldots, E^\alpha). \tag{2.2}$$

To construct the sections we use the results of Fornæss [3] which we briefly outline in the following and refer to [3] for details:

We first observe that for any strongly convex domain $G \subset \mathbb{C}^N$ with C^k, $k \geq 2$ boundary, there exist a C^k function μ with positive real Hessian, a constant $c > 0$ such that $G = \{\mu < 0\}$, and $d\mu \neq 0$ in a neighborhood of ∂G. Furthermore, if we define

$$\mathcal{H}(\xi, \eta) = \sum_1^N \frac{\partial \mu}{\partial \xi_j}(\xi)(\xi_j - \eta_j),$$

then it satisfies

$$\mathcal{H}(\xi, \eta) \geq \mu(\xi) - \mu(\eta) + c|\xi - \eta|^2$$

for all ξ,η in a small neighborhood of $\bar{G}$. The section $(\frac{\partial \mu}{\partial \xi_1}(\xi), \cdots, \frac{\partial \mu}{\partial \xi_N}(\xi))$ will serve the purpose of this paper in case the given domain is strongly convex.

On the other hand, Fornæss proved in [3] that any strongly pseudoconvex domain $X \subset \mathbb{C}^N$ with C^k, $k \geq 2$ boundary, admits an embedding into a bounded strongly convex domain $Y \subset \mathbb{C}^{N'}$ with C^k boundary for some N', such that the boundary of X is mapped into the boundary of Y, and the map is 1-1 holomophic in a neighborhood of $\bar{X}$ (cf. [3, Theorem 9] for explicit statements).

Now for any strongly pseudoconvex domain $X \subset \mathbb{C}^N$ with C^k, $k \geq 2$ boundary, the above oberservation and Fornæss' embedding theroem together imply the existence of $\mathcal{H}$ and the section for $Y \subset \mathbb{C}^{N'}$. Their pull-backs to $\mathbb{C}^N$ then give the following resuts (cf. [3, Theorem 16]):

There exist a C^k function ν which is strictly plurisubharmonic in a neighborhood of $\bar{X}$ with $X = \{\nu < 0\}$, a constant $\epsilon > 0$ and a function $H(\xi, \eta) \in C^{k-1}(X_\epsilon \times X_\epsilon)$, where $X_\epsilon = \{\eta \in \mathbb{C}^N, \nu(\eta) < \epsilon\}$ satisfying

$$H(\xi, \cdot) \quad \text{is holomorphic in} \quad X_\epsilon, \tag{2.3}$$

$$\exists\, n_j(\xi, \eta) \in C^{k-1}(X_\epsilon \times X_\epsilon),\ j = 1, \ldots, N,\ \text{holomorphic in}\ \eta,\ \text{such that} \tag{2.4}$$

$$H(\xi, \eta) = \sum_1^N n_j(\xi, \eta)(\xi_j - \eta_j),$$

$$\exists\, c > 0,\ \text{such that}\ \forall\, \eta \in \bar{X}, \xi \in \bar{X} \tag{2.5}$$

$$2Re\ H(\xi, \eta) \geq \nu(\xi) - \nu(\eta) + c|\xi - \eta|^2,$$

$$d_\xi H(\xi, \eta)|_{\xi = \eta} = \partial \nu(\xi). \tag{2.6}$$

Let ρ, σ be as in Section 1. In view of the above discussion, for the strongly pseudoconvex domain $\{\rho < 0\} \subset \mathbb{C}^{m+n}$ there exist $\mathfrak{r}_j$'s which correspond to the n_j's in (2.4), and we define the section $\mathfrak{r}(\zeta, z)$ as $(\mathfrak{r}_1, \dots, \mathfrak{r}_{m+n})$. Similarly for the domain $\{\sigma < 0\} \subset \mathbb{C}^m$ we define the section $\mathfrak{s}'(\zeta', z') = (\mathfrak{s}_1, \dots, \mathfrak{s}_m)$. We then use $\mathfrak{s}(\zeta, z)$ for the section $(\mathfrak{s}_1(\zeta', z'), \dots, \mathfrak{s}_m(\zeta', z'), \underbrace{0, \cdots, 0}_{n})$. Let $\mathfrak{r}^*(\zeta, z) = (\mathfrak{r}_1^*(\zeta, z), \dots, \mathfrak{r}_{m+n}^*(\zeta, z))$, where $\mathfrak{r}_j^*(\zeta, z) = -\mathfrak{r}_j(z, \zeta)$. Thus $\mathfrak{r}$, $\mathfrak{r}^*$ and $\mathfrak{s}$ are C^{k-1} in a neighborhood of $\bar{M} \times \bar{M}$.

The kernels $\Omega(\mathfrak{r}, \mathfrak{r}^*)$, $\Omega(\mathfrak{r}, \mathfrak{r}^*, \mathfrak{s})$, $\Omega(\mathfrak{r}^*, \mathfrak{s})$ etc., are defined according to formula (2.1).

For $f \in C^0_{(0,q)}(\bar{M})$, satisfying $\bar{\partial}_M f = 0$ in distribution sense on M with $1 \le q \le n + m - 1$, we recall the following basic representation formula from [1, p. 543]: for $z \in M$

$$(-1)^q f(z) = \bar{\partial}\left\{\int_M f(\zeta) \wedge \Omega_{q-1}(\mathfrak{r}, \mathfrak{r}^*)(\zeta, z) + (-1)^q \int_{\partial M} f(\zeta) \wedge \Omega(\mathfrak{r}, \mathfrak{r}^*, \mathfrak{s})(\zeta, z)\right\}$$

$$+ \int_{\partial M} f(\zeta) \wedge \Omega_q(\mathfrak{r}^*, \mathfrak{s})(\zeta, z). \tag{2.7}$$

The last integral in (2.7) is null when $q < n - 1$ by type consideration. For $f \in C^0_{(0,n-1)}(\bar{M})$, $\bar{\partial}_M f = 0$ in distribution sense, we define the following transform:

$$Tf(z) = (-1)^{n-1} \int_{\partial M} f(\zeta) \wedge \Omega(\mathfrak{r}^*, \mathfrak{s})(\zeta, z). \tag{2.8}$$

Remark 2.1. A $(0, n - 1)$ form f defined in a subset of $\mathbb{C}^{m+n}$ can be decomposed as follows:

$$f = \sum_{j=1}^{s} f_j \text{ where } f_j = \sum_{\substack{|\alpha|+|\beta|=n-1 \\ |\alpha|=j-1}} f_{j\alpha} d\bar{z}'^{\alpha} d\bar{z}''^{\beta} \text{ and } s = \min(m + 1, n). \tag{2.9}$$

Moreover, when f is $\bar{\partial}$-closed we have:

$$\bar{\partial}_{z''} f_1 = 0,\ \bar{\partial}_{z'} f_j = -\bar{\partial}_{z''} f_{j+1},\ j = 1, \dots, s - 1,\ \text{and } \bar{\partial}_{z'} f_s = 0. \tag{2.10}$$

The definition of T immediately gives

$$Tf = Tf_1 = (Tf)_1. \tag{2.11}$$

3. Properties of Tf

In this section we always assume that $f \in C^0_{(0,n-1)}(\bar{M})$ and $\bar{\partial}_M f = 0$ in distribution sense.

Lemma 3.1. *$Tf = 0$ if there exists $u \in C^0_{(0,n-2)}(\bar{M})$ such that $\bar\partial_M u = f$ on M.*

Proof. Suppose there exists $u \in C^0_{(0,n-2)}(\bar{M})$ satisfying $\bar\partial_M u = f$. By the definition of Tf, we have

$$Tf = (-1)^{n-1} \int_{\partial M} \bar\partial_\zeta u(\zeta) \wedge \Omega(\mathfrak{r}^*, \mathfrak{s})(\zeta, z) = \int_{\partial M} u \wedge \bar\partial_\zeta \Omega(\mathfrak{r}^*, \mathfrak{s})(\zeta, z)$$

by Stokes' theorem. Invoking (2.2) the last integral becomes

$$\int_{\partial M} u \wedge (\Omega(\mathfrak{r}^*) - \Omega(\mathfrak{s}) - \bar\partial_z \Omega(\mathfrak{r}^*, \mathfrak{s})) = 0$$

by type considerations. This proves the lemma. □

Lemma 3.2. *There is an open neighborhood $\mathcal{N}$ of M in $\{\sigma < 0\} \times \mathbb{C}^n$, depending on ρ only, such that $Tf \in C^1(\mathcal{N})$. On the set $U = \mathcal{N} \cap \{\rho \geq 0\}$ we have $\bar\partial(Tf) = 0$.*

Proof. Observe that in the definition of T the integration is just over ∂M. By (2.3)-(2.6), we see that Tf is well-defined for z in an open set $\mathcal{N}$, depending on ρ only, containing M in $\{\sigma < 0\} \times \mathbb{C}^n$ and is C^{k-2} there.

We show that $\bar\partial(Tf) = 0$ in the interior of U. Consider $m > 1$ first. The identity (2.2) and type considerations imply $\bar\partial_z \Omega(\mathfrak{r}^*, \mathfrak{s}) = -\bar\partial_\zeta \Omega(\mathfrak{r}^*, \mathfrak{s})$ in this case. Thus

$$\bar\partial(Tf) = (-1)^{n-1} \int_{\partial M} f \wedge \bar\partial_z \Omega(\mathfrak{r}^*, \mathfrak{s}) = (-1)^n \int_{\partial M} f \wedge \bar\partial_\zeta \Omega(\mathfrak{r}^*, \mathfrak{s})$$
$$= -\int_{\partial M} \bar\partial_\zeta (f \wedge \Omega(\mathfrak{r}^*, \mathfrak{s})) = -\int_{\partial M} d_\zeta (f \wedge \Omega(\mathfrak{r}^*, \mathfrak{s})) = 0$$

by Stokes' theorem. For $m = 1$, the section $\mathfrak{s}$ is the Cauchy kernel which is holomorphic in both ζ and z. Hence (2.2) and type consideration give

$$\bar\partial_z(Tf) = (-1)^{n-1} \int_{\partial M} f \wedge \Omega(\mathfrak{r}^*).$$

For $z \in U \setminus M$ we can apply Stokes' theorem to the above integral and get $\bar\partial_z(Tf) = 0$ as $\bar\partial f = \bar\partial_\zeta \mathfrak{r}^* = 0$.

Since $Tf \in C^{k-2}(\mathcal{N})$, we conclude that $\bar\partial(Tf) = 0$ on U by continuity. The lemma is proved. □

Remark 3.3.

(a) In (2.7) the form inside the parenthesis after $\bar\partial_M$ is in $C^1(M)$ provided that f is in $C^1(M)$, and so it can be extended to a C^1 form in $\{\sigma < 0\} \times \mathbb{C}^n$. By Lemma 3.2 the last form in (2.7) is actually in $C^{k-2}(U)$ and is $\bar\partial$-closed. Therefore, in this case, (2.7) becomes a "jump formula" for f. A more general jump formula can be found in [2], but we don't need it here.

(b) In view of (2.10),(2.11) and Lemma 3.2, we see that all the coefficients of Tf are holomorphic in z'.

Let $\tilde{\sigma}$ be a C^1 function defined in a neighborhood of $\{\sigma \le 0\}$ such that $\{\tilde{\sigma} < 0\} \subset \{\sigma < 0\}$. Set $\tilde{M} = \{\rho = 0\} \cap (\{\tilde{\sigma} < 0\} \times \mathbb{C}^n)$. Suppose $d\rho \wedge d\tilde{\sigma} \neq 0$ on $\partial\tilde{M}$. Denote by $b' = (\tilde{b}', \underbrace{0, \cdots, 0}_{n})$, where $\tilde{b}'$ is the section for the Bochner-Martinelli kernel in $\mathbb{C}^m$. For f in the domain of T, define

$$T'f(z) = (-1)^{n-1} \int_{\partial\tilde{M}} f \wedge \Omega(\mathfrak{r}^*, b')(\zeta, z).$$

Lemma 3.1 and the proof of Lemma 3.2 still hold for T' and consequently $T'f$ is a $\bar{\partial}$-closed form in $\tilde{U} = \mathcal{N} \cap \{\rho \ge 0\} \cap (\{\tilde{\sigma} < 0\} \times \mathbb{C}^n)$. The next lemma shows that T is "locally" defined over $\mathbb{C}^m$.

Lemma 3.4. *Let f be in the domain of T. Then $Tf = T'f$ on $\tilde{U}$.*

Proof. For $z \in \tilde{U}$, apply (2.2) to get

$$\begin{aligned} Tf &= (-1)^{n-1} \int_{\partial M} f \wedge \Omega(\mathfrak{r}^*, \mathfrak{s})(\zeta, z) \\ &= (-1)^{n-1} \int_{\partial M} f \wedge (\Omega(\mathfrak{r}^*, b') - \Omega(\mathfrak{s}, b') - \bar{\partial}_{\zeta,z}\Omega(\mathfrak{r}^*, \mathfrak{s}, b')) \\ &= (-1)^{n-1} \int_{\partial M} f \wedge \Omega(\mathfrak{r}^*, b') \text{ by Stokes' theorem and type considerations,} \\ &= (-1)^{n-1} \int_{\partial\tilde{M}} f \wedge \Omega(\mathfrak{r}^*, b') \\ &\quad + (-1)^{n-1} \int_{M \setminus \tilde{M}} d_\zeta(f \wedge \Omega(\mathfrak{r}^*, b')) \text{ by Stokes' theorem.} \end{aligned}$$

In the last integral we use (2.2) again to get

$$d_\zeta \Omega(\mathfrak{r}^*, b') = \bar{\partial}_\zeta \Omega(\mathfrak{r}^*, b') = -\bar{\partial}_z \Omega(\mathfrak{r}^*, b') + \Omega(\mathfrak{r}^*) - \Omega(b')$$

and the integral vanishes by type considerations. So

$$Tf = (-1)^{n-1} \int_{\partial\tilde{M}} f \wedge \Omega(\mathfrak{r}^*, b') = T'f.$$

This completes the proof. □

Lemma 3.4 has two implications. First, in Lemma 3.1 the assumption on u can be weakened by $u \in C^0_{(0,n-2)}(M)$. Next, it suggests that the strong pseudoconvexity of the base set can be relaxed when one defines Tf. This is seen more precisely in Lemma 3.5.

For $t \geq 0$, set

$$\begin{aligned} M_t &= \{\rho = t\} \cap (\{\sigma < 0\} \times \mathbb{C}^n) \\ \tilde{M}_t &= \{\rho = t\} \cap (\{\tilde{\sigma} < 0\} \times \mathbb{C}^n) \quad \text{and} \\ \Gamma_{z',t} &= \{\rho = t\} \cap (\{z'\} \times \mathbb{C}^n) \quad \text{for } z' \in \{\sigma < 0\}. \end{aligned}$$

When $t = 0$ we have $\Gamma_{z'} = \Gamma_{z',0}$, $\tilde{M} = \tilde{M}_0$, and $M = M_0$.

Now fix $z'_0 \in \{\sigma < 0\}$. As in Lemma 3.4, let $\tilde{\sigma} = |z' - z'_0|^2 - \epsilon^2$, where $\epsilon > 0$ is chosen so that $\{\tilde{\sigma} < 0\} \subset \{\sigma < 0\}$ and $d\rho \wedge d\tilde{\sigma} \neq 0$ on $\partial \tilde{M}$. For ϵ small enough there exist $t > 0$ small such that $\Gamma_{z'_0,t}$ is strongly pseudoconvex in $\{z'_0\} \times \mathbb{C}^n$ and

$$\{z' \mid |z' - z'_0| < \epsilon\} \times \{z'' \mid (z'_0, z'') \in \Gamma_{z'_0,t}\} \subset \tilde{U} \setminus M.$$

Fix such ϵ and t. By Lemma 3.4 we have for $z_0 \in \tilde{U}$ satisfying $\rho(z_0) > t$

$$\begin{aligned} Tf(z_0) &= (-1)^{n-1} \int_{\partial \tilde{M}} f(\zeta) \wedge \Omega(\mathfrak{r}^*, b')(\zeta, z_0) \\ &= (-1)^{n-1} \int_{\partial \tilde{M}} (Tf)(\zeta) \wedge \Omega(\mathfrak{r}^*, b')(\zeta, z_0) \end{aligned}$$

where the last equality follows from (2.7) and Lemma 3.1 for T'.

Since in the last integral

$$\Omega(\mathfrak{r}^*, b') = R(\zeta, z) \wedge \Omega_0(b')$$

where $R(\zeta, z)$ is a form holomorphic in ζ. Applying Stokes' theorem to the last integral in the above formula, we have

$$Tf(z_0) = (-1)^{n-1} \int_{S_\epsilon} (Tf)(\zeta) \wedge \Omega(\mathfrak{r}^*, b')(\zeta, z_0)$$

by type consideration and the fact that $\bar{\partial}_\zeta \Omega_0(b') = 0$, where $S_\epsilon = \{\zeta' \mid |\zeta' - z'_0| = \epsilon\} \times \{\zeta'' \mid (z'_0, \zeta'') \in \Gamma_{z'_0,t}\}$.

Let ϵ tend to zero. It follows from Lemma 1.14 of [Ky] that

$$Tf(z_0) = (-1)^{n-1} c_m \int_{\zeta'' \in \Gamma_{z'_0,t}} (Tf)(z'_0, \zeta'') \wedge \Omega(\tilde{\mathfrak{r}}^*_{z'_0,t})(\zeta'', z''_0)$$

where $c_m = \frac{(2\pi i)^m}{(m-1)!}$ and $\tilde{\mathfrak{r}}^*_{z'_0,t}$ is the section constructed from $\tilde{\rho}(z'') = \rho(z'_0, z'') - t$. We thus have the following lemma:

Lemma 3.5. *For any $z_0' \in \{\sigma < 0\}$, for any $z_0 = (z_0', z_0'') \in \tilde{U}$ and for any $t > 0$ such that $\rho(z_0) > t$ and $\{z'' \mid \rho(z_0', z'') < t\}$ is strictly pseudoconvex in $\tilde{U} \cap (\{z_0'\} \times \mathbb{C}^n)$, we have*

$$Tf(z_0) = (-1)^{n-1} c_m \int_{\zeta'' \in \Gamma_{z_0',t}} (Tf)(z_0', \zeta'') \wedge \Omega(\tilde{\mathfrak{r}}^*_{z_0',t})(\zeta'', z_0''). \tag{3.1}$$

In particular, if $\{z'' \mid \rho(z_0', z'') < 0\}$ is strictly pseudoconvex which holds for z_0' in a dense open set in $\{\sigma < 0\}$, we have

$$Tf(z_0) = (-1)^{n-1} c_m \int_{\zeta'' \in \Gamma_{z_0'}} f(z_0', \zeta'') \wedge \Omega(\tilde{\mathfrak{r}}^*_{z_0'})(\zeta'', z_0''). \tag{3.1$'$}$$

Proof. It remains to prove (3.1′). The denseness of such z_0' follows from Sard's theorem. In view of the fact that (3.1) holds for any $0 < t < \rho(z_0)$ with $\{z'' \mid \rho(z_0', z'') < t\}$ strictly pseudoconvex in $\tilde{U} \cap (\{z_0'\} \times \mathbb{C}^n)$, (3.1′) is obtained by taking limit as such t goes to 0 in (3.1) and by (2.7) and Stokes'theorem. □

Remark 3.6. Formula (3.1) (or ((3.1′))) is of fundamental importance. It shows that Tf can be defined by ρ only: the connectivity and the strong pseudoconvexity of the base set $\{\sigma < 0\}$ can be relaxed. Moreover, it shows that Tf has a continuous extension to the set $U_1 = \mathcal{N} \cap \{\rho \geq 0\} \cap (\{\sigma \leq 0\} \times \mathbb{C}^n)$.

Now instead of $\{\sigma < 0\}$ we take the base set to be an arbitrary open set B in $\mathbb{C}^m$. It is easy to see that all the preceding results about Tf still holds. Indeed, through (3.1) and ((3.1′)) the properties of Tf become even more transparent.

4. The moment operator $\mathcal{M}_h(g)$ and the proof of Theorem 1.2

Let ρ be the same as before and B be an arbitrary relatively compact open subset in $\mathbb{C}^m$. Set

$$\tilde{M} = \{\rho = 0\} \cap (B \times \mathbb{C}^n).$$

For an open neighborhood O of $\tilde{M}$, we set

$$\tilde{U} = O \cap \{\rho \geq 0\} \cap (B \times \mathbb{C}^n).$$

Let g be a C^1 $\bar{\partial}$-closed $(0, n-1)$ form in $\tilde{U}$. Let V be an open set in B and h be a function holomorphic in a neighborhood of $\{\rho = 0\} \cap (V \times \mathbb{C}^n)$. Define the moment operator of g with respect to h on V by

$$\mathcal{M}_h(g)(z') = \int_{\zeta'' \in \Gamma_{z'}} h(z', \zeta'') g(z', \zeta'') d\zeta'', \text{ where } d\zeta'' = d\zeta_{m+1} \wedge \cdots \wedge d\zeta_{m+n}. \tag{4.1}$$

Lemma 4.1. *$\mathcal{M}_h(g)(z')$ is a holomorphic function in V.*

Proof. First observe that

$$\mathcal{M}_h(g)(z') = \int_{\zeta''\in\Gamma_{z'}} h(z',\zeta'')g_1(z',\zeta'')d\zeta''$$

where g_1 is defined by (2.9). Fix $z'_0 \in V$. Choose a domain D in $\mathbb{C}^n$ with C^1 boundary and a neighborhood W of z'_0 in V such that

$$\Gamma_{z'_0} \subset \{z'_0\} \times D, \quad \{\rho \le 0\} \cap (W \times \mathbb{C}^n) \subset W \times D \quad \text{and } W \times \partial D \subset \tilde{U}.$$

For any function h holomorphic in $\overline{W \times D}$ it follows from Stokes' theorem and (2.9), (2.10) that

$$\mathcal{M}_h(g)(z') = \int_{\Gamma_{z'}} h(z',\zeta'')g_1(z',\zeta'')d\zeta'' = \int_{\partial D} h(z',\zeta'')g_1(z',\zeta'')d\zeta''$$

whenever $z' \in W$. Now by (2.10)

$$\bar\partial_{z'}\mathcal{M}_h(g)(z') = \int_{\partial D} h(z',\zeta'')\bar\partial_{z'}g_1(z',\zeta'')d\zeta'' = -\int_{\partial D} h(z',\zeta'')\bar\partial_{\zeta''}g_2(z',\zeta'')d\zeta''$$

whenever $z' \in W$ and so

$$\bar\partial\mathcal{M}_h(g)(z') = -\int_{\partial D} d_{\zeta''}(h(z',\zeta'')g_2(z',\zeta''))d\zeta'' = 0.$$

This completes the proof. □

Remark 4.2. Suppose f is a $\bar\partial$-closed $(0, n-1)$ form in $C^0(\bar{\tilde{M}})$. In view of Remark 3.6, Tf is well-defined in $\tilde{U} = \mathcal{N} \cap \{\rho \ge 0\} \cap (B \times \mathbb{C}^n)$ where $\mathcal{N}$ is given by Lemma 3.2. Locally, for each $z' \in B$ there is a small open neighborhood W of z' contained in B such that for $z \in \tilde{U} \cap (W \times \mathbb{C}^n)$ f can be represented as in (2.7), we see immediately that $\mathcal{M}_h(f) = \mathcal{M}_h(Tf)$ for any holomorphic function h. In other words, the moment of f is well-defined and is holomorphic in z'.

With Remark 4.2 we have:

Corollary 4.3. *Fix $z' \in B$. All the holomorphic moments of f on $\Gamma_{z'}$ vanish is equivalent to $Tf(z) = 0$ on $\tilde{U} \cap (\{z'\} \times \mathbb{C}^n)$. Moreover, suppose $\Gamma_{z'}$ is strictly pseudoconvex, equation $\bar\partial_{\Gamma_{z'}} u = f$ is solvable on $\Gamma_{z'}$ if and only if $Tf(z) = 0$ for all $z \in \tilde{U} \cap (\{z'\} \times \mathbb{C}^n)$.*

Proof. To prove the first statement, we need only show that $M_h(f)(z') = 0$ for all h holomorphic near $\Gamma_{z'}$ implies $Tf(z) = 0$ on $\tilde{U} \cap (\{z'\} \times \mathbb{C}^n)$. The proof of Lemma 4.1 and Remark 4.2 give

$$\begin{aligned} 0 = \mathcal{M}_h(f)(z') &= \int_{\Gamma_{z'}} h(z',\zeta'')(Tf)_1(z',\zeta'')d\zeta'' \\ &= \int_{\Gamma_{z',t}} h(z',\zeta'')Tf(z',\zeta'')d\zeta'', \quad t > 0, \end{aligned}$$

whenever $\Gamma_{z',t} \subset \tilde{U}$ and h is holomorphic near $\Gamma_{z',t}$. The last equality follows from (2.10), (2.11) and Stokes' theorem. Now suppose $\Gamma_{z',t}$ is strictly pseudoconvex. Let $h(z', \zeta'')d\zeta'' = \Omega(\tilde{\mathfrak{r}}^*_{z'})$. By Lemma 3.5 we have $Tf(z) = 0$ for $z \in \tilde{U} \cap (\{z'\} \times \mathbb{C}^n)$, $\rho(z) \geq t$. By Sard's theorem 0 is a limit point of such t so the first statement is proved. The second statement follows from the first statement and Remark 1.3(a). □

Theorem 4.4. *Let ρ be as in Theorem 1.2. Let B be a connected relatively compact open subset in $\mathbb{C}^m$. Set*

$$\tilde{M} = \{\rho = 0\} \cap (B \times \mathbb{C}^n).$$

Let f be a $(0, n-1)$ form in $C^0(\bar{\tilde{M}})$ with $\bar{\partial} f = 0$ on $\tilde{M}$ in distribution sense. Let A be a subset of B not contained in any subvariety of codimension one in B. If for every $z' \in A$ all the holomorphic moments of f on $\Gamma_{z'}$ vanish, then Tf vanishes identically on $\tilde{U} = \mathcal{N} \cap \{\rho \geq 0\} \cap (B \times \mathbb{C}^n)$ where $\mathcal{N}$ is defined in Lemma 3.2.

Proof. **Step 1.** The assumption on the set A implies that there is a point $p \in \bar{B}$ such that the intersection of A with any neighborhood of p is not contained in a subvariety of codimension one in B. Let V be any connected neighborhood of p in $\bar{B}$. Let h be any function holomorphic near $\tilde{M} \cap (V \times \mathbb{C}^n)$. By Lemma 4.1 $\mathcal{M}_h(f)(z')$ is a holomorphic function in $V \cap B$. Remark 4.2 and the assumption give that $\mathcal{M}_h(f)(z') = \mathcal{M}_h(Tf)(z') = 0$ for all $z' \in V \cap A$. By the choice of p, we must have $\mathcal{M}_h(f)(z')$ identically equal to zero on V.

Step 2. *Claim*: There is a neighborhood W of p in $\bar{B}$ such that Tf vanishes identically on $\tilde{U} \cap (W \times \mathbb{C}^n)$.

Proof of the Claim. By Corollary 4.3 it suffices to show that for every $z' \in W$, $\mathcal{M}_h(f)(z') = 0$ for all h holomorphic near $\Gamma_{z'}$. If there is no such neighborhood, then there exists a sequence $\{z'_j\}_1^\infty \subset B$ such that $z'_j \to p$ as $j \to \infty$ and $Tf(z'_j, \cdot)$ does not vanish identically for all $j = 1, 2, \ldots$.

Choose $t > 0$ so that $\Gamma_{p,t}$ is a C^k strictly pseudoconvex real hypersuface in $\tilde{U} \cap (\{p\} \times \mathbb{C}^n)$. Let W be a connected open neighborhood of p in B such that $W \times \{z'' | (p, z'') \in \Gamma_{p,t}\} \subset \tilde{U} \setminus \tilde{M}$.

For every $j = 1, 2, \ldots$ there is a function h_j holomorphic near $\Gamma_{z'_j}$ such that $\mathcal{M}_{h_j}(f)(z'_j) \neq 0$. For simplicity we may assume that $\Gamma_{z'_j}$ is strictly pseudoconvex in $\{z'_j\} \times \mathbb{C}^n$ (or we do as in the proof of Corollary 4.3).

Fix j so that $z'_j \in W$. Set $D_1 = \{z'' | \rho(p, z'') < t\}$ and $D_2 = \{z'' | \rho(z'_j, z'') < 0\}$. By our choice both D_1, D_2 are strictly pseudoconvex domains in $\mathbb{C}^n$ and $D_2 \Subset D_1$ for every $j = 1, 2, \ldots$. Since $\rho(z'_j, \cdot)$ is plurisubharmonic in D_1 if ρ is plurisubharmonic in $\tilde{U}$ and which can be assumed, it follows from Corollary 5.4.3 of [8] that D_1, D_2 form a Runge pair.

Thus $h_j(z'_j, \cdot)$ can be uniformly approximated on $\bar{D}_2$ by functions holomorphic on D_1. Let $\{g_k\}_{k=1}^\infty$ be such a sequence of holomorphic functions on D_1. By Step 1, for all $k \geq 1$ $\mathcal{M}_{g_k}(f)(z') = \mathcal{M}_{g_k}(Tf)(z') = 0$ for all $z' \in W$. Therefore we must have $\mathcal{M}_{h_j}(f)(z'_j) = 0$, contradicting our assumption on z'_j and h_j. This completes the proof of the claim.

Step 3. Set $\mathcal{S} \equiv \{z' | \; z' \in B, \; Tf(z', \cdot) \equiv 0 \text{ on } \tilde{U} \cap (\{z'\} \times \mathbb{C}^n)\}$. By Step 2 $\mathcal{S}$ has non-empty interior. In fact, the argument in Step 2 shows that the interior of $\mathcal{S}$ is both open and closed in B. Since B is connected, we conclude that $\mathcal{S} = B$. Theorem 4.4 is proved. □

From the proof of Theorem 4.4 we have the following:

Corollary 4.5. *Under the assumptions of Theorem* 4.4, *the following statements are equivalent:*

(a) *For all $z' \in A$, every holomorphic moment of f on $\Gamma_{z'}$ vanishes.*

(b) $Tf(z) \equiv 0$ *on* $\tilde{U}$.

(c) $Tf(z) \equiv 0$ *for all* $z \in \Gamma_{z'}$, $z' \in A$.

(d) $\bar{\partial}_{\Gamma_{z',t}} u = Tf(z', \cdot)$ *is solvable on* $\Gamma_{z',t} \subset \tilde{U}$ *for every* $z' \in A$ *and for every* $t \geq 0$, *provided that* $\Gamma_{z',t}$ *is strictly pseudoconvex in* $\{z'\} \times \mathbb{C}^n$.

Corollary 4.6. *Under the assumptions of Theorem 4.4, if the set* $\{z' | \; \Gamma_{z'} = \emptyset, \; z' \in B\}$ *is not contained in a subvariety of codimension one in* B, *then* $Tf \equiv 0$ *on* $\tilde{M}$. *In particular, if* $B = \{\sigma < 0\}$, *then* $\bar{\partial}_M$ *is solvable at* $q = n - 1$.

On the other hand, we have

Remark 4.7. Let $M = \{\rho = 0\} \cap (\{\tilde{\sigma} < 0\} \times \mathbb{C}^n)$ where ρ is as in the assumption of Theorem 4.4 and $\tilde{\sigma}$ is any C^1 function satisfying $d\rho \wedge d\tilde{\sigma} \neq 0$ on ∂M. Suppose M and $\{\tilde{\sigma} < 0\}$ are connected and $m < n$, then for $f \in C^0_{(0,q)}(\bar{M})$ satisfying $\bar{\partial}_M f = 0$ in distribution sense on M with $m \leq q \leq n + m - 1$, considering type, the following representation formula holds for $z \in M$

$$
\begin{aligned}
(-1)^q f(z) = \bar{\partial} \left\{ \int_M f(\zeta) \wedge \Omega_{q-1}(\mathfrak{r}, \mathfrak{r}^*)(\zeta, z) + (-1)^q \int_{\partial M} f(\zeta) \wedge \Omega(\mathfrak{r}, \mathfrak{r}^*, b')(\zeta, z) \right\} \\
+ \int_{\partial M} f(\zeta) \wedge \Omega_q(\mathfrak{r}^*, b')(\zeta, z).
\end{aligned}
$$

Note that the last integral vanishes for $m \leq q \leq n - 2$ by type consideration. In other words, $\bar{\partial}_M$ is always solvable for $m \leq q \leq n - 2$ in this case. Thus the tangential Cauchy-Riemann equation $(*)$ is solvable at $q = n - 1$ iff $T'f(z) = Tf(z) = 0$. In view of Corollary 4.6, if $\{\tilde{\sigma} < 0\}$ is not contained in $\pi(\{\rho = 0\})$, the projection of $\{\rho = 0\}$ to the $\mathbb{C}^m$ plane, then $(*)$ is solvable at $q = n - 1$.

Proof of Theorem 1.2. The solution u is obtained from Corollary 4.5 and formula (2.7). The regularity of u can be proved by routine procedure, see e.g. [14], and we omit the details here. □

5. Proof of Theorem 1.1

Let P_N denote the unit polydisc in $\mathbb{C}^N$ as before. We are going to define a sequence of subdomains exhausting Ω. First choose a sequence of C^∞ real-valued strictly convex functions $\{\rho_{1,j}\}_{j=1}^\infty$ in $\mathbb{C}^{m+n}$ satisfying:

(p_1) $\{\rho_{1,j} < 0\} \Subset \{\rho_{1,j+1} < 0\} \Subset P_{m+n}$ for each $j \geq 1$;

(p_2) $\cup_1^\infty \{\rho_{1,j} < 0\} = P_{m+n}$;

(p_3) for each $j = 1, 2, \ldots,$ $\rho_{1,j}(z) = \rho_{1,j}(|z_1|, \ldots, |z_{m+n}|)$ and is symmetric in $|z_k|, k = 1, \ldots, m+n$.

Next, for $j = 1, 2, \ldots$, set $C_j = \{z \in \mathbb{C}^{m+n} | \, |z_k| < 1 + \frac{1}{3j}, k = 1, \ldots, m, \; |z_k| < \frac{1}{2} + \frac{1}{3j}, k = m+1, \ldots, m+n\}$. Choose a sequence of C^∞ real-valued strictly convex functions $\{\rho_{2,j}\}_{j=1}^\infty$ satisfying:

(p_4) $\{\rho_{2,j} < 0\} \Subset C_j$ for $j = 1, 2, \ldots$;

(p_5) $\{\rho_{2,j} = 0\} \subset C_j \setminus \bar{C}_{j+1}$;

(p_6) for each $j = 1, 2, \ldots,$ $\rho_{2,j}(z) = \rho_{2,j}(|z_1|, \cdots, |z_{m+n}|)$ and is symmetric in $|z_k|$, $k = 1, \ldots, m$ and is also symmetric in $|z_k|$, $k = m+1, \ldots, m+n$ respectively.

Define

$$D_j = \{\rho_{1,j} < 0\} \cap \{\rho_{2,j} > 0\}, \text{ for } j = 1, 2, \ldots.$$

Clearly we have

$$D_j \Subset D_{j+1}, \text{ for } j = 1, 2, \ldots \text{ and } \cup_1^\infty D_j = \Omega.$$

Remark 5.1. Let π be the orthogonal projection from $\mathbb{C}^{m+n}$ into $\mathbb{C}^m$. Set

$$E_j = \{z \in \mathbb{C}^{m+n} | \, z' \in \pi(D_j), \rho_{1,j}(z) < 0\}, \; j = 1, 2, \ldots.$$

In addition to $(p_1) - (p_6)$, we choose $\rho_{1,j}$, $\rho_{2,j}$ so that E_j is a relatively compact convex set in $\mathbb{C}^{m+n}$ for each $j = 1, 2, \ldots$. Such functions $\rho_{1,j}$, $\rho_{2,j}$ are easy to construct.

For each $j \geq 1$, the boundary of D_j can be written as

$$\partial D_j = \partial D_{j1} \cup \partial D_{j2},$$

where $\partial D_{j1} = \{\rho_{1,j} = 0\} \cap \{\rho_{2,j} \geq 0\}$ and $\partial D_{j2} = \{\rho_{2,j} = 0\} \cap \{\rho_{1,j} \leq 0\}$.

Remark 5.2.

(a) For each $j = 1, 2, \dots,$ we can choose strictly plurisubharmonic function $\sigma_j \in C^\infty(\mathbb{C}^m)$ such that $\{\sigma_j < 0\} \Subset P_m$ and $\partial D_{j2} \Subset \{\sigma_j < 0\} \times \mathbb{C}^n$. Set

$$M_j = \{\rho_{2,j} = 0\} \cap (\{\sigma_j < 0\} \times \mathbb{C}^n), \; j = 1, 2, \dots.$$

It follows from [1] that $\bar\partial_{M_j} u = g$ is solvable on M_j for any L^p, $1 \le p \le \infty$, $\bar\partial$-closed $(0, q)$ form g on M_j, $1 \le q < n-1$. Furthermore, if $g \in C^k(\bar M_j)$ then $u \in C^k(M_j)$ for $j = 1, 2, \dots$.

(b) If the assumption of Theorem 1.1 is satisfied, then Corollary 4.5 implies that $Tf \equiv 0$ on $\tilde M_j = \{\rho_{2,j} = 0\} \cap (P_m \times \mathbb{C}^n)$ with $B = P_m$. Hence by Theorem 1.2 the conclusions in (a) for the solvability and regularity of $\bar\partial_{M_j} u = g$ on M_j also hold for $q = n-1$, $j = 1, 2, \dots,$ provided that $\bar\partial_{M_j} g = 0$ and $g \in C^k(\bar M_j)$, $k \ge 0$.

Lemma 5.3. *Let g be a $\bar\partial$-closed C^k $(0, q)$ form on Ω with k any nonnegative integer and $1 \le q < n-1$. For $j = 1, 2, \dots,$ the equation $\bar\partial v_j = g$ is solvable with $v_j \in C^k(D_j)$. In fact,*

$$v_j = -\int_{D_j} g \wedge \Omega(b) + \int_{\partial D_{j1}} g \wedge \Omega(b, \mathfrak{r}_{1,j}) + \int_{\partial D_{j2}} g \wedge \Omega(b, \mathfrak{r}^*_{2,j})$$
$$+ (-1)^{q+1} \int_{\partial D_{j1} \cap \partial D_{j2}} g \wedge \Omega(b, \mathfrak{r}_{1,j}, \mathfrak{r}^*_{2,j}) - \int_{\partial D_{j1} \cap \partial D_{j2}} u_j \wedge \Omega(\mathfrak{r}_{1,j}, \mathfrak{r}^*_{2,j})$$

where b is the Bochner-Martinelli section in $\mathbb{C}^{m+n}$; $\mathfrak{r}_{1,j}$,$\mathfrak{r}_{2,j}$ are sections corresponding to $\rho_{1,j}$, $\rho_{2,j}$; and u_j is the C^k solution to $\bar\partial_{M_j} u_j = g$ on M_j in view of Remark 5.2(a).

Proof. Since $\rho_{2,j}$ is a smooth strictly convex function in $\mathbb{C}^{m+n}$, the section $\mathfrak{r}^*_{2,j}(\zeta, z)$ is well-defined for all $z \in \{\rho_{2,j} > 0\}$ as long as $\zeta \in \{\rho_{2,j} \le 0\}$. As usual, one starts from the formula:

$$g(z) = -\bar\partial \left(\int_{D_j} g \wedge \Omega(b)(\zeta, z) \right) + \int_{\partial D_j} g \wedge \Omega(b)(\zeta, z).$$

The lemma is proved by repeated use of (2.2), Stokes' theorem and type considerations when interploating the integrals with $\Omega(\mathfrak{r}_{1,j})$, $\Omega(\mathfrak{r}^*_{2,j})$, etc.. We omit the routine computations. □

By part (b) of Remark 5.2 we have:

Corollary 5.4. *Let g be a C^k, $k \ge 0$, $\bar\partial$-closed $(0, n-1)$ form on Ω satisfying the assumption of Theorem 1.1 Then $\bar\partial_{D_j} v_j = g$ is solvable for every $j \ge 1$ with $v_j \in C^k_{(0,n-2)}(D_j)$.*

Proof of Theorem 1.1. By Corollary 5.4 it remains to construct a C^k solution v on Ω out of v_j, $j = 1, 2, \ldots$. The process below is a modification of [11, Lemma 3] which deals with $k = \infty$.

Consider the case $n > 2$ first. Let v_j be given by Corollary 5.4. Set $\tilde{v}_0 = v_3$. Obviously $v_3 - v_4$ is a $\bar\partial$-closed form in $C^k_{(0,n-2)}(D_3)$. By Lemma 5.3 there exists $w_1 \in C^k_{(0,n-3)}(D_2)$ so that $\bar\partial w_1 = v_3 - v_4$ in D_2. Let $\chi_1 \in C^\infty_0(D_2)$ such that $\chi_1 \equiv 1$ on $\bar{D}_1$. Set

$$\tilde{v}_1 = v_4 + \bar\partial(\chi_1 w_1) \in C^k_{(0,n-2)}(D_4).$$

Then $\tilde{v}_1 = v_3 = \tilde{v}_0$ on D_1 and $\bar\partial\tilde{v}_1 = f$ on D_4. We use induction to construct $\tilde{v}_j$ for $j > 1$. Suppose we already have $\tilde{v}_j \in C^k_{(0,n-2)}(D_{j+3})$ with $\bar\partial\tilde{v}_j = f$ on D_{j+3} and $\tilde{v}_j = \tilde{v}_{j-1}$ on D_j. Now $\tilde{v}_j - v_{j+4} \in C^k_{(0,n-2)}(D_{j+3})$ and $\bar\partial(\tilde{v}_j - v_{j+4}) = 0$ on D_{j+3}. By Lemma 5.3 there exists $w_{j+1} \in C^k_{(0,n-3)}(D_{j+2})$ so that $\bar\partial w_{j+1} = \tilde{v}_j - v_{j+4}$ on D_{j+2}. Choose $\chi_{j+1} \in C^\infty_0(D_{j+2})$ such that $\chi_{j+1} \equiv 1$ on $\bar{D}_{j+1}$. Set

$$\tilde{v}_{j+1} = v_{j+4} + \bar\partial(\chi_{j+1} w_{j+1}) \in C^k_{(0,n-2)}(D_{j+4}).$$

We have $\tilde{v}_{j+1} = \tilde{v}_j$ on D_{j+1} and $\bar\partial\tilde{v}_{j+1} = f$ on D_{j+4}. In this way we get $v = \lim_{j\to\infty} \tilde{v}_j$ in $C^k_{(0,n-2)}(D)$ and $\bar\partial v = f$ on D. This proves Theorem 1.1 when $n > 2$.

When $n = 2$, let E_j be as in Remark 5.1. As $n = 2 > 1$, every function holomorphic in D_j extends holomorphically to E_j by Hartogs theorem. Since E_j is a bounded convex set in $\mathbb{C}^{m+2}$; it is a Runge domain in $\mathbb{C}^{m+2}$ (see [7, Theorem 4.7.8]). So the assumption of [11, Lemm 3] is satisfied and the case $n = 2$ is proved. This completes the proof of Theorem 1.1. □

With Corollary 5.4 replaced by Lemma 5.3 in the proof of Theorem 1.1 we immediately have:

Corollary 5.5. *Let g be a C^k $\bar\partial$-closed $(0, q)$ form, $1 \le q \le n - 2$, on Ω, where k is any non-negative integer. Then there exists a C^k $(0, q - 1)$ form u on Ω such that $\bar\partial u = g$.*

Note that Corollary 5.5 is part of Frenkel's lemma if $k = \infty$. The case $n \le q \le m + n$ will be proved elsewhere.

There are many applications of the proof of Theorem 1.1, for example we have:

Corollary 5.6. *Let D be any bounded pseudoconvex domain in $\mathbb{C}^N$. Let k be any non-negative integer. For any $\bar\partial$-closed C^k $(0, q)$ form g on D, $1 \le q \le N$, there exists a C^k $(0, q - 1)$ form u on D such that $\bar\partial u = g$.*
Moreover, if $g \in L^p(D)$, $1 \le p \le \infty$, then there exists $u \in L^p_{loc}(D)$ such that $\bar\partial u = g$.

Corollary 5.7. *Let ρ be a C^k real-valued function on $\mathbb{C}^{m+n}$ which is strictly plurisubharmonic near $\{\rho \leq 0\}$, $3 \leq k \leq \infty$. Let B be a relatively compact pseudoconvex domain in $\mathbb{C}^m$. Set $M = \{\rho = 0\} \cap (B \times \mathbb{C}^n)$. Let f be a $C^{k'}$ $\bar{\partial}$-closed $(0, q)$ form on M, $0 \leq k' \leq k-3$. Then there exists $u \in C^{k'}_{(0,q-1)}(M)$ such that $\bar{\partial}u = f$ on M if $\quad 1 < q < n-1$.*

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