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INEQUALITIES FOR BOUNDARY VALUE PROBLEMS FOR SYSTEMS OF PARTIAL DIFFERENTIAL OPERATORS

by

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INTRODUCTION.

One of the classical techniques for studying homogeneous boundary value problems for elliptic operators was to rephrase them in terms of integro-differential sesquilinear forms. After it was discovered that there exist boundary problems with optimal regularity, but which cannot be expressed by sesquilinear forms, the method was abandoned by many people. However, others have stayed with it because it handles existence and uniqueness questions in a very convenient way, and they have applied it to more general differential operators, without worrying much about which boundary problems could be treated (there was always the Dirichlet problems, and other examples).

The first part of this lecture will be concerned with filling this gap, determining exactly which boundary problems enter in the variational framework. It will be seen that as soon as one requires some semiboundedness - more precisely, that

(I) $\quad \operatorname{Re}(Au,u) \geqslant -\text{const.}\|u\|_m^2$, all u satisfying boundary conditions, A being a differential operator of order $2m$ - then (and only then) the boundary condition falls within the class associated with sesquilinear forms on H^m. This holds not only for elliptic operators, but for quite general ones, e.g. the degenerate elliptic operators. Our treatment can be extended to systems of "mixed order", giving new information on these. The inequality (I) will be called "weak semiboundedness".

In the second part of the lecture we study "Gårding's inequality"

(II) $\quad \mathrm{Re}\,(Au,u) \geqq c_m \|u\|_m^2 - c_0 \|u\|_0^2$, all u satisfying bdry. cond.,

characterizing the boundary conditions for which it holds. Here and in the first part the methods are inspired from a general "abstract" theory for boundary problems [3], [4]. However, we thought it might be useful to show how (II) can be treated directly, without the whole machinery.

In the third part, the above results are applied, with some use of the abstract theory, to show that when A is strongly elliptic, the negative eigenvalues of the selfadjoint elliptic realizations A_B satisfy

(III) $\quad [\, N^-(A_B;t) = \,] \sum_{-t<\lambda_j<0} 1 \leqq c(A_B)\,t^{(n-1)/2m} + o(t^{(n-1)/2m})$

for $t\to\infty$, improving results of Agmon and Hörmander.

1. WEAKLY SEMIBOUNDED REALIZATIONS

Let $\bar\Omega$ be an n-dimensional compact riemannian manifold with boundary Γ and interior $\Omega = \bar\Omega \setminus \Gamma$. Let E be a C^∞ hermitian vector bundle over $\bar\Omega$, of fiber dimension $q \geqq 1$. Then the spaces of square integrable sections, $L^2(E)$ and $L^2(E|_\Gamma)$, and the Sobolev spaces $H^s(E)$, $H^s(E|_\Gamma)$ $(s \in \mathbb{R})$ and $H_0^s(E)$ $(s \geqq 0)$ may be defined; we denote inner products over $\bar\Omega$ by $(\ ,\ )$ and inner products (and dualities) over Γ by $\langle\ ,\ \rangle$. With a normal derivative D_n defined in E as in Hörmander [6], we have the trace operators γ_k defined by

$$\gamma_k : u \mapsto (D_n{}^k u)|_\Gamma \quad , \ k = 0,1,2,\dots \quad ,$$

they are continuous from $H^s(E)$ into $H^{s-k-1/2}(E|_\Gamma)$, for $s>k+1/2$, cf. [8].

We shall study boundary value problems defined as follows:

Given a linear differential operator A in E of order $2m$, m integer > 0. Denote

(1.1) $\quad M = \{0,\dots,2m-1\}$, $M_0 = \{0,\dots,m-1\}$, $M_1 = \{m,\dots,2m-1\}$,

and denote

(1.2) $\quad \varrho u = \{\gamma_0 u,\dots,\gamma_{2m-1}u\} = \{\gamma_k u\}_{k\in M}$, $\gamma u = \{\gamma_k u\}_{k\in M_0}$, $\nu u = \{\gamma_k u\}_{k\in M_1}$,

respectively the Cauchy data, Dirichlet data and Neumann data of u with respect to A (usually regarded as column vectors).

Given for each $j\in M$ a hermitian vector bundle F_j over Γ of fiber dimension $p_j \geqq 0$. For each pair $\{j,k\}\in M\times M$ there is given a differential operator B_{jk} from $E|_\Gamma$ to F_j of order $j-k$; we use the convention that differential operators of negative order are zero. Note that the operators B_{kk} are of order zero, so they are locally multiplication by a $p_k\times q$ - matrix, globally they may be viewed as vector bundle morphisms from $E|_\Gamma$ to F_k . Of course B_{jk} is 0 if $p_j = 0$ (i.e. $F_j = \Gamma\times\{0\}$). Altogether, the B_{jk} form a $2m\times 2m$ - matrix of differential operators, $B = (B_{jk})_{j,k\in M}$, it is <u>of type</u> $(-k,-j)_{j,k\in M}$ in the notation of [6], i.e. continuous from $\prod_{k\in M} H^{\alpha-k}(E|_\Gamma)$ to $\prod_{j\in M} H^{\alpha-j}(F_j)$, for all $\alpha\in\mathbb{R}$.

The boundary condition is now given by

(1.3) $$B\varrho u = 0 ,$$

and we shall study the realization A_B of A defined by

(1.4) $$A_B : u \mapsto Au ; \quad D(A_B) = \{u\in H^{2m}(E) \mid B\varrho u = 0\} .$$

(1.3) is a reformulation of the usual systems of boundary conditions, where we have grouped together conditions of the same normal order, and permitted the range space for each group to be a nontrivial bundle.

<u>We shall assume throughout that the following definition is satisfied:</u>

Definition 1.1 The boundary condition $B\rho u = 0$ - or the operator B - will be said to be normal when all the morphisms B_{kk} ($k\in M$) are surjective (so in particular $p_k \leq q$, all $k\in M$).

The assumption is partly justified by the observation of Seeley [11] that, for elliptic A_B, Agmon's necessary and sufficient condition for the existence of a "ray of minimal growth" implies normality; and when (II) holds (even with $c_m = 0$), there are many rays of minimal growth. We believe that normality is necessary for (I) under much more general circumstances. - The definition is a vector bundle version of that of Seeley [11]. It is more general than that of Geymonat [2] for matrices, which may be shown to hold exactly when $\ker B_{kk}$ is a trivial bundle, for all $k\in M$.[1]

B is a lower triangular matrix, since $B_{jk} = 0$ for $j < k$. It is the sum of its diagonal part B_d and its subtriangular part B_s

(1.5) $$B_d = (\delta_{jk}B_{jk})_{j,k\in M}, \quad B_s = B - B_d ;$$

we call matrices with zeroes in and on one side of the diagonal subtriangular. B_d is a surjective morphism from $\bigoplus_{k\in M} E|_\Gamma$ to $\bigoplus_{j\in M} F_j$.

Proposition 1.2 B_d has the right inverse

$$C_d = B_d^* (B_d B_d^*)^{-1} ,$$

and B has the right inverse

$$C = C_d - C_d B_s C_d + \dots + C_d(-B_s C_d)^{2m} ;$$

C is an injective triangular differential operator from $\bigoplus_{k\in M} F_k$ to $\bigoplus_{j\in M} E|_\Gamma$ of type $(-k,-j)_{j,k\in M}$.

In the proof it is used that $B_s C_d$ is a subtriangular differential operator in $\bigoplus_{j\in M} F_j$, thus nilpotent.

1) The definition clearly extends the original definition of Oronszajn and Milgram for the case where A is scalar (i.e.g. = 1).

Define for $\alpha \in \mathbb{R}$

$$(1.6) \qquad Z^{\alpha}(B) = \{\varphi \in \prod_{k\in M} H^{\alpha-k-1/2}(E|_{\Gamma}) \mid B\varphi = 0\} , \quad Z(B) = \bigcup_{\alpha\in\mathbb{R}} Z^{\alpha}(B) ,$$

$$(1.7) \qquad R^{\alpha}(I - CB) = (I-CB)\prod_{k\in M} H^{\alpha-k-1/2}(E|_{\Gamma}) , \quad R(I-CB) = \bigcup_{\alpha\in\mathbb{R}} R(I-CB) ,$$

and define similarly $R^{\alpha}(B^*)$, $R(B^*)$, $Z^{\alpha}((I-CB)^*)$ and $Z((I-CB)^*)$. One has immediately

<u>Lemma 1.3</u> <u>For all</u> $\alpha\in\mathbb{R}$

$$(1.8) \qquad Z^{\alpha}(B) = R^{\alpha}(I-CB) , \qquad Z(B) = R(I-CB) ,$$

$$(1.9) \qquad R^{\alpha}(B^*) = Z^{\alpha}((I-CB)^*) , \qquad R(B^*) = Z((I-CB)^*) .$$

We shall later need the following lemma:

<u>Lemma 1.4</u> <u>There exists a differential operator</u> Φ <u>from</u> $\bigoplus_{k\in M} Z_k$ <u>to</u> $\bigoplus_{j\in M} E|_{\Gamma}$ <u>of type</u> $(-k,-j)_{j,k\in M}$, <u>with injective diagonal part, such that</u>

$$Z^{\alpha}(B) = \Phi \prod_{k\in M} H^{\alpha-k-1/2}(Z_k) , \quad \underline{\text{all}}\ \alpha\in\mathbb{R}$$

<u>here</u> Z_k <u>denotes the bundle</u> $\ker B_{kk}$, $k\in M$.

By Proposition 1.2 applied to Φ^* , Φ has a left inverse Ψ, triangular differential operator with surjective diagonal part.

Next, we shall establish the necessary Green's formulae. The following version of Green's formula is well known (cf. [9], [6])

$$(1.10) \qquad (Au,v) - (u,A'v) = \langle \mathcal{a}\rho u, \rho v\rangle , \quad \text{all} \quad u\in H^{2m}(E) ;$$

here A' denotes the formal adjoint of A , and $\mathcal{a}$ is a skew-triangular $2m\times 2m$ - matrix $\mathcal{a} = (\mathcal{a}_{jk})_{j,k\in M}$ where $\mathcal{a}_{jk}$ is of order $2m-j-k-1$; the elements in the second diagonal, $\mathcal{a}_{j,2m-1-j}$, are of order 0 , and they are vector bundle isomorphisms if and only if Γ is noncharacteristic for A .

Split $\mathcal{A}$ in four blocks (recall (1.1))

$$\mathcal{A} = \begin{pmatrix} \mathcal{A}^{00} & \mathcal{A}^{01} \\ \mathcal{A}^{10} & 0 \end{pmatrix} , \quad \mathcal{A}^{\delta\varepsilon} = (\mathcal{A}_{jk})_{j\in M_\delta, k\in M_\varepsilon} \quad ;$$

then (1.10) becomes, in view of (1.2)

$$(1.11) \qquad (Au,v) - (u,A'v) = \langle \mathcal{A}^{00}\gamma u, \gamma v\rangle + \langle \mathcal{A}^{01}\nu u, \gamma v\rangle + \langle \mathcal{A}^{10}\gamma u, \nu v\rangle \; .$$

We shall also need the "halfways" Green's formula (proved in [5])

<u>Proposition 1.5</u> <u>Let</u> $a(u,v)$ <u>be a sesquilinear form on</u> $H^m(E)$ <u>associated with</u> A , <u>i.e. a form</u>

$$(1.12) \qquad a(u,v) = \sum_{i\in I} (Q_i u, P_i v) \quad ,$$

<u>with</u> Q_i <u>and</u> P_i <u>of order</u> $\leqq m$, <u>indexed by a finite index set</u> I , <u>and with</u> $\sum_{i\in I} P_i' Q_i = A$. <u>Then for all</u> $u\in H^{2m}(E)$, <u>all</u> $v\in H^m(E)$,

$$(1.13) \qquad (Au,v) = a(u,v) + \langle \mathcal{A}^{01}\nu u, \gamma v\rangle + \langle \mathcal{S}\gamma u, \gamma v\rangle \; ,$$

<u>where</u> $\mathcal{S} = (\mathcal{S}_{jk})_{j,k\in M_0}$ <u>is a differential operator of type</u> $(-k,-2m+1+j)_{j,k\in M_0}$ <u>in</u> $\bigoplus_{k\in M_0} E|_\Gamma$. <u>Conversely, for any such</u> $\mathcal{S}$ <u>there exists a sesquilinear form</u> $a(u,v)$ <u>associated with</u> A , <u>such that</u> (1.13) <u>holds</u>.

Now split B and C in blocks also :

$$B = \begin{pmatrix} B^{00} & 0 \\ B^{10} & B^{11} \end{pmatrix} , \qquad C = \begin{pmatrix} C^{00} & 0 \\ C^{10} & C^{11} \end{pmatrix} ,$$

where $B^{\delta\varepsilon} = (B_{jk})_{j\in M_\delta, k\in M_\varepsilon}$ and $C^{\delta\varepsilon} = (C_{jk})_{j\in M_\delta, k\in M_\varepsilon}$; then C^{00} and C^{11} are the right inverses of B^{00} resp. B^{11} by the construction in Proposition 1.2. Then the boundary condition may be written

$$(1.14) \qquad B^{00}\gamma u = 0 \; , \quad B^{10}\gamma u + B^{11}\nu u = 0 \; .$$

Moreover, we observe that by Lemma 1.3

Lemma 1.6 A section $u \in H^{2m}(E)$ *satisfies* (1.14) *if and only if*

$$\gamma u \in Z^{2m}(B^{00}) , \qquad \nu u + C^{11}B^{10}\gamma u \in Z^{2m}(B^{11}) ; \tag{1.15}$$

and if and only if

$$\gamma u \in R^{2m}(I-C^{00}B^{00}) , \ \nu u + C^{11}B^{10}\gamma u \in R^{2m}(I-C^{11}B^{11}) . \tag{1.16}$$

The theorem can now be stated.

Theorem I The following statement are equivalent:

(i) *For some* $c > 0$,

$$\operatorname{Re}\,(Au,u) \geqq -c\,\|u\|_m^2 \ , \ \textit{for all} \ u \in D(A_B) \tag{1.17}$$

i.e., A_B *is weakly semibounded.*

(ii) *The identity holds*

$$(I-C^{00}B^{00})^* \mathcal{A}^{01}(I-C^{11}B^{11}) = 0 . \tag{1.18}$$

(iii) *There exists a sesquilinear form* $a_B(u,v)$ *on* $H^m(E)$ *associated with* A , *for which*

$$(Au,v) = a_B(u,v) \ , \ \textit{for all} \ u,v \in D(A_B) . \tag{1.19}$$

(iv) *For some* $c > 0$,

$$|(Au,v)| \leqq c\,\|u\|_m\,\|v\|_m \ , \ \textit{for all} \ u,v \in D(A_B) .$$

A detailed proof is given in [5] ; let us just explain how (1.18) enters: In the formula (1.13), the only term on the right side that is not continuous on $H^m(E)\times H^m(E)$ is $\langle \mathcal{A}^{01}\nu u,\gamma v\rangle$. However, when u and $v \in D(A_B)$, then by (1.16)

$$\nu u + C^{11}B^{10}\gamma u = (I-C^{11}B^{11})\varphi , \quad \gamma v = (I-C^{00}B^{00})\psi$$

for suitable φ, ψ, so when (1.18) holds, then

$$\begin{aligned} \langle \mathcal{A}^{01}\nu u,\gamma v\rangle &= \langle \mathcal{A}^{01}(I-C^{11}B^{11})\varphi,(I-C^{00}B^{00})\psi\rangle - \langle \mathcal{A}^{01}C^{11}B^{10}\gamma u,\gamma v\rangle \\ &= \quad 0 \ - \ \langle \mathcal{A}^{01}C^{11}B^{10}\gamma u,\gamma v\rangle , \end{aligned} \tag{1.20}$$

and thus $|\langle \mathcal{A}^{01}\nu u,\gamma v\rangle| \leqq c\,\|u\|_m\,\|v\|_m$ for $u,v \in D(A_B)$.

Then (i) and (iv) are valid, and (iii) follows by use of the last part of Proposition 1.5. Conversely, (1.18) is seen to be necessary even for (i) , since φ varies independently of γu .

Let us give a few more remarks on (1.18), for the case where Γ is noncharacteristic for A . Then (1.18) implies $\sum_{j\in M} p_j \geq mq$, and when $\sum_{j\in M} p_j = mq$ (the case usually considered for elliptic operators), (1.18) is equivalent with the statement

$$(1.21) \qquad Z^{2m}(B^{00}) = (\mathfrak{A}^{01*})^{-1}R^{2m}(B^{11*}) \quad ,$$

and with

$$(1.22) \qquad B^{00}(\mathfrak{A}^{01*})^{-1}B^{11*} = 0 \quad .$$

Now $\gamma D(A_B) = Z^{2m}(B^{00})$, and it is easily seen that the formally adjoint realization $(A_B)'$ is defined by a normal boundary condition $B'\rho u = 0$, for which

$$Z^{2m}(B'^{00}) = (\mathfrak{A}^{01*})^{-1}R^{2m}(B^{11*}) \quad .$$

Thus (1.21) means

$$\gamma D(A_B) = \gamma D(A'_{B'}) \ .$$

This is <u>symmetric</u> in $\{A,B\}$ and $\{A',B'\}$; so when it holds, $A'_{B'}$ is also weakly semibounded.

Finally, we mention the treatment of systems of "mixed order" (details are given in [5]). Let $A = (A_{st})_{s,t=1,\dots,q}$, where each A_{st} is a differential operator on $\overline{\Omega}$ of order m_s+m_t , for a given set of nonnegative integers $\{m_1,\dots,m_q\}$. Denote $\max m_t = m$. Then one can establish a Green's formula

$$(1.23) \quad (Au,v) - (u,A'v) = \langle\tilde{\mathfrak{A}}^{00}\beta^0 u,\beta^0 v\rangle+\langle\tilde{\mathfrak{A}}^{01}\beta^1 u,\beta^0 v\rangle+\langle\tilde{\mathfrak{A}}^{10}\beta^0 u,\beta^1 v\rangle \ ,$$

where $\beta^0 u$ is a certain rearrangement of the Dirichlet traces

$\{\gamma_0 u_1, \ldots, \gamma_{m_1-1} u_1; \ \ldots \ ; \gamma_0 u_q, \ldots, \gamma_{m_q-1} u_q\}$, and $\beta^1 u$ is a rearrangement of the traces $\{\gamma_{m_1} u_1, \ldots, \gamma_{m_1+m-1} u_1; \ \ldots \ ; \gamma_{m_q} u_q, \ldots, \gamma_{m_q+m-1} u_q\}$, and $\tilde{\mathcal{A}}^{01}$ and $\tilde{\mathcal{A}}^{10}$ are skew-triangular matrices of matrix-valued differential operators on Γ . When Γ is noncharacteristic for A , $\tilde{\mathcal{A}}^{01}$ is <u>surjective</u> and $\tilde{\mathcal{A}}^{10}$ is <u>injective</u> (but they are bijective only if all m_t equal m). Proposition 1.5 extends, and normal boundary conditions can be defined. (This treatment seems new.) Now Theorem I can be proved in a version completely analogous to the 2m-order case, γu , νu and $\mathcal{A}^{01}$ replaced by $\beta^0 u$, $\beta^1 u$ and $\tilde{\mathcal{A}}^{01}$, respectively.

Moreover, we find that the boundary conditions satisfying the statements in Theorem I, are actually <u>normal boundary conditions on</u> $\beta^0 u$ <u>and</u> $\tilde{\mathcal{A}}^{01}\beta^1 u$ (when Γ is noncharacteristic, and the number of boundary conditions is $m_1+\ldots+m_q$, like in elliptic problems). For such conditions, it is possible to <u>extend the Lions - Magenes theory</u>, on the basis of (1.23).

2. GÅRDING'S INEQUALITY

Gårding has shown that A must be strongly elliptic in order for (II) to hold even for $u \in C_0^\infty(E)$, so we assume from now on that A <u>is strongly elliptic</u>. Define the "real" part of A

$$A^r = 1/2\ (A + A') \ , \tag{2.1}$$

it is also strongly elliptic. We can assume that a sufficiently large constant has been added to A so that (with $c_1, c_2 > 0$)

$$c_1 \|u\|_m^2 \leqq (A^r u, u) = (Au, u) \leqq c_2 \|u\|_m^2 \ , \ \text{for} \ \ u \in H_0^m(E) \cap H^{2m}(E) \ ;$$

then the Dirichlet realizations A_γ and A_γ^r defined by the boundary condition $\gamma u = 0$, i.e. with domains

$$D(A_\gamma) = D(A_\gamma^r) = H_0^m(E) \cap H^{2m}(E) \ ,$$

are bijective (onto $L^2(E)$).

Define for each $\alpha \in \mathbb{R}$

$$(2.2)\qquad \begin{aligned} Z^{\alpha}(A) &= \{u \in H^{\alpha}(E) \mid Au = 0\} , \\ Z^{\alpha}(A^r) &= \{u \in H^{\alpha}(E) \mid A^r u = 0\} . \end{aligned}$$

For $u \in H^{2m}(E)$, let

$$u^r_\gamma = (A^r_\gamma)^{-1} Au , \quad u^r_\zeta = u - u^r_\gamma ,$$

this defines a decomposition of $H^{2m}(E)$ (by $u = u^r_\gamma + u^r_\zeta$) :

$$H^{2m}(E) = D(A^r_\gamma) \dotplus Z^{2m}(A^r) \qquad \text{(topological direct sum)} .$$

It follows from the theory of Lions and Magenes [8], that

$$(2.3)\qquad \gamma : Z^{\alpha}(A^r) \to \prod_{k \in M_0} H^{\alpha-k-1/2}(E|_\Gamma)$$

is an <u>isomorphism</u> for all $\alpha \in \mathbb{R}$; we call it γ^r_Z . Then we can define the composed operator

$$(2.4)\qquad P^r = \nu \circ (\gamma^r_Z)^{-1} : \prod_{k \in M_0} H^{\alpha-k-1/2}(E|_\Gamma) \to \prod_{k \in M_1} H^{\alpha-k-1/2}(E|_\Gamma) ,$$

and it is a consequence of [6], [9] (see also [4]), that P^r is an elliptic pseudo-differential operator, of type $(-k,-j)_{j \in M_1, k \in M_0}$. Note that when $u \in H^{2m}(E)$, we have

$$(2.5)\qquad \begin{aligned} &\gamma u^r_\gamma = 0 , && \gamma u^r_\zeta = \gamma(u - u^r_\gamma) = \gamma u , \\ &\nu u^r_\zeta = P^r \gamma u^r_\zeta = P^r \gamma u , && \nu u^r_\gamma = \nu u - P^r \gamma u . \end{aligned}$$

The analogous formulae hold for A (omit r everywhere).

<u>Lemma 2.1</u> <u>Let</u> $u \in H^{2m}(E)$. <u>Then</u>

$$(2.6)\qquad \begin{aligned} \mathrm{Re}\,(Au,u) = (A^r u^r_\gamma, u^r_\gamma) &+ \mathrm{Re} \langle \mathcal{A}^{01} \nu u, \gamma u\rangle \\ &+ \langle \tfrac{1}{2} (\mathcal{A}^{00*} + (\mathcal{A}^{10*} - \mathcal{A}^{01}) P^r) \gamma u, \gamma u\rangle . \end{aligned}$$

<u>Proof</u>: Write $u = v + w$, where $v = u^r_\gamma$ and $w = u^r_\zeta$. Then

$$\begin{aligned}\operatorname{Re}(Au,u) &= \tfrac{1}{2}(Au,u) + \tfrac{1}{2}(u,Au)\\ &= \tfrac{1}{2}[(Av,v) + (Av,w) + (Aw,v) + (Aw,w)]\\ &\quad + \tfrac{1}{2}[(v,Av) + (v,Aw) + (w,Av) + (w,Aw)]\end{aligned}$$

By Green's formula (1.11), we find using that $\gamma v = 0$:

$$\begin{aligned}(v,Av) &= (A'v,v) \quad ,\\ (w,Av) &= (A'w,v) + \langle \gamma w, \mathcal{A}^{01}\nu v\rangle \quad ,\\ (w,Aw) &= (A'w,w) + \langle \gamma w, \mathcal{A}^{00}\gamma w + \mathcal{A}^{01}\nu w\rangle + \langle \nu w, \mathcal{A}^{10}\gamma w\rangle \ .\end{aligned}$$

This gives, using (2.1),

$$\begin{aligned}\operatorname{Re}(Au,u) &= (A^r v,v) + (A^r w,v) + (v,A^r w) + (A^r w,w) +\\ &+ \tfrac{1}{2}[\langle \gamma w, \mathcal{A}^{01}\nu v\rangle + \langle \mathcal{A}^{01}\nu v,\gamma w\rangle + \langle \gamma w, \mathcal{A}^{00}\gamma w + \mathcal{A}^{01}\nu w\rangle + \langle \nu w, \mathcal{A}^{10}\gamma w\rangle] \ .\end{aligned}$$

Here $A^r w = 0$. Using (2.5) we then find

$$\begin{aligned}\operatorname{Re}(Au,u) &= (A^r u^r_\gamma, u^r_\gamma) + \tfrac{1}{2}[\langle \gamma u, \mathcal{A}^{01}(\nu u - P^r\gamma u)\rangle +\\ &\quad + \langle \mathcal{A}^{01}(\nu u - P^r\gamma u),\gamma u\rangle + \langle \gamma u, \mathcal{A}^{00}\gamma u + \mathcal{A}^{01}P^r\gamma u\rangle + \langle P^r\gamma u, \mathcal{A}^{10}\gamma u\rangle]\\ &= (A^r u^r_\gamma, u^r_\gamma) + \operatorname{Re}\langle \mathcal{A}^{01}\nu u,\gamma u\rangle + \langle \tfrac{1}{2}(\mathcal{A}^{00*} + (\mathcal{A}^{10*} - \mathcal{A}^{01})P^r)\gamma u,\gamma u\rangle \ .\end{aligned}$$

<u>Corollary 2.2</u> <u>When</u> A_B <u>is weakly semibounded, then</u>

(2.7) $$\operatorname{Re}(Au,u) = (A^r u^r_\gamma, u^r_\gamma) + \operatorname{Re}\langle \mathcal{R}\gamma u,\gamma u\rangle \ , \ \text{where}$$

(2.8) $$\mathcal{R} = -\mathcal{A}^{01}C^{11}B^{10} + \tfrac{1}{2}\mathcal{A}^{00*} + \tfrac{1}{2}(\mathcal{A}^{10*} - \mathcal{A}^{01})P^r \ .$$

<u>Proof</u>: By the characterization of weak semiboundedness given in Theorem I, we have in particular from (1.20) :

$$\langle \mathcal{A}^{01}\nu u,\gamma u\rangle = -\langle \mathcal{A}^{01}C^{11}B^{10}\gamma u,\gamma u\rangle$$

for $u \in D(A_B)$; the corollary follows by inserting this in (2.6) .

We can now show

<u>Theorem II</u> <u>Let</u> A <u>be strongly elliptic, and let</u> A_B <u>be the realization of a normal boundary condition</u> $B\rho u = 0$. <u>Then</u> A_B <u>satisfies Gårding's inequality</u>

(II) $\quad \operatorname{Re}(Au,u) \geq c_m\|u\|_m^2 - c_0\|u\|_0^2$, all $u \in D(A_B)$

for some c_m and $c_0 > 0$, if and only if (i) and (ii) hold :

(i) $\quad (I - C^{00}B^{00})^*\mathcal{Q}^{01}(I - C^{11}B^{11}) = 0$;

(ii) $\operatorname{Re}\sigma^0(\mathcal{K}) = \frac{1}{2}[\sigma^0(\mathcal{K}) + \sigma^0(\mathcal{K})^*]$ is positive definite on $T^*(\Gamma)\setminus 0$, where

$$\mathcal{K} = \Phi^*\mathcal{R}\Phi\ , \tag{2.9}$$

Φ being the differential operator obtained by applying Lemma 1.4 to B^{00} ; here $\mathcal{K}$ is a pseudo-differential operator in $\bigoplus_{k\in M_0} Z_k$ of type $(-k,-2m+1+j)_{j,k\in M_0}$; its principal symbol is defined accordingly.

Proof: We know from Theorem I that (i) is necessary for (II), so we may assume it to hold. Then (2.7) holds on $D(A_B)$. Now it is proved just as in [4 , Theorems 3.3 and 4.3] that (II) is equivalent with

$$\operatorname{Re}\langle\mathcal{R}\gamma u,\gamma u\rangle \geq c_m'\|\gamma u\|^2_{\{m-k-1/2\}} - c_0'\|\gamma u\|^2_{\{-k-1/2\}},\ \text{all } \gamma u\in\gamma D(A_B), \tag{2.10}$$

where we denote the norm in $\prod_{k\in M_0} H^{\alpha-k-1/2}(E|_\Gamma)$ by $\|\cdot\|_{\{\alpha-k-1/2\}}$. Here

$$\gamma D(A_B) = Z^{2m}(B^{00}) = \Phi\prod_{k\in M_0} H^{2m-k-1/2}(Z_k)$$

by Lemmas 1.4 and 1.6. Inserting $\gamma u = \Phi\varphi$, we find, using the continuity of Φ and its left inverse Ψ , that (2.10) is equivalent with

$$\operatorname{Re}\langle\Phi^*\mathcal{R}\Phi\varphi,\varphi\rangle \geq c_m''\|\varphi\|^2_{\{m-k-1/2\}} - c_0''\|\varphi\|^2_{\{-k-1/2\}}, \tag{2.11}$$

$$\text{all } \varphi\in\prod_{k\in M_0} H^{2m-k-1/2}(Z_k)\ ,$$

$\|\cdot\|_{\{\alpha-k-1/2\}}$ now denoting the norm in $\prod_{k\in M_0} H^{\alpha-k-1/2}(Z_k)$. It is easily checked that

$\Phi^*\mathcal{R}\Phi = \mathcal{K}$ is continuous from $\prod_{k\in M_0} H^{\alpha-k}(Z_k)$ to $\prod_{k\in M_0} H^{\alpha-2m+k+1}(Z_k)$

for all α ; in particular it is of type $(m-\frac{1}{2}-k,-m+\frac{1}{2}+j)_{j,k\in M_0}$, so by a well-known theorem on pseudo-differential operators, (2.11) holds if and only if $\operatorname{Re} \sigma^0(\mathcal{K}) > 0$ on $T^*(\Gamma)\ 0$.

Remark 2.3 Consider the case where A is formally selfadjoint. Then $A = A^r$, $P = P^r$ etc. Moreover, $\mathcal{A}^* = -\mathcal{A}$, so $\mathcal{A}^{00*} = -\mathcal{A}^{00}$, $\mathcal{A}^{01*} = -\mathcal{A}^{10}$, $\mathcal{A}^{10*} = -\mathcal{A}^{01}$. Then

$$\mathcal{R} = -\mathcal{A}^{01}C^{11}B^{10} - \frac{1}{2}\mathcal{A}^{00} - \mathcal{A}^{01}\ P = -\mathcal{A}^{01}(C^{11}B^{10} + P) - \frac{1}{2}\mathcal{A}^{00} .$$

Assume now furthermore that $\sum_{j\in M} p_j = mq$. Then we have by (1.21)

$$Z^{2m}(B^{00}) = (\mathcal{A}^{01*})^{-1}B^{11*}\prod_{j\in M_1} H^{2m-j-1/2}(F_j) . \tag{2.12}$$

Thus, writing $\gamma u = (\mathcal{A}^{01*})^{-1}B^{11*}\psi$, and using that $\operatorname{Re}\langle \mathcal{A}^{00}\gamma u,\gamma u\rangle = 0$,

$$\begin{aligned}\operatorname{Re}\langle \mathcal{R}\gamma u,\gamma u\rangle &= \operatorname{Re}\langle -\mathcal{A}^{01}(C^{11}B^{10}+P)(\mathcal{A}^{01*})^{-1}B^{11*}\psi,(\mathcal{A}^{01*})^{-1}B^{11*}\psi\rangle \\ &= \operatorname{Re}\langle \mathcal{K}_1\psi,\psi\rangle ,\end{aligned}$$

where

$$\mathcal{K}_1 = -(B^{10} + B^{11}P)(\mathcal{A}^{01*})^{-1}B^{11*} , \tag{2.13}$$

a pseudo-differential operator in $\bigoplus_{j\in M_1} F_j$; and (II) holds if and only if $\operatorname{Re}\sigma^0(\mathcal{K}_1) > 0$. (This gives a somewhat simpler formula.)

3. NEGATIVE EIGENVALUES

From now on we assume that A is formally selfadjoint, besides being strongly elliptic. Let A_B be a selfadjoint, elliptic realization defined by a boundary condition $B\rho u = 0$ (necessarily normal); the general theory shows that A_B has the mentioned properties if and only if (1.18) holds and $\mathcal{K}_1$, defined by (2.13), is selfadjoint and elliptic. The spectrum of A_B (as an operator in $L^2(E)$) consist of the two sequences

$$0 \leqq \lambda_1^+ \leqq \lambda_2^+ \leqq \cdots\cdots ,$$

$$0 > \lambda_1^- \geqq \lambda_2^- \geqq \cdots\cdots ;$$

$\{\lambda_p^+\}$ goes to $+\infty$ and $\{\lambda_p^-\}$ is either finite or goes to $-\infty$; it is finite if and only if $\sigma^0(\mathcal{K}_1) > 0$. By adding a real constant to A if necessary, we obtain that A_B is invertible.

Denote (cf. (2.3) and Lemma 1.6)

$$(3.1) \qquad V^{2m} = \gamma_Z^{-1} Z^{2m}(B^{00}) = \{u_\zeta \mid u \in D(A_B)\} ,$$

and denote its closure in $L^2(E)$ by V. The general theory asserts that there corresponds to A_B an unbounded selfadjoint invertible operator T in V with domain $D(T) = V^{2m}$, satisfying

$$(3.2) \qquad (Au,u) = (Au_\gamma, u_\gamma) + (Tu_\zeta, u_\zeta) , \quad \text{for all} \quad u \in D(A_B) ;$$

$$(3.3) \qquad A_B^{-1} = A_\gamma^{-1} + T^{(-1)} \quad \text{on} \quad L^2(E) ,$$

where $T^{(-1)}f = T^{-1}\mathrm{proj}_V f$ (orthogonal projection). By (3.3), $T^{(-1)}$ is compact. (Formulae like (3.2) - (3.3) have been applied by Krein and by Birman to semibounded problems.) Note that it follows from (3.2), (2.7) and Remark 2.3 that

$$(3.4) \qquad (Tu_\zeta, u_\zeta) = \langle \mathcal{K}_1\psi, \psi\rangle, \text{ when } u_\zeta = \gamma_Z^{-1}(\mathcal{A}^{01*})^{-1}B^{11*}\psi .$$

Denote by $N^+(A_B;t)$ resp. $N^-(A_B;t)$ the number of positive resp. negative eigenvalues in $]-t,t[$ $(t \leqq \infty)$. It is known that

$$(3.5) \qquad N^+(A_B;t) - c(A)t^{n/2m} = \mathcal{O}(t^{(n-\theta)/2m}) \qquad \text{as} \quad t \to \infty ,$$

$$(3.6) \qquad N^-(A_B;t) = \mathcal{O}(t^{(n-\theta)/2m}) \qquad \text{as} \quad t \to \infty ,$$

for $\theta < \frac{1}{2}$ (Agmon [1]) (and seemingly for $\theta < 1$ in certain cases as consequence of Hörmander [7]). We shall now show that (3.6) holds with $\theta = 1$ (actually we give a more precise result).

(3.2) - (3.3) imply

Proposition 3.1 (i) $N^-(A_B;\infty) = N^-(T;\infty)$ (i.e., A_B and T have the same number of negative eigenvalues).

(ii) $N^-(A_B;t) \leq N^-(T;t)$ for all $t \in [0,\infty[$.

In order to apply this, we need

Proposition 3.2 There exists an elliptic pseudo-differential operator Λ in $\bigoplus_{j\in M_1} F_j$ of type $(0,-j-\frac{1}{2})_{j,k\in M_1}$, such that the composed operator

$$\gamma_Z^{-1}(\mathcal{Q}^{01*})^{-1}B^{11*}\Lambda$$

maps $L^2(\bigoplus_{j\in M_1} F_j)$ isometrically onto V.

The proof uses [4, Example 6.3].

Denote $\bigoplus_{j\in M_1} F_j = F^1$ and denote $(\mathcal{Q}^{01*})^{-1}B^{11*} = \Theta$, then clearly $\gamma D(A_B) = \Theta \Lambda H^{2m}(F^1)$. Now, when $u_\zeta = \gamma_Z^{-1}\Theta\Lambda\eta$, where $\eta \in H^{2m}(F^1)$, then by (3.4)

$$(Tu_\zeta,u_\zeta) = \langle\mathcal{K}_1\Lambda\eta,\Lambda\eta\rangle = \langle\mathcal{T}\eta,\eta\rangle ,$$

where

$$\mathcal{T} = \Lambda^*\mathcal{K}_1\Lambda$$

is seen to be an elliptic pseudo-differential operator in F^1 of order $2m$, bijective from $H^{2m}(F^1)$ onto $L^2(F^1)$. Moreover, by Proposition 3.1,

$$\frac{(Tu_\zeta,u_\zeta)}{\|u_\zeta\|^2_{L^2(E)}} = \frac{\langle\mathcal{T}\eta,\eta\rangle}{\|\eta\|^2_{L^2(F^1)}} , \text{ all } u_\zeta\in D(T) .$$

Thus, by the mini-max principle, applied to the inverses, T and $\mathcal{T}$ have the same eigenvalues.

Since Γ is $(n-1)$-dimensional, it follows from a theorem of Seeley [10] , that for a certain constant $c^-(\mathcal{T})$ depending on $\sigma^0(\mathcal{T})$,

$$N^-(\mathcal{T};t) - c^-(\mathcal{T})t^{(n-1)/2m} = o(t^{(n-1)/2m})$$

(with $o(t^{(n-1)/2m})$ replaced by $\mathcal{O}(t^{(n-2)/2m})$ when $\mathcal{T}$ is scalar or a certain root condition is satisfied so that [7] applies) . We have as an immediate application , using Proposition 3.1

Theorem III

$$N^-(A_B;\infty) = N^-(\mathcal{T};\infty) ,$$

and

$$N^-(A_B;t) \leqq N^-(\mathcal{T};t) = c^-(\mathcal{T})\, t^{(n-1)/2m} + R(t) \quad \text{for} \quad t\ [0,\infty[, \tag{3.7}$$

where $R(t)$ is in general $o(t^{(n-1)/2m})$ for $t \to \infty$, and is $\mathcal{O}(t^{(n-2)/2m})$ in certain cases.

Let us finally mention that one can also prove that, at least when the B_{jk} in B^{01} are permitted to be pseudo-differential operators, there exists for any $c > 0$ and any normal B^{00} an elliptic selfadjoint realization A_B satisfying (in addition to (3.7))

$$N^-(A_B;t) \geqq c\ t^{(n-1)/2m} .$$

Remark. Some of the results presented here have been announced in Comptes Rendus Acad. Sci. (Sér. A) 1972, p.319-323 and p. 409-412, and briefly explained in Séminaire Goulaouic-Schwartz 1971-1972 (exposés XIX et 19 bis). The complete details for section 1 are given in [5]; an article "Properties of normal boundary problems for elliptic systems" elaborating the results of sections 2 and 3 is under preparation.

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