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Algebraic Morava *K*-theory spectra over perfect fields

SIMONE BORGHESI

Abstract. In the paper [2] we constructed (co)homology theories on the category of smooth schemes which share some of the some of the defining properties of the (co)homology theories induced by the Morava k-theory spactra in classical homotopy theory. Some proofs used the topological realization functor (*cf.* [8]). The existence of that functor requires the base field k to be embedded in $\mathbb{C}$. In this manuscript we investigate up to what extent we can obtain the same results under the sole assumption of perfectness of the base field. The results proved here guarantee the existence of spectra $\mathbf{\Phi}_i$ satisfying the same properties as in [2], provided that the algebra of all the bistable motivic cohomology operations verifies an assumption involving the Milnor operation Q_t.

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1. Introduction

Let k be a field and Sm/k be the category of smooth schemes over k. In the paper [2], under the assumption that the base field is embeded in $\mathbb{C}$, we constructed a family of (co)homology theories $\mathbf{\Phi}_i$ satisfying certain properties. If $H^{*,*}(X, \mathbb{Z}/q)$ denotes the motivic cohomology groups of a smooth scheme X, as defined by V. Voevodsky in [10], then $\mathbf{\Phi}_i^{*,*}(X)$ are $i+1$-tuple extensions of the groups $H^{*-2j(q^t-1),*-j(q^t-1)}(X, \mathbb{Z}/q)$ which detect certain Chern numbers of the tangent bundle of X. The (co)homology theories $\mathbf{\Phi}_i(\quad)$ correspond to objects denoted $\mathbf{\Phi}_i$ of a certain category $\mathcal{SH}(k)$. This is called the *stable homotopy category of schemes over a base field* k, and it is strictly related to the usual stable homotopy category of topological spaces. There are two, equivalent, constructions of this category (*cf.* [4] and [9]), each having its own advantages. If we assume the base field k to be perfect, motivic cohomology is a representable functor, represented by an object $\mathbf{H}_{\mathbb{Z}} \in \mathcal{SH}(k)$, called the *Eilenberg-MacLane spectrum*. This is the only instance where we use perfectness of k. The word *spectrum* refers to an object of the category $\mathcal{SH}(k)$. This is the type of objects we will deal with in this manuscript.

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The purpose of the paper is to construct the spectra $\mathbf{\Phi}_i$ of [2] for a more general kind of base fields: we now only assume the base field k to be perfect instead of admitting an embedding $k \hookrightarrow \mathbb{C}$. The spectra $\mathbf{\Phi}_i$ constructed in [2] depend on a prime number q and on a positive integer t. There are two main properties these spectra have: the first is that they fit in exact triangles

$$\Sigma^{2(q^t-1),q^t-1}\mathbf{\Phi}_{i-1} \longrightarrow \mathbf{\Phi}_i \xrightarrow{\tau_i} \mathbf{H}_{\mathbb{Z}/q} \tag{1.1}$$

thus they are assembled by using motivic Eilenberg MacLane spectra. The second is that the way these spectra are put together is nontrivial. In particular $\mathbf{\Phi}_i$ are not simply a direct sum (wedge) of suspensions of motivic Eilenberg MacLane spectra. This property follows from the definition of $\mathbf{\Phi}_i$ given in Theorem 5.1 and that the cohomology classes $y_i \in H^{2(i+1)(q^t-1)+1,(i+1)(q^t-1)}(\mathbf{\Phi}_i, \mathbb{Z}/q)$ that appear in the exact triangle (5.1) are chosen so that $\delta y_i = Q_t\Sigma^{2i(q^t-1)+1,i(q^t-1)}\iota$. The program we will follow is exactly the same as the one already employed in [2]. More precisely we determine homotopy classes a_n of the *algebraic cobordism spectrum* **MGl** that have an appropriate divisibility property with respect to their images through the Hurewicz homomorphism (Theorem 3.1). These homotopy classes are then employed to produce a spectrum $k'(t)$ which will serve as "model" in the costruction of $\mathbf{\Phi}_i$ in the Theorem 5.1. To us the most important features of $k'(t)$ are the existence of a canonical **MGl** module structure on it and its motivic cohomology with coefficients in $\mathbb{Z}/q$. To check that the motivic cohomology is as in Theorem 4.6, we first have to compute motivic cohomology groups of the spectrum **MGl** as module over the motivic Steenrod algebra (Theorem 4.1) and then use this answer in the argument already described in [2] to prove Theorem 12. In that paper we used the assumption that the base field k admits an embedding in $\mathbb{C}$ three times, namely in the proofs of Theorem 10, Proposition 6 and few times in Section 4.3 to prove Theorem 13. In each of them the $\mathbb{C}$ *realization functor* (*cf.* [8]) has been employed and its existence relies on such assumption on the base field. This paper provides alternative proofs for those statements, requiring no assumption on the base field other than its perfectness, except in Section 5. Following our approach, the existence of a spectrum with motivic cohomology isomorphic to $\mathcal{A}^{**}/\mathcal{A}^{**}Q_t$ such as $k'(t)$ enables us to place the obstructions to the existence of $\mathbf{\Phi}_i$ in the most "exotic" part of the algebra of the bistable motivic cohomology operations, that is the kernel $\mathcal{B}^{**}$ of the canonical map $\mathcal{A}_m^{**} \to \mathcal{A}^{**}$ (see the part following the Theorem 4.6). In Section 5 we prove that such obstructions are contained in certain homogeneous degrees of $\mathcal{B}^{**}/\mathcal{B}^{**}Q_t$ (see the condition and (5) and (5)). In Proposition 5.2 we show that the Margolis homology groups of $\mathcal{B}^{**}$ vanish if all the motivic cohomology operations in $\mathcal{A}_m^{**}$ satisfy a Cartan formula. The issue whether $\mathcal{B}^{**}$ satisfies such conditions is quite involved. $\mathcal{B}^{**}$ is expected to be zero for all perfect fields, in which case the conditions would be fulfilled trivially. The proof that $\mathcal{B}^{**}$ is zero for characteristic zero fields, recently appeared in the paper [16].

We will repeatedly use the vanishing of the motivic cohomology groups $H^{i,j}(X, A)$, with coefficients in an Abelian group A, of a smooth scheme X in the following range:

$$j < 0, \qquad i > 2j, \qquad i - j > \text{Krull dimension of X}$$

in the case the base field k is perfect, there is a pair of adjoint functors

$$DM^{-}_{\text{eff}}(k) \underset{L}{\overset{R}{\rightleftarrows}} \mathcal{H}(k) \tag{1.2}$$

where $DM^{-}_{\text{eff}}(k)$ is the derived category of effective motives and $\mathcal{H}(k)$ is the (unstable) homotopy category of smooth schemes. By means of this adjunction it is possible to prove that the motivic cohomology is represented by the class in $\mathcal{H}(k)$ of certain simplicial sheaves [12]. These simplicial sheaves can be assembled together to give a spectrum in $\mathcal{SH}(k)$ which represents motivic cohomology in $\mathcal{SH}(k)$, as well.

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2. Notations and basic facts

Much like the topological stable homotopy category, $\mathcal{SH}(k)$ can be seen as the homotopy category associated to two model categories: one of which has *symmetric spectra* as objects and the other S modules in the the sense of [3] (*cf.* [4] and [9], respectively). The associated homotopy categories are canonically equivalent. The main difference from the topological case is that here the category $\Delta^{\text{op}}(Shv(Sm/k)_{\text{Nis}})$ of simplicial Nisnevich sheaves of sets over the site of smooth schemes over a base field k plays the role of the category of topological spaces and the former is not *locally contractible* and some appropriate version of the Whitehead theorem is not known, at this date. We have distinguished pointed simplicial sheaves: S^1_s is the constant simplicial sheaf $\Delta^1/\partial\Delta^1$ pointed by $\partial\Delta^1$, where Δ^n is the sheaf of simplicial sets constantly equal to the simplicial set Δ^n. $\mathbb{P}^1_k$ is the constant simplicial sheaf $\mathcal{U} \to \text{Hom}_{Sm(k)}(\mathcal{U}, \mathbb{P}^1_k)$, for all smooth schemes $\mathcal{U}$ pointed by $\mathcal{U} \to \text{Hom}_{Sm(k)}(\mathcal{U}, \{\infty\})$ with $\infty \in \mathbb{P}^1_k$. If $\mathcal{X}$ is a pointed sheaf, then we will denote $\Sigma^\infty \mathcal{X}$ its *suspension spectrum*, that is the spectrum consisting of $(\mathbb{P}^1_k)^{\wedge i} \wedge \mathcal{X}$ at nonnegative level i and identities as structure morphisms. Let G be an Abelian group, then $\mathbf{H}_G$ is the *motivic Eilenberg-MacLane spectrum* with coefficients in G as described in [2, Definition 3]. As usual, if $\mathbf{A}$ and $\mathbf{B}$ are spectra we define the following groups:

Definition 2.1.

(1) $\mathbf{A}^{i,j}(\mathbf{B}) = [\mathbf{B}, \Sigma^\infty(S_s^{i-2j} \wedge (\mathbb{P}^1_k)^{\wedge j}) \wedge \mathbf{A}]_{\mathcal{SH}(k)}$,
(2) $\mathbf{A}_{i,j}(\mathbf{B}) = [\Sigma^\infty(S_s^{i-2j} \wedge (\mathbb{P}^1_k)^{\wedge j}), \mathbf{A} \wedge \mathbf{B}]_{\mathcal{SH}(k)}$
(3) $\pi_{i,j}(\mathbf{A}) = [\Sigma^\infty(S_s^{i-2j} \wedge (\mathbb{P}^1_k)^{\wedge j}), \mathbf{A}]_{\mathcal{SH}(k)}$.

We will use the following notation: $\Sigma^\infty(S_s^{i-2j} \wedge (\mathbb{P}^1_k)^{\wedge j}) \wedge \mathbf{A} = \Sigma^{i,j}\mathbf{A}$. If the simplicial sheaf X has a point $x : \operatorname{Spec} k \to X$, the group $\tilde{\mathbf{A}}^{i,j}(X)$ is defined as $[\Sigma^\infty X, \Sigma^{i,j}\mathbf{A}]_{\mathcal{SH}(k)}$ with X pointed by x. Similarly, we define $\tilde{\mathbf{A}}_{i,j}(X)$. In particular, $H^{i,j}(X, G)$ denotes the group $[\Sigma^\infty X_+, \Sigma^{i,j}\mathbf{H}_G]$. If $\mathbf{E}$ is a ring spectrum, $\tilde{\mathbf{A}}^{i,j}(\mathbf{E})$ is the kernel of the map $u^* : \mathbf{A}^{i,j}(\mathbf{E}) \to \mathbf{A}^{i,j}(\mathbf{S^0})$ induced by the unit $u : \mathbf{S^0} \to \mathbf{E}$. We recall that the spectrum **MGl** is constructed levelwise by means of Thom spaces exactly in the same way as it is done in the classical stable homotopy category. It has a *strict multiplication* satisfying the axioms making it a ring spectrum [9] and it is oriented in the sense that there exists a class $c \in \widetilde{\mathbf{MGl}}^{2,1}(\mathbb{P}^\infty_k)$ pulling back to the canonical class in $\widetilde{\mathbf{MGl}}^{2,1}(\mathbb{P}^1_k)$ via the canonical inclusion $\mathbb{P}^1_k \hookrightarrow \mathbb{P}^\infty_k$. Since its proof is valid in complete generality (*cf.* [2, Corollary 1]), we will freely use the Thom isomorphism theorem:

Theorem 2.2. *Let* $\mathbf{E}$ *be an oriented ring spectrum,* X *be a connected smooth scheme and* $\nu \to X$ *be a n dimensional vector bundle over* X*. Then we have isomorphisms* $\mathbf{E}^{i-2n,j-n}(X) \xrightarrow{\cong} \tilde{\mathbf{E}}^{i,j}(\mathrm{Th}(\nu))$ *for all i and j. More precisely,* $\mathbb{P}(\nu \times \mathbb{A}^1) \xrightarrow{p} X$ *and* $\mathbb{P}(\nu \times \mathbb{A}^1) \xrightarrow{q} \mathrm{Th}(\nu)$ *induce a commutative diagram*

$$\begin{array}{ccc} \mathbf{E}^{*,*}(X) & \xhookrightarrow{p^*} & \mathbf{E}^{*,*}(\mathbb{P}(\nu \times \mathbb{A}^1)) \\ & \underset{(\ )\cup c^n}{\searrow^{\cong}} & \uparrow q^* \\ & & \tilde{\mathbf{E}}^{*+2n,*+n}(\mathrm{Th}(\nu)) \cong \mathbf{E}^{*,*}(X) \cdot c^n. \end{array} \tag{2.1}$$

Moreover, we will regularly refer to the properties of the *motivic Steenrod algebra* listed in [2] and proved in [13].

3. Choice of homotopy classes of MGl

Our first goal is to construct an **MGl** module spectrum $k'(t)$ with motivic cohomology as stated in Theorem 4.6. This can be achieved by starting with the spectrum **MGl** and then "killing" the homotopy classes of **MGl** which we are going to describe in this section. The existence of these special homotopy classes is established by the following result

Theorem 3.1. *Let* $h : \pi_{*,*}(\mathbf{MGl}) \to H_{*,*}(\mathbf{MGl}, \mathbb{Z})$ *be the Hurewicz homomorphism, defined by associating to* $\phi : \Sigma^{r,s}(\mathbb{P}^1_k) \to \mathbf{MGl}$ *the composition* $\Sigma^{r,s}(\mathbb{P}^1_k) \cong$

$\mathbf{S}^0 \wedge \Sigma^{r,s}(\mathbb{P}^1_k) \stackrel{u\wedge\phi}{\to} \mathbf{H}_{\mathbb{Z}} \wedge \mathbf{MGl}$, *where* $u : \mathbf{S}^0 \to \mathbf{H}_{\mathbb{Z}}$ *is the unit of the canonical ring structure of* $\mathbf{H}_{\mathbb{Z}}$. *Then there exist classes* $a_n \in \pi_{2n,n}(\mathbf{MGl})$ *for* $n = 0, 1, 2, \cdots$ *such that*

(1) *if* $n = q^t - 1$ *for some prime number* q *and positive integer* t *then* $h(a_n)$ *is divisible by* q, *but not by* q^2;
(2) *in all the other cases no prime number* q *divides* $h(a_n)$.

Proof. The strategy is to find homotopy classes in a way that the coefficients of their images through the Hurewicz homomorphism do not depend on the base field k. More precisely we will show that for certain homotopy classes a_n, the coefficients of $h(a_n)$ are the degree of the 0 cycle represented by homogeneous polynomials in the Chern classes of the tangent bundle of some smooth projective algebraic varieties. Such polynomials have constant integral coefficients making the coefficients of $h(a_n)$ independent on the base field on which these varieties are definable. In turn, these projective algebraic varieties can be taken to be disjoint unions of smooth degree q hypersurfaces of dimension $q^t - 1$ for all prime numbers q, and degree $(1, 1)$ smooth hypersurfaces in $\mathbb{P}^n_k \times \mathbb{P}^m_k$ for appropriate n and m thus they can be defined on any field k. We are going to employ the [14, Theorem 2.11]. □

Theorem 3.2 (Voevodsky). *Let* M *be a connected smooth projective variety of pure dimension* d *over a field* k. *Then there exists an integer* n *and a vector bundle* V *of rank* n *on* M *such that*:

(1) $V + T_M = \mathcal{O}_M^{n+d}$, *where* T_M *is the tangent bundle on* M;
(2) *Let* V *as above. Then there exists a morphism*

$$f_V : (\mathbb{P}^1_k)^{\wedge n+d} \to \mathrm{Th}_M(V)$$

in the category $\mathcal{H}_\bullet(k)$ *with the property that, if* k *is perfect, the composition*

$$\begin{aligned} H^{2d,d}(M, \mathbb{Z}) &\stackrel{t'}{\to} \tilde{H}^{2(n+d),n+d}(\mathrm{Th}_M(V), \mathbb{Z}) \stackrel{f_V^*}{\to} \\ &\to \tilde{H}^{2(n+d),n+d}((\mathbb{P}^1_k)^{\wedge n+d}, \mathbb{Z}) \cong \mathbb{Z} \end{aligned} \tag{3.1}$$

can be described as sending a zero cycle

$$\sum_i m_i[\mathrm{Spec}\, L_i] \in CH^d(M) \cong H^{2d,d}(M, \mathbb{Z})$$

to the number $\sum_i m_i[L_i : k]$, *known in the literature as the* degree *of the zero cycle in question. Here,* t' *is the cohomological Thom isomorphism* (see [2]) *and the last is the* (*stable*) *suspension isomorphism.*

We introduce the following notation:

- let M is a smooth projective variety of pure dimension d and V one of the vector bundles whose existence is asserted by the previous theorem,
- Gr_n is the infinite Grassmannian of n dimensional planes,
- $v : M \to \mathrm{Gr}_n$ is the map in $\mathcal{H}(k)$ classifying V,
- t is the homology Thom isomorphism,
- $i : \mathrm{Gr}_d \to \mathrm{Gr} := \lim_h \mathrm{Gr}_h$ is the canonical map,
- and $j : \Sigma^\infty MGl_n \to \Sigma^{2n,n}\mathbf{MGl}$ the composition in $\mathcal{SH}(k)$

$$\Sigma^\infty MGl_n \to \mathbf{MGl}[n] \overset{\sigma^{-n}}{\to} \Sigma^{2n,n}\mathbf{MGl}$$

where $MGl[n]_i = MGl_{i+n}$ (*cf.* [4, Lemma 3.19]).

To each such M, we can associate the homotopy class $a_d^V \in \pi_{2d,d}(\mathbf{MGl})$, represented by the composition in $\mathcal{SH}(k)$

$$\Sigma^\infty(\mathbb{P}^1_k)^{\wedge n+d} \overset{f_V}{\to} \Sigma^\infty \mathrm{Th}(V) \overset{\mathrm{Th}(v)}{\to} \Sigma^\infty MGL_n \overset{j}{\to} \Sigma^{2n,n}\mathbf{MGl}. \tag{3.2}$$

Essentially by definition we have that $j_* \circ \mathrm{Th}(v)_* \circ (f_V)_*(\Sigma^{2(n+d),n+d}\iota) = h(a_d^M)$ where ι is the generator of $\widetilde{H}_{2(n+d),n+d}((\mathbb{P}^1_k)^{\wedge n+d}, \mathbb{Z})$. We will now show that the coefficients of $h(a_d^M)$ are the degrees of the zero cycles represented by certain polynomials of degree d in the Chern classes. Consider the following diagram:

$$\begin{array}{ccccc}
\widetilde{H}_{2(n+d),n+d}((\mathbb{P}^1_k)^{\wedge n+d}, \mathbb{Z}) & & & & \\
\downarrow (f_V)_* & & & & \\
\widetilde{H}_{2(n+d),n+d}(\mathrm{Th}(V),\mathbb{Z}) & \xrightarrow{\mathrm{Th}(v)_*} & \widetilde{H}_{2(n+d),n+d}(MGl_n,\mathbb{Z}) & \xrightarrow{j_*} & \widetilde{H}_{2d,d}(\mathbf{MGl},\mathbb{Z}) \\
\cong\downarrow t & & \cong\downarrow t & & \cong\downarrow t \\
H_{2d,d}(M,\mathbb{Z}) & \xrightarrow{v_*} & H_{2d,d}(\mathrm{Gr}_n,\mathbb{Z}) & \xrightarrow{i_*} & H_{2d,d}(\mathrm{Gr},\mathbb{Z}).
\end{array} \tag{3.3}$$

The diagram commutes by naturality of Thom isomorphism (vertical maps). In Section 4, Step 3, we will show that we can choose canonical polynomial generators β_u and b_u of the motivic homology of $\mathbf{MGl}$ and Gr respectively, with the property that the homological Thom isomorphism t is described by

$$\begin{aligned}\tilde{H}_{*,*}(\mathbf{MGl}, \mathbb{Z}) \cong H_{*,*}(\mathrm{Spec}\, k, \mathbb{Z})[\beta_1, \beta_2, \cdots] \overset{t}{\to} \\ \to H_{*,*}(\mathrm{Spec}\, k, \mathbb{Z})[b_1, b_2, \cdots] \cong H_{*,*}(\mathrm{Gr}, \mathbb{Z})\end{aligned} \tag{3.4}$$

with $t(\beta_1^{\alpha_1} \cdots \beta_r^{\alpha_r}) = b_1^{\alpha_1} \cdots b_r^{\alpha_r}$. Therefore the coefficients of $h(a_d^M)$ can be computed via the Kronecker products $\langle (i \circ v)_*(t((f_V)_*\iota)), s_\alpha\rangle$ having denoted s_α the dual class to $b_1^{\alpha_1}\cdots b_r^{\alpha_r}$ with $\alpha=(\alpha_1,\cdots,\alpha_r)$. By adjunction, this is the same as $\langle (t((f_V)_*\iota), (i\circ v)^*(s_\alpha)\rangle$ or even $\langle \iota, f_V^* t'(i\circ v)^*(s_\alpha)\rangle$ having set t' to be the cohomological Thom isomorphism. Because of Theorem 3.2, $f_V^* t'(i\circ v)^*(s_\alpha)$ is precisely the degree of the zero cycle $(i\circ v)^*(s_\alpha)$ in M. This is some homogeneous polynomial of degree d in the Chern classes of V. On the other hand, since any such vector bundle V satisfies the equation $T_M\oplus V=\mathcal{O}_M^{n(V)+d}$, we see inductively that the Chern classes of all these V are the same and can be expressed in fuction of Chern classes of T_M. To prove Theorem 3.1.1. it suffices to present smooth projective varieties M^{q^t-1} of dimension q^t-1 such that q divides all the Chern classes $c_k(T_{M^{q^t-1}})$, but q^2 does not divide $\deg((i\circ v)^*(s_\alpha))$ for $\alpha=(0,\cdots,\overset{q^t-1}{1})$, where the latter notation stands for a string of zeroes with a 1 placed in the $(q^t-1)-th$ position. Notice that $s_r \overset{\text{def}}{=} s_{\underset{(0,\cdots,1)}{r}}$ are primitive classes, therefore they are additive in the sense that if V and W are vector bundles, $s_r(V\oplus W)=s_r(V)+s_r(W)$. In particular $s_r(T_M)=-s_r(V)$. We claim that any degree q smooth projective hypersurface in $\mathbb{P}^{q^t}$ can be taken to be M^{q^t-1}. In general, if $i: M\hookrightarrow \mathbb{P}^N$ is an hypersurface, we have that $i^*x=x\cup c_1(\mathcal{O}(M))$ for any cycle $x\in CH^*(\mathbb{P}^N)$. If M is of degree h, then $c_1(\mathcal{O}(M))=h\cdot L$, where L is the class of an hyperplane. On the other hand, we have an exact sequence of vector bundles:

$$0\to T_M\to i^*T_{\mathbb{P}^N}\to i^*\mathcal{O}(M)\to 0 \tag{3.5}$$

yielding $c_k(T_M)=c_k(i^*T_{\mathbb{P}^N})-c_{k-1}(T_M)c_1(i^*\mathcal{O}(M))$ for any k. By naturality of Chern classes, $c_k(T_M)=i^*c_k(T_{\mathbb{P}^N})-c_{k-1}(T_M)i^*c_1(\mathcal{O}(M))$ so that h divides $c_k(T_M)$ and any polynomial in the Chern classes of the tangent bundle of M. In particular h divides all the $(i\circ v)^*(s_\alpha)$. For any positive integer r, $s_r(T_M)$ is computed by using that this type of class is additive and the answer is $s_d(T_M)=h(d+2-h^d)$ for a smooth hypersurface M of dimension d and degree h (see [7, Problem 16D], for instance). If we take $h=q$ and dimension $d=q^t-1$ we have that q^2 does not divide $s_d(M)$. This proves part (3.1) of Theorem 3.1. To settle part (5.13) we argue as follows: let $H_{n,m}$ be a smooth hypersurface of degree $(1,1)$ in $\mathbb{P}^n\times\mathbb{P}^m$. Then ([7, Problem 16E])

$$s_{n+m-1}(T_{H_{n,m}})=\frac{-(n+m)!}{n!m!}.$$

If $u\neq q^t-1$ for some prime number q and positive integer t, let α be an integer such that $\alpha q<u+1\leq(\alpha+1)q$. We can check that

$$s_u(H_{(u+1-\alpha q),\alpha q})=\frac{(u+1)!}{(u+1-\alpha q)!\alpha q!}$$

is not divisible by q. Let now $H_{n(q_i),m(q_i)}=H_{(u+1-\alpha q_i),\alpha q_i}$ for each prime number $q_i<u+1$. Then, there exist integers $\lambda_1,\cdots\lambda_{\Phi(u)}$ so that

$$\lambda_1 s_u(H_{n(2),m(2)})+\lambda_2 s_u(H_{n(3),m(3)})+\cdots\lambda_{\Phi(u)}s_u(H_{n(p),m(p)})=1$$

where $\Phi(u)$ is the number of the prime numbers less that u and p is the greatest such prime. It follows that the homotopy class

$$a_u := \sum_{i=1}^{\Phi(u)} \lambda_i a_u^{H_{n(q_i),m(q_i)}}$$

has the property that the coefficient of b_u in $h(a_u)$ equals 1, hence q does not divide $h(a_u)$, as sought.

4. Motivic cohomology of MGl

Theorem 3.1 gives us the classes necessary to proceed with the construction of the spectrum that has been denoted with $k'(t)$ in [2]. We fix a prime number q and recall that $k'(t)$ has been defined in [2] as the homotopy colimit of the diagram

$$\mathbf{MGl} \xrightarrow{\eta_0} \mathbf{E}_0 \xrightarrow{\eta_1} \mathbf{E}_1 \xrightarrow{\eta_2} \cdots \tag{4.1}$$

with $\mathbf{E}_i$ defined inductively by the exact triangle

$$\Sigma^{2i,i}\mathbf{E}_{i-1} \stackrel{\cdot a_i}{\to} \mathbf{E}_{i-1} \stackrel{\eta_i}{\to} \mathbf{E}_i \tag{4.2}$$

where $\cdot a_i$ is the map induced by the **MGl**-module multiplication by a_i: $(\mathbb{P}^1_k)^{\wedge i} \wedge \mathbf{E}_{i-1} \stackrel{a_i \wedge id}{\to} \mathbf{MGl} \wedge \mathbf{E}_{i-1} \stackrel{m}{\to} \mathbf{E}_{i-1}$. The morphism $\cdot a_0$ is taken to be the *multiplication by q*. Here we use that the category of **MGl** modules spectra is closed under homotopy colimits. In classical homotopy theory we can prove that such $k'(t)$ is the homotopy limit of a tower of fibrations whose fibers are appropriate suspensions of Eilenberg-MacLane spectra with coefficients in $\mathbb{Z}/q$. Our purpose here is to build such tower and then we will take its homotopy limit as our model for $k(t)$. This spectra have been employed to prove the so called *higher degree formulas* (*cf.* [1]) and for that it was essential to have *motivic* Eilenberg-MacLane spectra appearing in the tower (see also [2, page 411]). The purpose of this section is to compute the motivic cohomology of the spectra $k'(t)$. It will turn out that it is "as expected" (Theorem 4.6). This result will be extensively used in the next section's computations. We start by expressing the motivic cohomology of **MGl** as graded left module over the ring $\mathcal{A}^{*,*}$. This is the sub left $H^{*,*}(\mathrm{Spec}\, k, \mathbb{Z}/q)$ module generated by the motivic Steenrod operations β, P^i defined in [13]. It happens to be closed under the multiplication and comultiplication of $\mathcal{A}_m^{*,*} := [\mathbf{H}_{\mathbb{Z}/q}, \Sigma^{*,*}\mathbf{H}_{\mathbb{Z}/q}]_{\mathcal{SH}(k)}$ so it inherits these structures. As usual we will denote $H^{*,*}(\mathrm{Spec}\, k, \mathbb{Z}/q) =: H^{*,*}$. In [13] V. Voevodsky has shown that the motivic Steenrod algebra $\mathcal{A}^{**}$ is isomorphic to $H^{*,*} \otimes_{\mathbb{Z}/q} \mathcal{A}^{*,*}_{\mathrm{top}}$ as left $H^{*,*}$ module. As $\mathbb{Z}/q$ vector space, $\mathcal{A}^{*,*}_{\mathrm{top}}$ is generated by the operations $Q_0^{\epsilon_0} Q_1^{\epsilon_1} \cdots Q_m^{\epsilon_m}(r_1, \cdots, r_n)$ for nonnegative integers r_i and $\epsilon_j \in \{0, 1\}$ for all j. These operations are entirely analogous to their topological counterparts except for the bidegree and that at the prime $q = 2$, the comultiplication is more involved than the usual one. We recall that the dual of the motivic Steenrod algebra $\mathcal{A}_{*,*}$ is isomorphic to a quotient of

$$H_{*,*}[\xi_1, \xi_2, \cdots] \otimes \Lambda_{H_{*,*}}[\tau_0, \tau_1, \cdots].$$

The operations Q_t and $(r_1, \cdots, r_n)$ are defined as the duals to the classes τ_t and $\xi_1^{r_1} \cdots \xi_n^{r_n}$ and have bidegree $(2q^t - 1, q^t - 1)$ and

$$\left(\sum_{i=1}^{n} 2r_i(q^i - 1), \sum_{i=1}^{n} r_i(q^i - 1)\right)$$

respectively. For more details we refer to [13].

Theorem 4.1. *For a fixed prime number q, denote by $T(q)$ the set of integers $\{q^n - 1 : n \in \mathbb{Z}^+\}$, where $\mathbb{Z}^+$ is the set of positive integers. Let $I = (i_1, \cdots, i_k)$ with $i_j \geq 0$ and $i_j \in \mathbb{Z}^+ - T(q)$. Then $H^{*,*}(\mathbf{MGl}, \mathbb{Z}/q)$ is isomorphic to*

$$\frac{\mathcal{A}^{**}}{\mathcal{A}^{**}(Q_0, \cdots, Q_t, \cdots)} \cdot \{d'_I, I = (i_1, \cdots, i_k)\} \tag{4.3}$$

as graded left $\mathcal{A}^{**}$ module. The basis elements d'_I are assigned the bidegree $(2(\sum_j ji_j), \sum_j ji_j)$.

Remark 4.2. Using the commuting rules of the [2, Corollary 4], we see that the $H^{*,*}$ subalgebra of $\mathcal{A}^{**}$ generated by $\{(r_1, \cdots, r_n)\}$ for all $r_i \geq 0$ is isomorphic, as left $H^{*,*}$ module, to $\frac{\mathcal{A}^{**}}{\mathcal{A}^{**}(Q_0, \cdots, Q_t, \cdots)}$, although it may happen that $\frac{\mathcal{A}^{**}}{\mathcal{A}^{**}(Q_0, \cdots, Q_t, \cdots)}$ is not a ring (for instance if $q = 2$, $\sqrt{-1} \notin k$ and $\mathrm{char}(k) \neq 2$, simultaneously).

Proof. The statement will follow by duality, from:

Theorem 4.3. *For any prime number q, there is a left comodule algebra isomorphisms over the dual motivic Steenrod algebra $\mathcal{A}_{*,*}$:*

$$\tilde{H}_{*,*}(\mathbf{MGl}, \mathbb{Z}/q) \cong H_{*,*}[\xi_1, \cdots, \xi_n, \cdots] \otimes_{H_{*,*}} H_{*,*}[d_i, i \neq q^n - 1, \forall n] \tag{4.4}$$

with $|d_i| = (2i, i)$ and $H_{*,*}[\xi_1, \cdots, \xi_n, \cdots] \hookrightarrow \mathcal{A}^{**}$. The $\mathcal{A}_{*,*}$ comodule structure of the tensor product is $\psi(p(\xi_i) \otimes r(d_j)) = \psi(p(\xi_i)) \otimes r(d_j)$ for $p(\xi_i) \in H_{*,*}[\xi_1, \cdots, \xi_n, \cdots]$ and $r(d_j) \in H_{*,*}[d_i, i \neq q^n - 1, \forall n]$ where ψ is the coproduct of $\mathcal{A}_{*,*}$. We set ξ_0 to be 1.

Let us prove first that Theorem 4.3 implies Theorem 4.1. If $\theta \in \mathcal{A}^{**}_m$ and $\sigma \in H^{*,*}(\mathbf{X}, \mathbb{Z}/q)$ for some spectrum $\mathbf{X}$, we will occasionally write $\theta \overset{\mathcal{A}}{\cdot} \sigma$ to denote the class in $H^{*,*}(\mathbf{X}, \mathbb{Z}/q)$ obtained by applying the cohomological operation θ to the cohomology class σ, when the simple notation $\theta\sigma$ may appear confusing. [2, Theorem 3] and the fact that $H^{u,v}(\mathrm{Spec}\ k, \mathbb{Z}/q) = 0$ for $v > u$ implies that

$Q_i \overset{\mathcal{A}}{\cdot} c = 0$ for any class in $c \in H^{2*,*}(\mathbf{MGl}, \mathbb{Z}/q)$ for all $i \geq 0$ simply by degree considerations. Thus, the left $\mathcal{A}^{**}$ action on the classes d'_I factors through $\frac{\mathcal{A}^{**}}{\mathcal{A}^{**}(Q_0,\cdots,Q_t,\cdots)}$. We now prove that $\frac{\mathcal{A}^{**}}{\mathcal{A}^{**}(Q_0,\cdots,Q_t,\cdots)} \cdot d'_I \cong \frac{\mathcal{A}^{**}}{\mathcal{A}^{**}(Q_0,\cdots,Q_t,\cdots)}$. Recall that ξ_k have the following $\mathcal{A}_{*,*}$ coproduct:

$$\psi(\xi_k) = \sum_{i=0}^{k} \xi_{k-i}^{q^i} \otimes \xi_i \tag{4.5}$$

(*cf.* [13] or [5]). Assume Theorem 4.3 holds true. The duality between $\mathcal{A}^{**}$ action and $\mathcal{A}_{*,*}$ coaction means that $\langle \theta \overset{\mathcal{A}}{\cdot} y, x\rangle = \langle \theta \otimes y, \psi(x)\rangle$ for all $\theta \in \mathcal{A}^{**}$, $y \in H^{*,*}(\mathbf{MGl}, \mathbb{Z}/q)$ and $x \in H_{*,*}(\mathbf{MGl}, \mathbb{Z}/q)$. For any $I = (i_1, \cdots, i_k)$, we let $d_I = d_1^{i_1} \cdots d_k^{i_k}$ and d'_I be the Kronecker dual to d_I and $1 \otimes d'_I \in H^{*,*}(\mathbf{MGl}, \mathbb{Z}/q)$ the Kronecker dual to $1 \otimes d_I \in H_{*,*}(\mathbf{MGl}, \mathbb{Z}/q)$. Using the previous rules, we prove that, for I fixed, the class $(r_1, \cdots, r_n) \overset{\mathcal{A}}{\cdot} (1 \otimes d'_I)$ is always nonzero. Indeed,

$$\begin{aligned} &\langle (r_1, \cdots, r_n) \overset{\mathcal{A}}{\cdot} (1 \otimes d'_I), \xi_1^{r_1} \cdots \xi_n^{r_n} \otimes x\rangle \\ &= \left\langle (r_1, \cdots, r_n) \otimes (1 \otimes d'_I), \prod_{k=1}^{n} \psi(\xi_k)^{r_k} \otimes x \right\rangle \\ &= \left\langle (r_1, \cdots, r_n) \otimes (1 \otimes d'_I), \prod_{k=1}^{n} \left(\sum_{j=0}^{k} \xi_{k-j}^{q^j} \otimes \xi_j \right)^{r_k} \otimes x \right\rangle. \end{aligned} \tag{4.6}$$

By definition of $(r_1, \cdots, r_n)$ and d'_I, this integer is nonzero and equals 1 only if $x = d_I$, since it equals to $\langle (r_1, \cdots, r_n), \xi_1^{r_1} \cdots \xi_n^{r_n}\rangle \langle 1 \otimes d'_I, 1 \otimes d_I\rangle$. Because of the Remark 4.2, this proves Theorem 4.1.

Proof of Theorem 4.3. We will proceed by steps. □

Step 1. Let γ_1 be the universal line bundle on $\mathbb{P}^\infty \cong \mathrm{Gr}_1$ so that we have $t : H^{*,*}(\mathrm{Gr}_1, \mathbb{Z}/q) \overset{\cong}{\to} \widetilde{H}^{*+2,*+1}(\mathrm{Th}(\gamma_1), \mathbb{Z}/q)$, the Thom isomorphism. We wish to describe the left $\mathcal{A}^{**}$ action on $\tilde{H}^{*,*}(\mathrm{Th}(\gamma_1), \mathbb{Z}/q)$. In $H^{*,*}(\mathbb{P}(\gamma_1 \times \mathbb{A}^1_k), \mathbb{Z}/q)$ this homomorphism can be described as $- \cup c^{\mathrm{rank}(\gamma_1)} = - \cup c$, where $c \in H^{2,1}(\mathbb{P}(\gamma_1 \times \mathbb{A}^1_k), \mathbb{Z}/q)$ is the pull back of the polynomial generator x of $H^{*,*}(\mathbb{P}^\infty, \mathbb{Z}/q) \cong H^{*,*}[x]$ via the map classifying the universal line bundle over $\mathbb{P}(\gamma_1 \times \mathbb{A}^1_k)$. In our case, by the Projective Bundle theorem (*cf.* [2]) and definition of Chern classes c_i, the powers of c are subject to the relation

$$c^2 = c_1(\gamma_1 \times \mathbb{A}^1_k)c - c_2(\gamma_1 \times \mathbb{A}^1_k)$$

which simplifies to $c^2 = xc$, where

$$H^{*,*}(\mathrm{Gr}_1, \mathbb{Z}/q) = H^{*,*}(\mathbb{P}^\infty, \mathbb{Z}/q) \cong H^{*,*}[x]$$

and $x = c_1(\gamma_1)$. In other words,[1]

$$H^{*,*}(\mathbb{P}(\gamma_1 \times \mathbb{A}^1), \mathbb{Z}/q) \cong H^{*,*}[c, x]/(c^2 - xc).$$

The canonical map $p : \mathbb{P}(\gamma_1 \times \mathbb{A}^1_k) \to \mathrm{Th}(\gamma_1)$, in motivic cohomology, identifies isomorphically the group $\widetilde{H}^{*+2,*+1}(\mathrm{Th}(\gamma_1), \mathbb{Z}/q)$ with the subgroup $H^{*,*}(\mathrm{Gr}_1, \mathbb{Z}/q) \cdot c$ of $H^{*,*}(\mathbb{P}(\gamma_1 \times \mathbb{A}^1_k), \mathbb{Z}/q)$. Define y_i as

$$\widetilde{H}^{2i,i}(\mathrm{Th}(\gamma_1), \mathbb{Z}/q) \ni y_i := (p^*)^{-1}(x^{i-1}c) \tag{4.7}$$

for $i > 0$. By the Thom isomorphism, $\widetilde{H}^{*,*}(\mathrm{Th}(\gamma_1), \mathbb{Z}/q) \cong \oplus_i H^{*,*} y_i$ as left $H^{*,*}$ module and $\widetilde{H}_{*,*}(\mathrm{Th}(\gamma_1), \mathbb{Z}/q) \cong \oplus_i H_{*,*} y_i'$, where y_i' are the classes dual to y_i. We want to understand the $\mathcal{A}_{*,*}$ coaction $\psi(y_i')$ of $y_i' \in \widetilde{H}_{*,*}(\mathrm{Th}(\gamma_1), \mathbb{Z}/q)$ by means of the $\mathcal{A}^{**}$ action on y_i. By the very definition of y_i (equation (4.7)), and the Cartan formulae ([13, Proposition 9.6]) this reduces to the $\mathcal{A}^{**}$ action on $x \in H^{2,1}(\mathbb{P}^\infty, \mathbb{Z}/q)$ and on c. Because of [13, Lemma 9.7] and Cartan formulae, we see that if $q = 2$, $\theta \overset{\mathcal{A}}{\cdot} x \neq 0$ if and only if $\theta = aSq^{2^{n-1}} Sq^{2^{n-2}} \cdots Sq^4 Sq^2$ for some $a \in H^{*,*}(\mathrm{Spec}\ k, \mathbb{Z}/q)$ and n positive integer and in that case $\theta \overset{\mathcal{A}}{\cdot} x = ax^{2^n}$. If q is odd then we conclude similarly that θ must be of the kind $aP^{q^{n-1}} P^{q^{n-2}} \cdots P^q P^1$ and the result is ax^{q^n}. If $i = 1$ and $q = 2$, $\theta \overset{\mathcal{A}}{\cdot} y_1 \neq 0$ if and only if $\theta = aSq^{2^{n-1}} \cdots Sq^4 Sq^2$ (and $\theta y_1 = ay_{2^n}$), whereas if q is odd, $\theta = aP^{q^{n-1}} P^{q^{n-2}} \cdots P^q P^1$ (and $\theta y_1 = ay_{q^n}$). By employing the equality $y_i = (p^*)^{-1}(x^{i-1}c)$ and the relation $c^2 = xc$ we see that: $\theta \overset{\mathcal{A}}{\cdot} y_n = \sum_i a_i y_i$ *if and only if* $\theta \overset{\mathcal{A}}{\cdot} x^n = \sum_i a_i x^i$.

Step 2. We need a better understanding of the $\mathcal{A}_{*,*}$ coaction on $\tilde{H}_{*,*}(\mathrm{Th}(\gamma_1), \mathbb{Z}/q)$:

Lemma 4.4. *Let* $\psi : \tilde{H}_{*,*}(\mathrm{Th}(\gamma_1), \mathbb{Z}/q) \to \mathcal{A}_{*,*} \otimes_{H_{*,*}} \tilde{H}_{*,*}(\mathrm{Th}(\gamma_1), \mathbb{Z}/q)$ *the* $\mathcal{A}_{*,*}$ *coaction. Then for any prime number q*

$$\mathrm{Im}(\psi) \subset H_{*,*}[\xi_1, \xi_2, \cdots, \xi_n, \cdots] \otimes_{H_{*,*}} \tilde{H}_{*,*}(\mathrm{Th}(\gamma_1), \mathbb{Z}/q).$$

Proof. Since ψ is $H_{*,*}$ linear, it suffices to consider $\psi(y_i')$ for all i. We recall that $a_i - 2b_i = 1$ if $|Q_i| = (a_i, b_i)$. This implies that $Q_i \overset{\mathcal{A}}{\cdot} x = 0$ and consequently $Q_i \overset{\mathcal{A}}{\cdot} y_1 = 0$ for all $i \geq 0$. In view of the duality between the action of $\mathcal{A}^{**}$ and the coaction of the dual and motivic cohomology of $\mathrm{Th}(\gamma_1)$ and the fact that $Q_i(r_1, \cdots, r_n)$ is dual to $\tau_i \xi_1^{r_1} \cdots \xi_n^{r_n}$, to prove the lemma it suffices to show that $Q_n \overset{\mathcal{A}}{\cdot} y_i = 0$ for all n and i. As in Step 1. $Q_n \overset{\mathcal{A}}{\cdot} y_i$ can be computed by means of

[1] Probably $x = c$, but we will not need this fact in the computations.

$Q_n \overset{\mathcal{A}}{\cdot} x^i$. More precisely, $Q_n \overset{\mathcal{A}}{\cdot} x^i = \sum_{j=0}^{i-1} x^j Q_n \overset{\mathcal{A}}{\cdot} x x^{i-j-1} + \sum \rho^h p(x)$, where $p(x)$ are polynomials whose monomials contain at least a factor of the kind $Q_s \overset{\mathcal{A}}{\cdot} x$ (*cf.* [2, Corollary 5]), therefore the whole expression is zero. By the last remark in Step 1 this implies $Q_n \overset{\mathcal{A}}{\cdot} y_i = 0$. □

Step 3. Now we will exibit a different $H_{*,*}$ polynomial algebra basis for $H_{*,*}(\mathrm{Gr_n}, \mathbb{Z}/q)$ than the one described in [2], which will be more useful in our argument. Let $\mu : \prod_{i=1}^{n} \mathrm{Gr}_1 \to \mathrm{Gr_n}$ classifying $\prod_{i=1}^{n} \gamma_{1,i}$. μ can be employed in the usual way to obtain a map, which we will call $\mu_* : \otimes_{i=1}^{n} H_{*,*}(\mathrm{Gr}_1, \mathbb{Z}/q) \to H_{*,*}(\mathrm{Gr_n}, \mathbb{Z}/q)$. In what follows we set $z_{k_i} \in H_{2k_i,k_i}(\mathrm{Gr}_1, \mathbb{Z}/q)$ to be the Kronecker dual to $c_1^{k_i}(\gamma_{1,i})$, that is the k_i-th power of the first Chern class of the universal line bundle on the i-th copy of Gr_1 in $\mu : \prod_{i=1}^{n} \mathrm{Gr}_1 \to \mathrm{Gr_n}$. We wish to show that all the elements $b_{k_1} \cdots b_{k_m} := \mu_*(z_{k_1} \otimes \cdots \otimes z_{k_m})$ form an $H_{*,*}$ basis of $H_{*,*}(\mathrm{Gr_n}, \mathbb{Z}/q)$ for $k_1 \geq \cdots \geq k_m \geq 0$ and $m \leq n$. To do this, notice that we can order lexicographically from right to left such b_I (*e.g.* $b_6 b_5 b_0 < b_2 b_1 b_1$) so that it suffices to give, for each monomial in the Chern classes $f(c_i) = c_1^{e_1}, \cdots c_n^{e_n}$, a class b_f with the properties that $\langle f(c_i), b_f \rangle = 1$, $\langle f(c_i), b_I \rangle = 0$ for $b_I < b_f$ and $b_f \neq b_{f'}$ if $f \neq f'$. Because we know that the Chern classes are polynomial generators of the motivic cohomology of $\mathrm{Gr_n}$, this proves the claim. Given two vector bundles ν and η over X we have that $c_h(\nu \oplus \eta) = \sum_{i=0}^{h} c_i(\nu) \cup c_{h-i}(\eta)$ from which it follows that $c_h(\nu \times \eta) = \sum_{i=0}^{h} c_i(\nu) \times c_{h-i}(\eta) \in H^{*,*}(X \times_{\mathrm{Spec}\, k} X, \mathbb{Z}/q)$. This shows that $c_h(\prod_{i=1}^{n} \gamma_{1,i}) = \sum_{j_1 < \cdots < j_h} x_{j_1} \otimes \cdots \otimes x_{j_h}$, that is the elementary h-th symmetric polynomial in the variables $x_i := c_1(\gamma_{1,i})$. We have the chain of equalities

$$\begin{aligned}\langle c_1^{e_1} \cdots c_n^{e_n}, \mu_*(z_{k_1} \otimes \cdots \otimes z_{k_n}) \rangle &= \langle \mu^*(c_1^{e_1} \cdots c_n^{e_n}), z_{k_1} \otimes \cdots \otimes z_{k_n} \rangle \\ &= \langle \mu^*(c_1)^{e_1} \cdots \mu^*(c_n)^{e_n}, z_{k_1} \cdots z_{k_n} \rangle\end{aligned}$$

but $\mu^*(c_h) = c_h(\prod_{i=1}^{n} \gamma_{1,i})$ and the smallest, according to the order given above, homology class b_I which has nonzero (actually equals to 1) Kronecker pairing with it comes from $z_1 \otimes \overset{h}{\cdots} \otimes z_1 \otimes z_0 \cdots \otimes z_0$ where we let $z_0 = 1$. Similarly we see that the smallest class with nonzero (and equals to 1) Kronecker pairing with $c_1^{e_1} \cdots c_n^{e_n}$ is

$$b_{e_1 + \cdots + e_n} b_{e_2 + \cdots + e_n} \cdots b_{e_{n-1} + e_n} b_{e_n}.$$

This is the class we denoted b_f above.

Step 4. (*end of the proof of Theorem* 4.3) We donote by t_i the homology Thom isomorphisms on the homology of Gr_i. By the commutativity of the diagram (which is consequence of the naturality of the Thom isomorphism)

$$\begin{array}{ccc}
H_{*,*}(\mathrm{Gr_n},\mathbb{Z}/q)\otimes H_{*,*}(\mathrm{Gr_m},\mathbb{Z}/q) & \xrightarrow{t_n\otimes t_m} & \widetilde{H}_{*+2m,*+m}(\mathrm{Th}(\gamma_m),\mathbb{Z}/q)\otimes\widetilde{H}_{*+2n,*+n}(\mathrm{Th}(\gamma_n),\mathbb{Z}/q) \\
\downarrow & & \downarrow \\
H_{*,*}(\mathrm{Gr_n}\times_k \mathrm{Gr_m},\mathbb{Z}/q) & & \widetilde{H}_{*+2(n+m),*+n+m}(\mathrm{Th}(\gamma_n)\wedge\mathrm{Th}(\gamma_m),\mathbb{Z}/q) \\
\downarrow{\scriptstyle \mu_{n,m*}} & & \downarrow{\scriptstyle \mathrm{Th}(\mu_{n,m})_*} \\
H_{*,*}(\mathrm{Gr_{n+m}},\mathbb{Z}/q) & \xrightarrow{t_{n+m}} & \widetilde{H}_{*+2(n+m),*+n+m}(\mathrm{Th}(\gamma_{n+m}),\mathbb{Z}/q)
\end{array} \tag{4.8}$$

we see that the equality in $H_{*,*}(\mathbf{MGl},\mathbb{Z}/q)$

$$t_n(b_{k_1}\cdot b_{k_s})\cdots t_m(b_{h_1}\cdots b_{h_r}) = t_{n+m}(b_{k_1}\cdots b_{k_s}b_{h_1}\cdots b_{h_r})$$

holds, thus $t_h(b_{k_1}\cdots b_{k_v}) = t_1(b_{k_1})\cdots t_1(b_{k_v})$. The ring structure in the motivic homology of both Gr and **MGl** is the Pontryagin product. We will take the monomials $t_1(b_{k_1})\cdots t_1(b_{k_l})$ for all $l>0$ as left $H_{*,*}$ basis for $\widetilde{H}_{*,*}(\mathbf{MGl},\mathbb{Z}/q)$. By naturality of the $\mathcal{A}_{*,*}$ coaction ψ and because of Lemma 4.4, we conclude that

$$\psi(\tilde{H}_{*,*}(\mathbf{MGl},\mathbb{Z}/q))\subset H_{*,*}[\xi_1,\cdots\xi_n,\cdots]\otimes_{H_{*,*}}\tilde{H}_{*,*}(\mathbf{MGl},\mathbb{Z}/q).$$

Consider the composition

$$\begin{aligned}\lambda:\tilde{H}_{*,*}(\mathbf{MGl},\mathbb{Z}/q)&\stackrel{\psi}{\to} H_{*,*}[\xi_1,\cdots\xi_n,\cdots]\otimes_{H_{*,*}}\tilde{H}_{*,*}(\mathbf{MGl},\mathbb{Z}/q)\stackrel{1\otimes f}{\to}\\ &\to H_{*,*}[\xi_1,\cdots\xi_n,\cdots]\otimes_{H_{*,*}}H_{*,*}[d_i, i\neq q^n-1,\forall n]\end{aligned} \tag{4.9}$$

the map f being the ring homomorphism defined by $f(t_1(1))=1=d_0$, $f(t_1(b_i))=d_i$ if $i\neq q^k-1$ for any k and $f(t_1(b_{q^k-1}))=0$.

Since both the domain and the target are polynomial algebras with one generator in bidegree $(2i,i)$ for all $i>0$. To prove that λ is an isomorphism, it suffices to show that the image through this map of each polynomial generator is indecomposable. In Step 1 we have shown that the Thom isomorphism in motivic cohomology sends $x^i\in H^{2i,i}(\mathrm{Gr}_1,\mathbb{Z}/q)=H^{2i,i}(\mathbb{P}^\infty,\mathbb{Z}/q)$ to $y_{i+1}\in\tilde{H}^{2(i+1),i+1}(\mathrm{Th}(\gamma_1),\mathbb{Z}/q)$, thus letting b_i to be the dual to x^i, $t_1(b_i)=y'_{i+1}$. If $i=q^k-1$, $\psi(t_1(b_i))=\psi(y'_{q^k})=1\otimes y'_{q^t}+\xi_k\otimes y'_1+\cdots$ and since $y'_1=t_1(1)$, we have that $\lambda(t_1(b_i))=\xi_k\otimes 1+\cdots$ hence it is indecomposable. If $i\neq q^k-1$, $\psi(t_1(b_i))=1\otimes t_1(b_i)+\cdots$ and $\lambda(t_1(b_i))=1\otimes d_i+\cdots$ which is still indecomposable. Thinking of λ as a morphism in the category of $\mathcal{A}_{*,*}$ comodule algebras by endowing the tensor products of the $\psi\otimes id$ coaction, we conclude the last part of Theorem 4.3.

At this point the construction proceeds as in [2]. Thanks to the previous theorem we can kill the homotopy classes described in Theorem 3.1 and keep track of the changements that occur in motivic cohomology when we pass from $\mathbf{E}_{i-1}$ to $\mathbf{E}_i$ for each i of the diagram (4.1). For the explicit calculations, we refer to the corresponding results in the proof of [2, Theorem 12]. They can be employed in our present context without any changements.

Definition 4.5. Denote $k'(t)$ the spectrum

$$\mathrm{hocolim}_i\{\mathbf{MGl} \stackrel{\eta_0}{\to} \mathbf{E}_0 \stackrel{\eta_1}{\to} \mathbf{E}_1 \cdots \to \mathbf{E}_n \stackrel{\eta_{n+1}}{\to} \cdots\}. \tag{4.10}$$

Let $\mathcal{A}^{**}$ be the left $H^{*,*}$ submodule of $[\mathbf{H}_{\mathbb{Z}/q}, \Sigma^{i,j}\mathbf{H}_{\mathbb{Z}/q}]_{\mathcal{SH}(k)}$ generated by monomials in Sq^i, if $q = 2$ and by P^i and the Bockstein β, if q is odd (see [2, 13]).

Theorem 4.6. *The spectra $k'(t)$, as defined above, have the property that*

$$H^{*,*}(k'(t), \mathbb{Z}/q) \cong \frac{\mathcal{A}^{**}}{\mathcal{A}^{**}Q_t}\tau \cong \frac{\mathcal{A}^{**}}{\mathcal{A}^{**}Q_t}$$

as left $\mathcal{A}^{**}$ module and $\tau \in H^{0,0}(k'(t), \mathbb{Z}/q)$ is the canonical class corresponding to the Thom class in $H^{0,0}(\mathbf{MGl}, \mathbb{Z}/q)$ via the canonical morphism $\mathbf{MGl} \to k'(t)$. Moreover, there exists an exact triangle $\Sigma^{2(q^t-1),q^t-1}k'(t) \stackrel{v_t}{\to} k'(t) \to C_0$, with $H^{*,*}(C_0, \mathbb{Z}/q) \cong \mathcal{A}^{**}\iota'$, $\iota' \in H^{0,0}(C_0, \mathbb{Z}/q)$ being the class corrsponding to τ.

Proof. See [2, Theorem 12]. □

Let $\mathcal{B}^{**}$ be the kernel of $(\iota')^*$ where $\iota' : C_0 \to \mathbf{H}_{\mathbb{Z}/q}$ is the class as in the statement of the theorem. Then, as left $\mathcal{A}^{**}$ module,

$$[\mathbf{H}_{\mathbb{Z}/q}, \Sigma^{*,*}\mathbf{H}_{\mathbb{Z}/q}]_{\mathcal{SH}(k)} =: \mathcal{A}^{**}_m \cong \mathcal{A}^{**} \oplus \mathcal{B}^{**}.$$

Corollary 4.7. *$b \in \mathcal{B}^{**}$ if and only if $b \stackrel{\mathcal{A}}{\cdot} \iota' = 0$. Furthermore, if $b \in \mathcal{B}^{**}$, $b \stackrel{\mathcal{A}}{\cdot} \tau = 0$ and hence $(bQ_t) \stackrel{\mathcal{A}}{\cdot} \iota' = b \stackrel{\mathcal{A}}{\cdot} (Q_t \stackrel{\mathcal{A}}{\cdot} \iota') = 0$ as well.*

Proof. The first two statements follow from the definition of action of a cohomological operation on a class. The third is proven via the commutativity of the square

$$\begin{array}{ccc} \mathbf{H}_{\mathbb{Z}/q} & \xrightarrow{Q_t} & \Sigma^{2(q^t-1)+1,q^t-1}\mathbf{H}_{\mathbb{Z}/q} \\ \uparrow{\scriptstyle \iota'} & & \uparrow{\scriptstyle \Sigma^{2(q^t-1)+1,q^t-1}\iota'} \\ C_0 & \longrightarrow & \Sigma^{2(q^t-1)+1,q^t-1}k'(t) \end{array} \tag{4.11}$$

and the second statement that implies $b \stackrel{\mathcal{A}}{\cdot} \tau = 0$. □

5. The spectra $\mathbf{\Phi}_i$

After the construction of $k'(t)$, the reason for proceeding further, lies in the fact that we do not know if these spectra are built out of suspensions of motivic Eilenberg-MacLane spectra $\Sigma^{*,*}\mathbf{H}_{\mathbb{Z}/q}$. Such property is valid for the spectra $\mathbf{\Phi}_i$ described below and is needed to derive the *degree formulae* of [1], because we know that

Eilenberg MacLane spectra represent the Chow groups functor under the assumption of perfectness of the base field. We will use what proved so far on the spectra $k'(t)$ to prove the main result. The $\mathbb{Z}/q$ algebra $\mathcal{A}^{**}_m$ has a left and right action by $\mathcal{A}^{**}$ induced by the multiplicative product of $\mathcal{A}^{**}_m$ and the canonical (split) inclusion $\mathcal{A}^{**} \hookrightarrow \mathcal{A}^{**}_m$ described at the end of Section 4. Recall that the *Margolis homology* $HM(M, Q_t\cdot)$ (resp. $HM(M, \cdot Q_t)$) of a left (resp. right) module M over $\mathcal{A}^{**}$ is defined as the first homology group of the complex $\{M \overset{Q_t\cdot}{\to} M \overset{Q_t\cdot}{\to} M\}$ (resp. of $\{M \overset{\cdot Q_t}{\to} M \overset{\cdot Q_t}{\to} M\}$). Let us denote by P the following condition: if a be a class in $\mathcal{B}^{**}/\mathcal{B}^{**}Q_t$ of degree $(2j(q^t-1)+2(j-1), j(q^t-1))$, then there exists a class $b \in \mathcal{B}^{**}/\mathcal{B}^{**}Q_t$ such that $a = Q_t b$, for $3 \le j$, where Q_t acts by left multiplication as an element of $\mathcal{A}^{**}_m$. We are going to make one of the following assumptions on the left $\mathcal{A}^{**}$ module $\mathcal{B}^{**}$:

(1) either P and the same Q_t-divisibility property hold for all classes $a \in \mathcal{B}^{**}/\mathcal{B}^{**}Q_t$ of degree $(2(j-2)(q^t-1)+(j-2), (j-2)(q^t-1))$,
(2) or P and the same Q_t-divisibility property hold for all classes $a \in HM(\mathcal{B}^{**}, \cdot Q_t)$ of degree $(2(j-2)(q^t-1)+(j-2), (j-2)(q^t-1))$.

Theorem 5.1. *Let k be any perfect field and q a fixed prime number. Suppose that $\mathcal{A}^{**}_m$ satisfies the assumption listed above. Then, for each $1 \le i \le r-1$, there exists a spectrum $\mathbf{\Phi}_i$ in the stable category $\mathcal{SH}(k)$ fitting in exact triangles*

$$\mathbf{\Phi}_i \longrightarrow \mathbf{\Phi}_{i-1} \xrightarrow{y_{i-1}} \Sigma^{2i(q^t-1)+1, i(q^t-1)} \mathbf{H}_{\mathbb{Z}/q} \ . \tag{5.1}$$

Let $\iota = 1 \in H^{0,0}(\mathbf{H}_{\mathbb{Z}/q}, \mathbb{Z}/q) = \mathcal{A}^{**}_m$ and

$$\delta : H^{*,*}(\mathbf{\Phi}_i, \mathbb{Z}/q) \to H^{*+1,*}(\Sigma^{2i(q^t-1), i(q^t-1)} \mathbf{H}_{\mathbb{Z}/q}, \mathbb{Z}/q)$$

be the coboundary operator induced by the exact triangle defining $\mathbf{\Phi}_i$ in function of y_{i-1} and $\mathbf{\Phi}_{i-1}$. Then the classes y_i are subject to the relation $\delta y_i = Q_t \Sigma^{2i(q^t-1), i(q^t-1)} \iota$.

We mention that [2, Lemma 14] can be employed to derive another kind of exact triangle involving the spectra $\mathbf{\Phi}_i$:

$$\Sigma^{2(q^t-1), q^t-1} \mathbf{\Phi}_{i-1} \longrightarrow \mathbf{\Phi}_i \xrightarrow{\tau_i} \mathbf{H}_{\mathbb{Z}/q} \ . \tag{5.2}$$

Proof of Theorem 5.1. Let $d = q^t - 1$. We first prove an algebraic result concerning certain left $\mathcal{A}^{**}$ modules. More precisely, we will recursively solve an algebraic problem following steps which correspond to the index i as in the spectrum $\mathbf{\Phi}_i$. Such algebraic picture is motivated by the homotopical approach to the construction of the $\mathbf{\Phi}_i$ as depicted in the diagram (5.15). At each step, we will use the assumptions (1) and (2) to prove the existence of a class y'_i satisfying few properties. Its existence allows us to proceed inductively to the next step. □

Step 0. Consider the short exact sequence of left $\mathcal{A}_m^{**}$ modules

$$0 \to \ker(p_0) \to \mathcal{A}_m^{**} \xrightarrow{p_0} \frac{\mathcal{A}^{**}}{\mathcal{A}^{**}Q_t} \to 0 \tag{5.3}$$

where $p_0(1) = 1$. Writing $\mathcal{A}_m^{**} \cong \mathcal{A}^{**} \oplus \mathcal{B}^{**}$ as left $\mathcal{A}^{**}$ modules and because of Corollary 4.7, we conclude that $\ker(p_0) \cong \mathcal{A}^{**} \cdot Q_t \oplus \mathcal{B}^{**}$ as left $\mathcal{A}^{**}$ modules. Since the $\cdot Q_t$ Margolis homology of $\mathcal{A}^{**}$ vanishes [2, Theorem 9], $\ker(p_0) \cong (\frac{\mathcal{A}^{**}}{\mathcal{A}^{**}Q_t} \cdot y_0') \oplus \mathcal{B}^{**}$, where we let y_0' to be Q_t.

Step 1. Consider now the new exact sequence of graded left $\mathcal{A}_m^{**}$ modules

$$0 \to \ker(p_1) \to \mathcal{A}_m^{**}\Sigma^{2d+1,d}\iota \xrightarrow{p_1} \ker(p_0) \cong \frac{\mathcal{A}^{**}}{\mathcal{A}^{**}Q_t} \cdot y_0' \oplus \mathcal{B}^{**}y_0' \tag{5.4}$$

where p_1 is defined by $p_1(\Sigma^{2d+1,d}\iota) = y_0' = Q_t$. Because of Corollary 4.7, multiplication to the right by Q_t is an homomorphism both $\mathcal{A}^{**} \to \mathcal{A}^{**}$ and $\mathcal{B}^{**} \to \mathcal{B}^{**}$, thus we have

$$\ker(p_1) = \frac{\mathcal{A}^{**}}{\mathcal{A}^{**}Q_t}(Q_t\Sigma^{2d+1,d}\iota) \oplus \ker\{\mathcal{B}^{**} \xrightarrow{\cdot Q_t} \mathcal{B}^{**}\}\Sigma^{2d+1,d}\iota$$

as left $\mathcal{A}^{**}$ modules. Later on, we are going to represent this exact sequence as part of a motivic cohomology long exact sequence between objects in $\mathcal{SH}(k)$. Thus, we wish to continue it on the left as if it were part of a long exact sequence of graded left $\mathcal{A}_m^{**}$ modules:

$$\cdots \xrightarrow{g_1} \mathcal{C}^{**} \xrightarrow{f_1} \mathcal{A}_m^{**}\Sigma^{2d+1,d}\iota \xrightarrow{p_1} \frac{\mathcal{A}^{**}}{\mathcal{A}^{**}Q_t} \cdot y_0' \oplus \mathcal{B}^{**} \tag{5.5}$$

where $\mathcal{C}^{**}$ fits in a short exact sequence of graded left $\mathcal{A}_m^{**}$ modules

$$\begin{aligned} &0 \to \Sigma^{-1,0}(\mathcal{B}^{**}/\mathcal{B}^{**}Q_t) \xrightarrow{g_1} \mathcal{C}^{**} \xrightarrow{f_1} \\ &\to \frac{\mathcal{A}^{**}}{\mathcal{A}^{**}Q_t}(Q_t\Sigma^{2d+1,d}\iota) \oplus \Sigma^{2d+1,d}\ker_{\mathcal{B}^{**}}(\cdot Q_t) = \ker(p_1) \to 0 \end{aligned} \tag{5.6}$$

where we denoted $\ker\{\mathcal{B}^{**} \xrightarrow{\cdot Q_t} \mathcal{B}^{**}\}$ by $\ker_{\mathcal{B}^{**}}(\cdot Q_t)$. Let y_1' be a lift of $Q_t\Sigma^{2d+1,d}\iota$ to $\mathcal{C}^{**}$. Composing f_1 with the projection $\ker(p_1) \to \ker(p_1)/\mathcal{A}_m^{**}(Q_t\Sigma^{2d+1,d}\iota)$, we see that $\mathcal{C}^{**}$ fits in the exact sequence of graded left $\mathcal{A}_m^{**}$ modules

$$0 \to \Sigma^{-1,0}(\mathcal{B}^{**}/\mathcal{B}^{**}Q_t) + \mathcal{A}_m^{**}y_1' \to \mathcal{C}^{**} \to \Sigma^{2d+1,d}\ker_{\mathcal{B}^{**}}(\cdot Q_t)/M \to 0. \tag{5.7}$$

Since $\cdot Q_t$ respects the splitting $\mathcal{A}_m^{**} = \mathcal{A}^{**} \oplus \mathcal{B}^{**}$, we get that $M = \mathcal{B}^{**}Q_t$. We wish to get a better understanding of the module $\mathcal{A}_m^{**}y_1'$. Since $\mathcal{A}_m^{**} \cong \mathcal{A}^{**} \oplus \mathcal{B}^{**}$

we conclude that $\mathcal{A}_m^{**}y_1' \cong \mathcal{A}^{**}y_1' + \mathcal{B}^{**}y_1'$. Now, suppose that the property (5) holds. Then we have that $Q_t y_1' = g_1(Q_t z)$ for some $z \in (\mathcal{B}^{**}/\mathcal{B}^{**}Q_t)^{4d+3,2d}$. But $g_1(Q_t z) = Q_t g_1(z)$ and letting $\hat{y}_1 := y_1' - g_1(z)$ we see that $Q_t \hat{y}_1 = 0$. Summing up, the property (5) implies that we can choose y_1' in such a way that $Q_t y_1' = 0$. Suppose now that $ay_1' = by_1'$ for $a \in \mathcal{A}^{**}$ and $b \in \mathcal{B}^{**}$, by applying p_1 we see that $aQ_t = bQ_t = 0$ because right multiplication by Q_t preserve the splitting of $\mathcal{A}_m^{**}$. Since $HM(\mathcal{A}^{**}, \cdot Q_t) = 0$ we have that $a = a'Q_t$, whence $ay_1' = 0$ by our assumption. This shows that

$$\mathcal{A}_m^{**}y_1' \cong \frac{\mathcal{A}^{**}}{\mathcal{A}^{**}Q_t}y_1' \oplus \mathcal{B}^{**}y_1'. \tag{5.8}$$

Using these new information, the sequence (5.7) becomes

$$\begin{aligned} 0 \to \Sigma^{-1,0}\frac{\mathcal{B}^{**}}{\mathcal{B}^{**}Q_t} + (\frac{\mathcal{A}^{**}}{\mathcal{A}^{**}Q_t}y_1' \oplus \mathcal{B}^{**}y_1') \\ \to \mathcal{C}^{**} \to \Sigma^{2d+1,d}\ker_{\mathcal{B}^{**}}(\cdot Q_t)/\mathcal{B}^{**}Q_t = \Sigma^{2d+1,d}HM(\mathcal{B}^{**}, \cdot Q_t) \to 0. \end{aligned} \tag{5.9}$$

We will use the description of the module $\mathcal{A}_m^{**}y_1'$ on the next step.

Step 2. The next short exact sequence is

$$0 \to \ker(p_2) \to \mathcal{A}_m^{**}\Sigma^{4d+2,2d}\iota \stackrel{p_2}{\to} \mathcal{C}^{**} \tag{5.10}$$

with $p_2(\Sigma^{4d+2,2d}\iota) = y_1'$. We extend it to a long exact sequence

$$\to \mathcal{D}^{**} \stackrel{f_2}{\to} \mathcal{A}_m^{**}\Sigma^{4d+2,2d}\iota \stackrel{p_2}{\to} \mathcal{C}^{**}. \tag{5.11}$$

The following exact sequences give a better understanding of $\mathcal{D}^{**}$.

(1) $\mathcal{D}^{**}$ fits in the short exact sequence of graded left $\mathcal{A}^{**}$ modules

$$0 \to \Sigma^{-1,0}\mathrm{coker}(p_2) \to \mathcal{D}^{**} \stackrel{f_2}{\to} \ker(p_2) \to 0 \tag{5.12}$$

(2) $\mathrm{coker}(p_2)$ fits in the short exact sequence

$$0 \to \Sigma^{-1,0}\frac{\frac{\mathcal{B}^{**}}{\mathcal{B}^{**}Q_t}}{M_1} \to \mathrm{coker}(p_2) \to HM(\mathcal{B}^{**}, \cdot Q_t)\Sigma^{2d+1,d}\iota \to 0 \tag{5.13}$$

for some module M_1;

(3) $\ker(p_2) \cong \mathcal{A}^{**}Q_t\Sigma^{4d+2,2d}\iota \oplus \mathrm{Ann}_{\mathcal{B}^{**}}(\cdot y_1')\Sigma^{4d+2,2d}\iota$ where the annihilator $\mathrm{Ann}_{\mathcal{B}^{**}}(\cdot y_1')$ fits in $\mathcal{B}^{**}Q_t \subset \mathrm{Ann}_{\mathcal{B}^{**}}(\cdot y_1') \subset \ker_{\mathcal{B}^{**}}(\cdot Q_t)$, by definition of y_1'. To prove this last isomorphism we use the computation of $\mathcal{A}_m^{**}y_1'$ done in the previous step.

We use this identification of $\ker(p_2)$ to define the class y_2' as a lifting of $Q_t\Sigma^{4d+2,2d}\iota$ to $\mathcal{D}^{**}$. For this we must have known that $Q_t\Sigma^{4d+2,2d}\iota$ belongs to $\ker(p_2)$. We conclude that $\mathcal{D}^{**}$ fits in the short exact sequence

$$0 \to \Sigma^{-1,0}\mathrm{coker}(p_2) + \mathcal{A}_m^{**}y_2' \to \mathcal{D}^{**} \xrightarrow{f_2} \frac{\mathrm{Ann}_{\mathcal{B}^{**}}(\cdot y_1')}{\mathcal{B}^{**}Q_t}\Sigma^{4d+2,2d}\iota \to 0. \quad (5.14)$$

Notice that the previous observation, $\mathrm{Ann}_{\mathcal{B}^{**}}(\cdot y_1')/(\mathcal{B}^{**}Q_t) \subset HM(\mathcal{B}^{**}, \cdot Q_t)$. As before, to set things for the next step and eventually define a y_3' we need more information on the module $\mathcal{A}_m^{**}y_2'$. We therefore use our assumption on $\mathcal{B}^{**}$ to show the existence of a lifting y_2' such that $Q_ty_2' = 0$. To do this we must ensure that $\Sigma^{-1,0}\mathrm{coker}(p_2)$ is divisible by $Q_t\cdot$. By the exact sequence (3.1) we see that it suffices to assume that $(\mathcal{B}^{**}/\mathcal{B}^{**}Q_t)^{8d+6,4d}$ and $HM(\mathcal{B}^{**}, \cdot Q_t)^{6d+3,3d}$ are divisible by $\cdot Q_t$ or $(\mathcal{B}^{**}/\mathcal{B}^{**}Q_t)^{8d+6,4d}$ and $(\mathcal{B}^{**}/\mathcal{B}^{**}Q_t)^{6d+3,3d}$ are divisible by $\cdot Q_t$.

Step n. Having a class $y_{n-1}' \in \mathcal{D}_{n-1}^{**}$ such that $Q_ty_{n-1}' = 0$ and $f_{n-1}(y_{n-1}') = Q_t\Sigma^{2(n-1)d+(n-1),n-1}\iota$, we define $\mathcal{D}_n^{**}$ as a module fitting in the long exact sequence

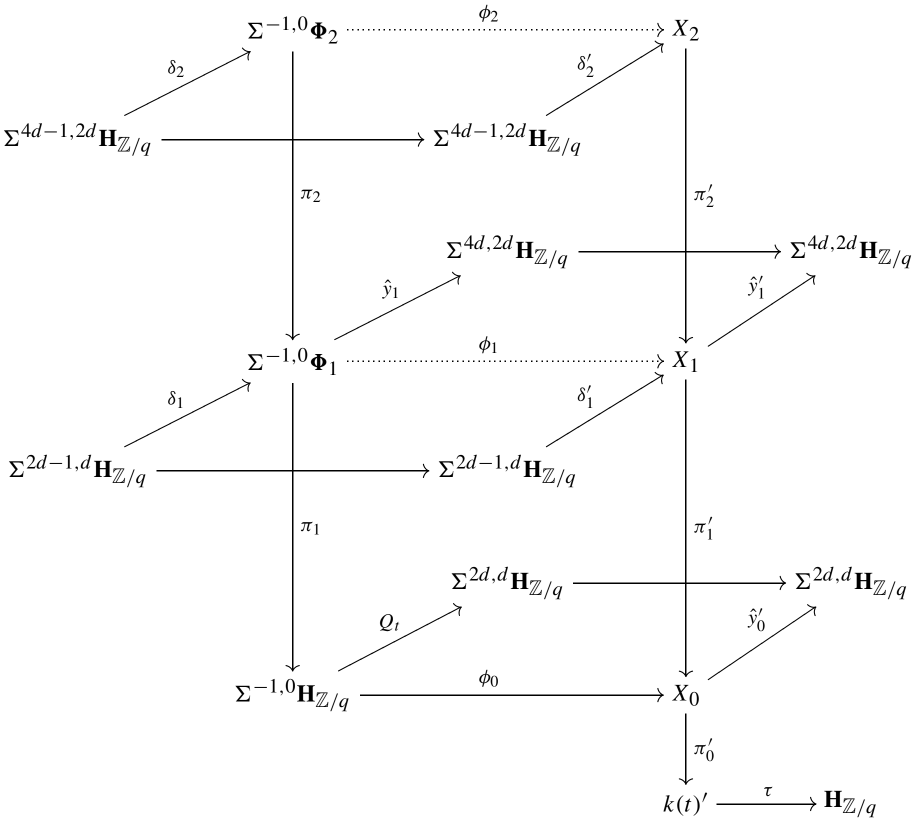
$$\to \mathcal{D}_n^{**} \to \mathcal{A}_m^{**}\Sigma^{2nd+n,nd}\iota \xrightarrow{p_n} \mathcal{D}_{n-1}^{**} \to \cdots$$

It follows that $\mathcal{D}_n^{**}$ lies in the short exact sequence

$$0 \to \Sigma^{-n+1,0}\mathrm{coker}(p_n) \to \mathcal{D}_n^{**} \to \ker(p_n) \to 0$$

the structure of $\ker(p_n)$ is relevant to the existence of y_n', whereas $Q_t\cdot$ divisibility of $\mathrm{coker}(p_n)$ and of $\ker(p_{n-1})$ are relevant to the existence of a y_n' with the property that $Q_ty_n' = 0$. We have that $\ker(p_n) \cong \mathcal{A}^{**}Q_t\Sigma^{2nd+n,n}\iota \oplus \mathrm{Ann}_{\mathcal{B}^{**}}(\cdot y_{n-1})\Sigma^{2nd+n,n}\iota$, thus we can define y_n' as a lifting to $\mathcal{D}_n^{**}$ of $Q_t\Sigma^{2nd+n,n}\iota$. Finally $\mathrm{coker}(p_n)$ is a multiple extension of suspended quotients of $\mathcal{B}^{**}/\mathcal{B}^{**}Q_t$ and of $HM(\mathcal{B}^{**}, \cdot Q_t)$ hence of $\mathcal{B}^{**}/\mathcal{B}^{**}Q_t$ (the latter statement follows from the remark that $\mathrm{Ann}_{\mathcal{B}^{**}}(\cdot y_{n-1}')/(\mathcal{B}^{**}Q_t) \subset HM(\mathcal{B}^{**}, \cdot Q_t) \subset \mathcal{B}^{**}/\mathcal{B}^{**}Q_t$). The assumptions on the module $\mathcal{B}^{**}$ set before the Theorem 5.1 imply the divisibility by $Q_t\cdot$ of $\Sigma^{-n+1,0}\mathrm{coker}(p_n)$ in degree $(2(n+2)+n+2, n+2)$, hence the existence of a choice of y_n' such that $Q_ty_n' = 0$.

We now perform constructions in the category $\mathcal{SH}(k)$ that will induce the exact sequences (5.3), (5.5), (5.11), …up to some degree shifting. The classes y_i' will be used to the define the y_i appearing in the statement of Theorem 5.1. Consider the following commutative diagram:



(5.15)

The triples $(\hat{y}_{i-1}, \pi_i, \delta_i)$ and $(\hat{y}'_{i-1}, \pi'_i, \delta'_i)$ form exact triangles as well as (τ, π'_0, ϕ_0). The exact triangle (τ, π'_o, ϕ_0) induces the sequence (5.3), the triangle $(\hat{y}'_0, \pi'_1, \delta'_1)$ induces the sequence (5.5), the triangle $(\hat{y}'_1, \pi'_2, \delta'_2)$ induces the sequence (5.11). This suffices to finish the proof of the theorem. To see this define $\hat{y}_i$ as $\phi_i^*(\hat{y}'_i)$, where ϕ_i are the fill-in maps between the relevant exact triangles. Since $(\delta'_i)^*\hat{y}'_i = Q_t\Sigma^{2id,id}\iota$, by commutativity of the square, we conclude that $\delta_i^*\hat{y}_i = Q_t\Sigma^{2id,id}\iota$ which is the statement of Theorem 5.1 since we can take $y_i := \sigma(\hat{y}_i)$, where σ is the canonical suspension isomorphism $\sigma : H^{*-1,*}(\Sigma^{-1,0}\mathbf{E}, \mathbb{Z}/q) \cong H^{*,*}(\mathbf{E}, \mathbb{Z}/q)$.

5.1. Remarks on the condition (2)

We end this manuscript with some remarks about the conditions we have assumed to prove Theorem 5.1. From [16, Theorem 1.4] it follows that $\mathcal{B}^{**} = 0$ if $\mathrm{char}(k) = 0$. In that case the conditions are trivially verified, but for such fields we have resolution of singularities and the spectra $\mathbf{\Phi}_i$ have already been constructed in [2].

It is not necessary for $\mathcal{B}^{**}$ to vanish in order for the condition (2) to be satisfied. We are going to prove that a reasonable assumption such as the existence for all the stable motivic cohomology operations in $\mathcal{A}_m^{**}$ of some "Cartan formula" respecting the splitting $\mathcal{A}_m^{**} \cong \mathcal{A}^{**} \oplus \mathcal{B}^{**}$ suffices for the vanishing of the Margolis homology of $\mathcal{A}_m^{**}$, thus of $\mathcal{B}^{**}$, as well. A Cartan formula is a formula that describes the way a cohomology operation θ acts on the cup product of two cohomology classes of a space. It is known that the cohomology operations lying in $\mathcal{A}^{**}$ posses such formula, that is described explicitely in [13, Proposition 9.7].

Proposition 5.2. *Let k be a field for which for any $b \in \mathcal{B}^{**}$ we have $b(x \cup y) = \sum_i b_i' x \cup b_i'' y$. Then*

(1) *there exists a left $\mathcal{A}^{**}$ module isomorphism $h : \mathcal{A}_m^{**} \to \mathcal{A}^{**} \otimes_{H^{**}} C$, where C is the left H^{**} module $H^{**} \otimes_{\mathcal{A}^{**}} \mathcal{A}_m^{**}$;*
(2) *$HM(\mathcal{A}_m^{**}, Q_t \cdot) = HM(\mathcal{A}_m^{**}, \cdot Q_t) = 0$, thus $HM(\mathcal{B}^{**}, Q_t \cdot) = HM(\mathcal{B}^{**}, \cdot Q_t) = 0$.*

Proof. We prove that statement (1) implies (2). Let $b \in \mathcal{A}_m^{**}$ be $b = a \otimes c$ and assume that $Q_t b = Q_t(a \otimes c) = (Q_t a) \otimes c = 0$. We see that this implies that $Q_t a = 0$. In [2, Theorem 9], $HM(\mathcal{A}^{**}, Q_t \cdot)$ has been shown to vanish, therefore $a = Q_t a'$, and $b = Q_t(a' \otimes c)$, thus $HM(\mathcal{A}_m^{**}, Q_t \cdot) = 0$. To prove that $HM(\mathcal{A}_m^{**}, \cdot Q_t) = 0$ we endow $\mathcal{A}_m^{**}$ of an Hopf algebroid structure over H^{**} by setting $\Delta_{\mathcal{A}_m^{**}}(b) = \sum_i b_i' \otimes b_i''$, where b_i' and b_i'' are the classes appearing in the hypothesis of the Proposition 5.2 and $\Delta_{\mathcal{A}_m^{**}}(a) = \Delta_{\mathcal{A}^{**}}(a)$ for $a \in \mathcal{A}^{**}$, where $\Delta_{\mathcal{A}^{**}}$ is the comultiplication of $\mathcal{A}^{**}$. Using the construction of [6, Section 8], we can define a (canonical) *antiautomorphism* $\chi : \mathcal{A}_m^{**} \to \mathcal{A}_m^{**}$, with the property that $\chi \circ \chi = id_{\mathcal{A}_m^{**}}$. Explicitely,

$$\chi(b) = -b - \sum_{|b_i'|, |b_i''| \neq (0,0)} b_i' \chi(b_i''). \tag{5.16}$$

In particolar, $\chi|_{\mathcal{A}^{**}}$ is the canonical antiautomorphism of $\mathcal{A}^{**}$. Suppose that $\alpha Q_t = 0$ in $\mathcal{A}_m^{**}$. Then $\chi(\alpha Q_t) = \chi(Q_t)\chi(\alpha) = 0$. In the proof of [2, Corollary 7], it has been shown that $\chi(Q_t) = -Q_t$, thus $\chi(\alpha) = Q_t \beta$ because of what we proved above. Applying χ to this equality we conclude that $\alpha = \chi(Q_t \beta) = -\chi(\beta) Q_t$.

To prove the statement (5.2), we slightly modify the proof of [6, Theorem 4.4]. In the same notation, we let $K = H^{*,*}$, $A = \mathcal{A}^{**}$ and $B = \mathcal{A}_m^{**}$. However, $\mathcal{A}_m^{**}$ with $\mathcal{A}^{**}$ acting on the left by ring multiplication is not a left $\mathcal{A}^{**}$ module coalgebra This would be the case if $\Delta_{\mathcal{A}_m^{**}}$ were a ring homomorphism, but the issue of endowing $\mathcal{A}_m^{**} \otimes_{H^{**}} \mathcal{A}_m^{**}$ of a ring structure is tricky as H^{**} does not lie in the center of $\mathcal{A}_m^{**}$. Anyways, we do have a submodule of $\mathcal{A}_m^{**} \otimes_{H^{**}} \mathcal{A}_m^{**}$ that is a ring and its product structure is the one induced by the one on $\mathcal{A}_m^{**} \otimes_{\mathbb{Z}/q} \mathcal{A}_m^{**}$ (*i.e.* $(x \otimes y) \cdot (u \otimes v) = (-1)^{\deg(y)\deg(u)} xu \otimes yv$). This ring, denoted by $(\mathcal{A}_m^{**} \otimes_{H^{**}} \mathcal{A}_m^{**})_r$, contains the image of $\Delta_{\mathcal{A}_m^{**}}$ which therefore can be thought of as a ring homomorphism $\mathcal{A}_m^{**} \to (\mathcal{A}_m^{**} \otimes_{H^{**}} \mathcal{A}_m^{**})_r$. For more details on this matter, we refer to the [13, page 42 and

Lemma 11.8]. We now follow that argument used to prove the [6, Theorem 4.4] except that we replace the compositions of the kind

$$B \xrightarrow{\Delta_B} B \otimes_K B \xrightarrow{\alpha \otimes \beta} U \otimes W \tag{5.17}$$

with the compositions

$$\mathcal{A}_m^{**} \overset{\Delta_{\mathcal{A}_m^{**}}}{\to} (\mathcal{A}_m^{**} \otimes_{H^{**}} \mathcal{A}_m^{**})_r \xrightarrow{(\alpha\otimes\beta)\circ i} U \otimes W \tag{5.18}$$

where $i : (\mathcal{A}_m^{**} \otimes_{H^{**}} \mathcal{A}_m^{**})_r \hookrightarrow \mathcal{A}_m^{**} \otimes_{H^{**}} \mathcal{A}_m^{**}$ is the canonical inclusion. For instance, the map that we wish to invert is composition h

$$\mathcal{A}_m^{**} \overset{\Delta_{\mathcal{A}_m^{**}}}{\to} (\mathcal{A}_m^{**} \otimes_{H^{**}} \mathcal{A}_m^{**})_r \xrightarrow{(g\otimes\pi)\circ i} \mathcal{A}^{**} \otimes_{H^{**}} (H^{**} \otimes_{\mathcal{A}^{**}} \mathcal{A}_m^{**}) \tag{5.19}$$

where g is the retraction $\mathcal{A}_m^{**} \to \mathcal{A}^{**}$ and $\pi : \mathcal{A}_m^{**} \to H^{**} \otimes_{\mathcal{A}^{**}} \mathcal{A}_m^{**}$ is the canonical surjection $x \to 1 \otimes x$. The composition is left $\mathcal{A}^{**}$ linear, provided that we endow $\mathcal{A}^{**} \otimes_{H^{**}} (H^{**} \otimes_{\mathcal{A}^{**}} \mathcal{A}_m^{**})$ of the "diagonal" action of $\mathcal{A}^{**}$. This action is defined by $a(\alpha \otimes c) = \sum_i (-1)^{\deg(a_i') \deg(\alpha)} a_i'\alpha \otimes a_i''c$, where $\Delta_{\mathcal{A}^{**}} a = \sum_i a_i' \otimes a_i''$. We prove that h is an isomorphism of left $\mathcal{A}^{**}$ modules by finding a left inverse $\hat{\Delta}$ which is an isomorphism in turn. The morphism $\hat{\Delta}$ is the composition

$$\mathcal{A}^{**} \otimes_{H^{**}} (H^{**} \otimes_{\mathcal{A}^{**}} \mathcal{A}_m^{**}) \xrightarrow{id_{\mathcal{A}^{**}} \otimes f} \mathcal{A}^{**} \otimes_{H^{**}} \mathcal{A}_m^{**} \overset{\phi}{\to} \mathcal{A}_m^{**}. \tag{5.20}$$

We prove that it is an isomorphism of left $\mathcal{A}^{**}$ modules essentially in the same way as in [6, Proposition 1.7]. In this case the map Δ of that proposition is the composition

$$\mathcal{A}_m^{**} \xrightarrow{\Delta_{\mathcal{A}_m^{**}}} (\mathcal{A}_m^{**} \otimes_{H^{**}} \mathcal{A}_m^{**})_r \xrightarrow{(id_{\mathcal{A}_m^{**}} \otimes\pi)\circ i} \mathcal{A}_m^{**} \otimes_{H^{**}} (H^{**} \otimes_{\mathcal{A}^{**}} \mathcal{A}_m^{**}) \tag{5.21}$$

where the left $\mathcal{A}^{**}$ action on the latter module is just on $\mathcal{A}_m^{**}$ factor on the left. □

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Università degli Studi di Milano Bicocca
Dipartimento di Matematica e Applicazioni
via Cozzi, 53
20125 Milano, Italia