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DETERMINING FACTORS AND HYPOELLIPTICITY OF ORDINARY DIFFERENTIAL OPERATORS WITH DOUBLE " CHARACTERISTICS "

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1 - INTRODUCTION.-

Let L be an ordinary differential operator with C^∞ coefficients in a neighborhood of the origin, and assume that the leading coefficient of L has a zero of finite multiplicity at the origin. It is a well known result of the classical theory of irregular singular points of ordinary differential equations ([1],[3]) that there exists a system of m linearly independent formal solutions $u_1(x), \ldots, u_m(x)$ of $Lu = 0$ (m is the order of L) with

$$(1.1) \qquad u_i(x) = e^{Q_i(x)} x^{\rho_i} v_i(x)$$

where the $Q_i(x)$ are polynomials in x^{-1/q_i} and

$$(1.2) \qquad v_i(x) = \Sigma_{j=0}^{m_i} \quad v_{i,j}(x)(\log x)^j$$

$$(1.3) \qquad v_{i,j}(x) \sim \Sigma_{n=0}^{\infty} v_{i,j,n} x^{n/q_i}$$

for $0 \leq j \leq m_i$, $1 \leq i \leq m$. Here m_i, q_i are integers, $m_i \geq 0$, $q_i > 0$ and the series in (1.3) do not converge, in general, even if the coefficients of L are analytic. (The equations $Lu_i = 0$ hold in the sense of formal power series ; the coefficients of L are replaced by their formal Taylor expansions at the origin). The functions $Q_i(x)$ are called " determining factors ". Clearly, we can assume that the constant terms vanish in the determining factors. (In some texts, the $e^{Q_i(x)}$ - rather than the $Q_i(x)$ - are called determining factors).

The determining factors may be computed explicitely in many cases. Thus, let $Lu = \Sigma_{j=0}^{m} a_j(x)d^j u/dx^j$, and set $p(x,\xi) = \Sigma_{j=0}^{m} a_j(x)(i\xi)^j$ - the (complete) symbol of L. It is well known that there exist m complex valued functions $\zeta_1(x), \dots, \zeta_m(x)$, which are continuous in a neighborhood of the origin exeept at $x = 0$, such that $p(x,\xi) = a_m(x)\pi_{j=1}^{m}(\xi-\zeta_j(x))$. The functions $\zeta_1(x), \dots, \zeta_m(x)$ are called branches of roots of p. If the " characteristics " of L are " simple ", in the sense that whenever $\zeta_i(x)$ is an unbounded branch of roots of p such that $|x| \quad |\zeta_i(x)|$ is also bounded and $\zeta_i(x)/\zeta_j(x) \to 1$ as $x \to 0$ then $i = j$, then the determination of the determining factors is relatively easy ([2],[4]). In fact, let

$$(1.4) \qquad \zeta_j(x) \sim \Sigma_{k=N(j)}^{\infty} \alpha_{j,k} x^{k/q}$$

be the formal Puiseux expansion ; here q is a positive integer, $N(j)$ is a finite integer (positive,negative,or zero) with $\alpha_{j,N(j)} \neq 0$ unless $\alpha_{j,k} = 0$ for all k. (The series (1.4) does not converge, in general, unless the coefficients of L are holomorphic.) The derivative of the determining factor Q_j is given by the simple formula

$$(1.5) \qquad \frac{dQ_j}{dx} = i \, \Sigma_{i=N(j)}^{-q-1} \alpha_{j,k} \, x^{k/q} \qquad j = r+1,\dots,m.$$

(Note that if $\zeta_j(x) = 0(\frac{1}{x})$ when $x \to 0$ then $Q_j(x) = 0$; this is true in general, even if " simplicity " of the " characteristics " is not assumed).

It is the aim of the present paper to present formulas for the determining factors. In section 2, we shall discuss briefly the main properties of the determining factors. In section 3, we state and prove a formula for the determining factors if the " characteristics " are at most " double ", in the sense that if $\zeta_i(x)$ is a branch of roots of p for which $x\zeta_i(x)$ is unbounded as $x \to 0$ and if $\lim_{x\to 0} \zeta_i(x)/\zeta_j(x) = \lim_{x\to 0} \zeta_i(x)/\zeta_k(x) = 1$, then either $i = j$, or $i = k$, or $j = k$. In section 4, we apply the results of section 3 to characterize all hypoelliptic ordinary differential operators with at most " double " " characteristics ". For this, we apply the characterization of hypoelliptic ordinary differential operators given in [4] ; this characterization is given essentially in terms of determining factors. In section 5, we compute the determining factors and characterize hypoellipticity for general third order ordinary differential operators.

2 - PROPERTIES OF DETERMINING FACTORS. -

For the discussion in this section, it will be convenient to consider operators whose coefficients are elements of $F[x]$ - the quotient field of the ring (formal series of powers of $x^{1/q}$, where q is an arbitrary positive integer (compare [1]). Thus, let $Lu = \Sigma_{j=0}^{m} a_j(x)d^ju/dx^j$ be a formal differential operator with

$$(2.1) \qquad a_j(x) \sim \Sigma_{k=n_jq}^{\infty} \quad a_{j,k}(x^{1/q})^k$$

where n_jq is an integer, $a_{j,k}$ are complex numbers, and $a_{j,n_jq} \neq 0$ unless $a_j(x) = 0$; in the latter case, we set

$n_j = +\infty$. Hence n_j is the multiplicity of the zero or minus the multiplicity of the pole of $a_j(x)$ at the origin. We shall also use the notation $n_j = O(a_j)$. Note that the integer q can be replaced in (2.1) by any integral multiple of it. We shall always assume that $n_m < \infty$.

Recall that the characteristic index [2] or class [3] of L is defined to be $m - r$, where the integer r, $0 \le r \le m$, is specified by the

$$
(2.2) \qquad \begin{array}{ll} n_j - j > n_r - r & \text{for } j > r \\ n_j - j \ge n_r - r & \text{for } j < r \end{array}
$$

The equation $Lu = 0$ possesses an indicial equation of order r. This indicial equation is obtained by equating to zero the coefficient of the lowest order term in the formal expansion of $x^{-\rho}L(x^{\rho})$ (the order of the lowest power of x in that expansion is $n_r - r$). Note that for $r = 0$, the indicial equation has no roots, and if $q = 1$ and the functions $a_j(x)$ are holomorphic at the origin, then $r = m$ corresponds to a regular singular point at $x = 0$. Using the roots of the indicial equation, one obtains (by equating coefficients) r linearly independent formal log-fractional powers series solutions of $Lu = 0$.

In order to find the remaining $m - r$ formal solutions of the homogeneous equation $Lu = 0$, one looks for functions $Q(x)$ of the form

$$
(2.3) \qquad Q(x) = \Sigma_{k=1}^{s} c_k x^{-k/q}
$$

(here q is an integral multiple of the integer occurring in (2.1)) such that the formal differential operator M defined by $Mv = e^{-Q}L(e^{Q}v)$ has a characteristic index $m - j < m$, so that the equation $Mv = 0$ possesses an indicial equation of order $j \geq 1$. Hence, there exist j linearly independent formal (log) fractional powers series solution of $Mv = 0$. Such a function $Q(x)$ is called a determining factor (of the operator L). In this manner, the existence of a system (1.1) - (1.3) of m linearly independent formal solutions of $Lu = 0$ is established in the classical theory. The r formal log-fractional powers series solutions which are determined by the indicial equation of L correspond to r identically vanishing determining factors. We shall assume, from now on, that $Q_1(x) = \ldots = Q_r(x) = 0$.

We shall need some expression for the coefficients of M. Set $e^{-Q(x)}(e^{Q(x)})^{(n)} = S(n,x)$. Then $S(0,x) = 1$ and $S(n+1,x) = Q'(x)S(n,X)+dS(n,x)/dx$. Using Leibnitz's rule, we see that

$$Mv = \Sigma_{j=0}^{m} a_j \, \Sigma_{h=0}^{j} \binom{j}{h} e^{-Q}(e^{Q})^{(j-h)} v^{(h)}(x) =$$

$$(2.4) \qquad = \Sigma_{h=o}^{m} \left(\Sigma_{j=h}^{m} \binom{j}{h} a_j(x) S(j-h,x)\right) v^{(h)}(x)$$

Differentiating (2.3), we see that

$$(2.5) \qquad Q'(x) = \Sigma_{k=q+1}^{t} e_k \, x^{-k/q}$$

where $e_k = -k\, c_k/q$, $t = s+q$. It can be verified by easy induction that

$$(2.6) \qquad o(S(n,x) - Q'^{n}) \geq -(n-1)t/q-1$$

Hence the lowest terms in the coefficient of v in **M** are contained in the sum

$\Sigma_{j=0}^{m} a_{j,n_j q} \, x^{n_j} (e_t)^j x^{-tj/q}$. Set

$\min\limits_{0\le j\le m} n_j - tj/q = w, Mv = \Sigma_{h=0}^{m} b_h(x)d^h v/dx^h$. Then (if e_t is arbitrary)

$w = O(b_0(x))$. Clearly,

$$O(b_h) \geq \min_{h\le j\le m} [O(a_j) + o(S(j-h,x))] \geq \min_{0\le j\le m} [n_j-(j-h)t/q] =$$

$$= w + ht/q > o(b_0(x)) + h.$$

It follows that the characteristic index of M can be strictly less than m only if $\sum\limits_{j\in I} a_{j,n_j q}(e_t)^j = 0$, where $I = \{j : n_j - t_j/q = w\}$. This well known result implies that the branches of roots can be labelled in such a way that the lowest order term c_s in the determining factor $Q_j(x)$ $(j > r)$ is given by $e_t = i\,\alpha_{j,-t}$ and $N(j) = -t$. (The formal Puiseux expansion (1.4) exists even if the coefficients $a_j(x)$ are only elements of $F[x]$; the integer q appearing in (1.4) is an integral multiple of the q which appears in (2.1).)

Unfortunately, the higher terms which appear in the formula (2.3) are less easily obtained, in the most general case. Only if the " characteristics " are " simple " (in the sense that if $N(j)/q < -1$ and $N(j) = N(k)$, $j \neq k$ then $\alpha_{j,N(j)} \neq \alpha_{k,N(k)}$) one can read the complete determining factor off the Puiseux expansion (1.4) ([4]). (If $a_j(x) \in C^\infty$ then the " characteristics " are " simple " in the sense of section 1 if and only if they are simple in the sense described above, see section 6 in [4] .)

Set $\beta(j) = N(j)/_q$ if $\alpha_{j,N(j)} \neq 0$ and $\beta(j) = \infty$ if $\zeta_j(x) = 0$. We shall assume, from now on, without explicitly mentioning it, that the factors in (2.1) are labelled in such a way that the sequence $\{\beta(j)\}$ is non-increasing. We note that $\beta(r) \geq -1$ and $\beta(r+1) < -1$. This follows immediately, either by equating the coefficients of $x^{n_j}\xi^j$ in the equation

$$(2.7) \qquad p(x,\xi) = a_m(x)\Pi_{j=1}^{m} (\xi - \xi_j(x))$$

(compare [4, section 2] for the case $q = 1$) or by the fact that the lowest terms of the determining factors $Q_{r+1},\dots,Q_m$ are given, with the correct multiplicities, by the lowest terms of the branches $\zeta_j(x)$.

The following lemma will simplify somewhat the proofs in sections 3 and 5.

Lemma 2.1 :

Let $P(x)$ be an element of $F[x]$, $P(x) = \Sigma_{k=s}^{\infty} a_k x^{k/q}$, such that the characteristic index of the operator $e^{-P(x)}L(e^{P(x)}v)$ is strictly less than m. Then $Q(x) = \Sigma_{k=s}^{-1} a_k x^{k/q}$ is a determining factor for L.

Proof :

Set $Mv = e^{-Q(x)}L(e^{Q(x)}v)$. Then
$Mv = e^{P(x)-Q(x)}e^{-P(x)}L(e^{P(x)}e^{Q(x)-P(x)}v)$.

It follows from the assumptions that there exists an element w of $F[x]$ with $e^{-P}L(e^{P}w) = 0$. But $e^{P(x)-Q(x)} \in F[x]$ which implies that $e^{P(x)-Q(x)}w \in F[x]$. Set $v = e^{P(x)-Q(x)}w$. Then $Mv = 0$. Hence M possesses an indicial equation and the lemma follows.

3 - DETERMINING FACTORS WHEN " CHARACTERISTICS " ARE A MOST " DOUBLE ".

In the process of obtaining formulas for the determining factors when the " characteristics " are not " simple ", we shall need the following simple lemma :

Lemma 3.1 :

Let $Lu = \Sigma_{j=0}^{m} a_j(x)d^ju/dx^j$ be a formal differential operator whose coefficients are formal power series, let $n_j = o(a_j(x))$, and let $m - r$ be the characteristic index of L. Let c_j be an arbitrary complex number, $0 \leq j \leq m - 1$. Then the characteristic index of the operator $Nu = a_m(x)d^mu/dx^m + \Sigma_{j=0}^{m-1}(a_j(x) + c_j\, a_{j+1}(x))d^ju/dx^j$ is also equal to $m - r$. Moreover, $n_r = o(a_r(x)+c_j\, a'_{r+1}(x))$ if $r < m$.

Proof :

Set $a_{m+1}(x) = 0$, $c_m = 1$, and $p_j = o(a_j(x) + c_j\, a'_{j+1}(x))$, $0 \leq j \leq m$. Clearly, $p_j \geq \min\,(o(a_j(x)), o(a'_{j+1}(x)) \geq \min(n_j,n_{j+1} - 1)$. But according to (2.2), $o(a'_{r+1}(x)) \geq n_{r+1} - 1 > n_r$, so that $p_r = o(a_r) = n_r$. Applying (2.2) once more we see that if $j > r$ then $p_j - j > p_r - r$. If $j < r$ then $j + 1 \leq r$. Hence we obtain from the second inequality of (2.2) that in this case $p_j - j \geq p_r - r$.

Recall the standard notations

$$p_{(\beta)}^{(\alpha)}(x,\xi) = \frac{\partial^{\alpha+\beta}p(x,\xi)}{\partial\xi^{\alpha}\partial x^{\beta}}$$

and $q(x,D)u = \Sigma_{j=0}^{m} c_j(x)(-i)^jd^ju/dx^j$ for $q(x,\xi) = \Sigma_{j=0}^{m} c_j(x)\,\xi^j$.

The main result of the present paper is

Theorem 1 :

Let the coefficients $a_j(x)$ of the differential operator $L = \Sigma_{j=0}^{m} a_j(x)d^j/dx^j$ be C^∞ (complex valued) functions in a neighborhood of the origin and let $n_m < \infty$. Assume that the branches $\xi_1(x),\ldots,\xi_m(x)$ of roots of $p(x,\xi) = \Sigma_{j=0}^{m} a_j(x)(i\xi)^j$

satisfy the following condition. If $x.\zeta_i(x)$ is unbounded as $x \to 0$, and $\lim_{x\to 0} \zeta_i(x)/\zeta_j(x) = \lim_{x\to 0} \zeta_i(x)/\zeta_k(x) = 1$, then either $i = j$ or $i = k$ or $j = k$. Set

$$(3.1) \qquad q(x,\xi) = p(x,\xi) + \frac{i}{2} p_{(1)}^{(1)}(x,\xi)$$

and let $m - r$ be the common characteristic index of $L = p(x,D)$ and $q(x,D)$. Then the derivatives of the non-zero determining factors $Q_{r+1}(x),\dots,Q_m(x)$ of L are given by the formula

$$(3.2) \qquad \frac{dQ_j}{dx} = i \, \Sigma_{k=M(j)}^{-q-1} \, \beta_{j,k} \, x^{k/q} \qquad j = r+1,\dots,m$$

where $\Sigma_{k=M(j)}^{\infty} \beta_{j,k}x^{k/q}$, $q \leq j \leq m$, is the formal Puiseux expansion of the branch $\eta_j(x)$ of roots of $q(x,\xi)$, labelled so that the sequence $\{M(j)\}$ is non-increasing.

Proof :

We show first that $M(j) = N(j)$ and $\beta_{j,N(j)} = \alpha_{j,N(j)}$ for $r < j \leq m$. Set $w = \min_{0\leq h\leq m} n_h + hN(j)/q$, $I = \{h : n_h + hN(j)/q = w\}$, so that

$o(\Sigma_{h \in I} a_h(x)\alpha_{j,N(j)}x^{N(j)/q})^h) > w$, and

$o(\Sigma_{h \in I} a_h(x)(\alpha_{j,N(j)}x^{N(j)/q})^h) > w$, but

$o(\Sigma_{h \in I} a_h(x)\ (\alpha x^{N(j)/q})^h) = w$ for all but finitely many values of α. If $h \in I$, then $n_{h+1} - 1 =$

$n_{h+1} + (h+1)N(j)/q - h\,N(j)/q - (1+N(j)/q \geq$

$\geq w - hN(j)/q - (1+N(j)/q) = n_h - (1+N(j)/q) > n_h$.

Hence $p_h = n_h$, where we set $J_h = o(a_h - \frac{1}{2}(h+1)a'_{h+1})$ and $a_{m+1}(x) = 0$. For all h, $0 \leq h \leq m$,

$$p_h + hN(j)/q \geq \min\,(n_h + hN(j)/q, n_{h+1} - 1 + hN(j)/q \geq$$

$$\geq \min\,(w, n_{h+1} + (h+1)\,N\,(j)/q - (1 + N(j)/q)) = w.$$

Hence $p_h + hN(j)/q = w$ precisely for $h \in I$ and $\alpha_{j,N(j)}x^{N(j)q}$ is the lowest term in a Puiseux expansion for a branch η_k of roots of $q(x,\xi)$, since

$$o(\Sigma_h\ {}_I\left[a_n - \tfrac{1}{2}(h+1)a'_{h+1}\right]\,(\alpha_{j,N(j)}x^{N(j)/q})^h) > w, \text{ and if } h \in I \text{ then}$$

$$o(a_h(x)(\alpha_{j,N(j)}x^{N(j)/q})^h) < o(a'_{h+1}(x)(\alpha_{j,N(j)}x^{N(j)/q})^h).$$

By lemma 3.1, $m \geq k > r$, and it follows that all lowest order terms of the expansions of the $m - r$ branches $\eta_{r+1}(x), \ldots, \eta_m(x)$ are obtained in this way.

We want to prove that the function $Q_j(x)$, whose derivative is given in (3.2), is a determining factor. By lemma 2.1, it suffices to show that the characteristic index of the operator $e^{-T_j(x)}L(e^{T_j(x)}v)$ is strictly less than m, where $dT_j(x)/dx = i\ {}_j(x)$. Thus suppressing the subindex j, consider the operator $Mv = e^{-T(x)}L(e^{T(x)}v) = \Sigma_{h=0}^{m} b_h(x)d^hv/dx^h$. Then (compare (2.4))

(3.3) $$b_0(x) = \Sigma_{k=0}^{m} a_k(x)S(k,x)$$

(3.4) $$b_1(x) = \Sigma_{k=1}^{m} k\, a_k(x)S(k-1,x)$$

(3.5) $$b_2(x) = \tfrac{1}{2}\,\Sigma_{k=1}^{m} k(k-1)a_k(x)S(k-2,x)$$

where

(3.6) $$S(n,x) = e^{-T(x)}(e^{T(x)})^{(n)}$$

But $S(0,x) = 1$ and $S(n+1,x) = T'(x)S(n,x) + S'(n,x)$. It follows easily by induction that

(3.7) $$o(R(n,x)) \geq (n-2)N(j)/q - 2$$

where

(3.8) $R(n,x) = S(n,x) - [T'(x)]^n + \frac{n(n-1)}{2}[T'(x)]^{n-2}T''(x).$

Substituting in (3.3), we see that

(3.9) $$b_0(x) = \Sigma_{k=0}^m a_k(x)[T'(x)]^k + \Sigma_{k=2}^n \frac{k(k-1)}{2} a_k(x)[T'(x)]^{k-2}T''(x) + \\ + \Sigma_{k=0}^m a_k(x)R(k,x).$$

Note that

$$\frac{1}{2}\frac{d}{dx} \Sigma_{k=1}^m k\, a_k(x)[T'(x)]^{k-1} = \Sigma_{k=1}^m \frac{k}{2} a_k'(x)[T'(x)]^{k-1} + \\ + \Sigma_{k=2}^m \frac{k(k-1)}{2} a_k(x)[T'(x)]^{k-2}T''(x) = \\ \Sigma_{k=0}^m a_k(x)[T'(x)]^k + \Sigma_{k=2}^m \frac{k(k-1)}{2} a_k(x)[T'(x)]^{k-2}T''(x) - q(x,-iT'(x))$$

But $q(x, -iT'(x)) = 0$. Hence

(3.10) $$b_0(x) = \frac{1}{2}\frac{d}{dx}\left(\Sigma_{k=1}^m k\, a_k(x)[T'(x)]^{k-1}\right) + \Sigma_{k=0}^m a_k(x)R(k,x).$$

It follows from (3.7) that $o(\Sigma_{k=0}^m a_k(x)R(k,x)) \geq w - 2N(j)/q - 2.$

We distinguish now between two cases. Either

(i) $o(p^{(1)}(x, -iT'(x))) < w - 2N(j)/q - 1$, or

(ii) $\sigma(p^{(1)}(x, -iT'(x))) \geq w - 2N(j)/q - 1.$

(i) In this case

$$o\left(\frac{d}{dx} \Sigma_{k=1}^m k\, a_k(x)[T'(x)]^{k-1}\right) \geq o(p^{(1)}(x, -iT'(x))) - 1$$

and $o(p^{(1)}(x, -iT'(x))) - 1 < w - 2N(j)/q - 2.$

Hence $o(b_0(x)) \geq o(p^{(1)}(x, -iT'(x))) - 1$. Note that (2.6) may be applied to (3.4), with T replacing Q and $-N(j)$ replacing t, yielding the estimate

$$o(b_1(x) - \Sigma_{k=1}^{m} k\, a_k(x)[T'(x)]^{k-1}) \geq \min_{1\leq k\leq m} [n_k + (k-2)N(j)/q-1]$$

$\geq w - 2N(j)/q - 1$. It follows that

$$o(b_1(x)) = o(\Sigma_{k=1}^{m} k\, a_k[T'(x))^{k-1}) = o(p^{(1)}(x, -iT'(x))).$$

Hence the characteristic index of M cannot be larger than $m-1$.

(ii) In this case, the branch $\eta_j(x)$ is not " simple ", i.e., there exists an index k, r $k \leq m$, $k \neq j$, such that $N(j) = N(k)$ and $\beta_{j,N(j)} = \beta_{k,N(k)}$. By assumption, if $N(\ell) = N(j)$ and $\beta_{j,N(j)} = \beta_{\ell,N(\ell)}$, then either $j = \ell$ or $k = \ell$. An easy computation shows that

$o(\Sigma_{h=2}^{m} \frac{h(h-1)}{2} a_h(x)[T'(x)]^{h-2}) = o(p^{(2)}(x,\eta_j(x))$ has to be equal to $w = 2N(j)/q$. Applying (2.6) (modified as in case (i)) to (3.5), we see that

$$o(b_2(x) - \Sigma_{h-2}^{m} \frac{h(h-1)}{2} a_h(x)\ [T'(x)]^{h-2}) \geq \min_{2\leq h\leq m} [n_h + (h-3)N(j)/q-1]$$

$\geq w - 3N(j)/q - 1 > w - 2N(j)/q$.

Hence $o(b_2(x)) = w - 2N(j)/q$. On the other hand, it follows from (3.7) and (3.10) that in our present case (ii) $o(b_0(x)) \geq w - 2N(j)/q - 2$. Hence the characteristic index of M is less than or equal to $m - 2$.

If j and ℓ are two distinct indices such that $N(j) = N(\ell)$ and $\beta_{j,k} = \beta_{\ell,k}$ for $N(j) \leq k \leq -q-1$ (so that the right hand sides of (3.2) coincide for j and ℓ, then case (ii) occurs (it is easy to see that

$o(q^{(1)}(x, -iT')) \geq w - 2N(j) - 1$ and

$o(p^{(2)}_{(1)}(x, -iT')) \geq w - 2N(j) - 1)$ and the indicial equation of M is of the second order. Hence $Q_j(x) = Q_\ell(x)$ is indeed a double determining factor. By assumption there can be no third index f with $N(j) = N(f)$ and $\beta_{j,N(j)} = \beta_{f,N(f)}$. Hence the formula (3.2) is completely established, with the correct multiplicities.

4 - HYPOELLIPTICITY WHEN " CHARACTERISTICS " ARE AT MOST " DOUBLE ".

In this section we apply theorem 1 to the problem of characterizing hypoelliptic ordinary differential operators whose " characteristics " are at most " double ". One of the main results of [4] is that if L is an ordinary differential operator of order m with C^∞ coefficients in a neighborhood of the origin and if $n_m < \infty$, then L is hypoelliptic in a neighborhood of the origin if and only if (i) $a_r(0) \neq 0$ and (ii) $|\mathrm{Re}\, Q_j(x)| \to \infty$ as $x \to 0$ for $r < j \leq m$. This result was applied in [4] for the characterization of hypoelliptic ordinary differential operators whose " characteristics " are at most " simple " (Theorem 2 of [4]). Similarly, we shall prove

Theorem 2.-

Let the coefficients $a_j(x)$ of the differential operator $L = \Sigma_{j=0}^m a_j(x) d^j/dx^j$ be C^∞ (complex valued) functions in

a neighborhood of the origin, and let $n_m < \infty$. Assume that the branches $\zeta_1(x), \ldots, \zeta_m(x)$ of roots of $p(x,\xi) = \Sigma_{j=0}^m a_j(x)(i\xi)^j$ satisfy the following condition. If $x\zeta_i(x)$ is unbounded as $x \to 0$, and $\lim_{x\to 0} \zeta_i(x)/\zeta_j(x) = \lim_{x\to 0} \zeta_i(x)/\zeta_k(x) = 1$, then either $i = j$ or $i = k$ or $j = k$. Set $q(x,\xi) = p(x,\xi) + \frac{i}{2} p_{(1)}^{(1)}(x,\xi)$.
A necessary and sufficient condition for the hypoellipticity of L in a neighborhood U of the origin is that there exists a constant $C > 0$ such that for $x \in U$ and ζ a complex number satisfying $q(x,\zeta) = 0$, either $|\zeta| < C$ or $|x|\ |\mathrm{Im}\zeta| \to \infty$ as $x \to 0$.

Proof :

Sufficiency : Set $Nu = q(x,D)u = \Sigma_{j=0}^m b_j(x)d^ju/dx^j$. By lemma 3.1, the characteristic index $m - r$ of L is equal to the characteristic index of N , and $n_r = o(b_r(x))$. The operator Nu satisfies all the assumptions of Lemma 6.1 bis of [4]; hence the conclusions of that lemma apply to N. In particular, $o(b_r(x)) = 0$, which implies that condition (i) is satisfied.

By theorem 1, the derivatives of the determining factors of L are given by (3.2). Choose a branch of $x^{1/q}$ which is positive for $x > 0$ (this implies a certain choice of the constants $\beta_{j,k}$). For every j, $r < j \leq m$, there exists at least one $\beta_{j,k}$ which is not real, with $N(j) \leq k \leq -q-1$ (otherwise $|x|\ |\mathrm{Im}\zeta|$ would not tend to ∞ as $x \to 0_+$, since $\eta_j(x) - \Sigma_{k=N(j)}^{-1} \beta_{j,k}x^{k/q}$ is bounded as $x \to 0$, by lemma 6.2 of [4]). Hence $|\mathrm{Re}\ Q_j(x)| \to \infty$ as $x \to 0_+$. Similarly, we can prove that $|\mathrm{Re}\ Q_j(x)| \to \infty$ as $x \to 0_-$. This condition (ii) is also satisfied.

Necessity : It is easy to prove (as is done at the end of section 6 in [4]) that if there exists an unbounded branch $\eta_j(x)$ of zeros of $q(x,\xi)$ with $1 \leq j \leq r$ then $b_r(0) = 0$ and, by Lemma 3.1, $n_r > 0$. Similarly, $q(0,\zeta) = 0$ can hold only for finitely many values of ζ. Otherwise $q(0,\zeta) = 0$, $b_j(0) = 0$ for all j, $0 \leq j \leq m$ and in particular $b_r(0) = 0$ and $a_r(0) = 0$. If $|x|\ |\mathrm{Im}\ \eta_j(x)|$ does not tend to ∞ as $x \to 0$ (even if it does so from one side only) for some index j, $r < j \leq m$, then Lemma 6.2 of [4] implies once again that $\beta_{j,k}$ is real for $N(j) \leq k \leq - q - 1$ (for a suitable choice of the branch of $x^{1/q}$). Thus $Q_j(x)$ is purely imaginary (at least from one side) by (3.2), and L cannot be hypoelliptic near the origin. Hence $p(x,\zeta) = 0$ only if either $|\zeta|$ is bounded or $|x|\ |\mathrm{Im}\zeta| \to \infty$ as $x \to 0$.

5. DETERMINING FACTORS AND HYPOELLIPTICITY FOR OPERATORS OF THE THIRD ORDER. -

Consider the operator $Lu = \Sigma_{j=0}^{3}\ a_j(x)d^ju/dx^j$, where the coefficients $a_j(x)$ are C^∞ functions at a neighborhood of the origin and $n_3 < \infty$. Hence $a_3(x) \neq 0$ for $x \neq 0$ if x is sufficiently small. Set

$$(5.1)\qquad b_j(x) = \frac{a_j(x)}{a_3(x)}\ ,\ 0 \leq j \leq 3.$$

Then every function $b_j(x)$, $0 \leq j \leq 3$, has a formal Laurent expansion, with $o(b_j) > -\infty$. The determining factors of L are obviously unaffected by this division. Moreover $3 - r$ is both the characteristic index of L and of $\frac{1}{a_3}\ L$.

Set $p(x,\xi) = \Sigma_{j=0}^{m} b_j(x)(i\xi)^j$ and

(5.2) $$q(x,\xi) = p(x,\xi) + \frac{i}{2} p_{(1)}^{(1)} (x,\xi) - \frac{1}{1^2} p_{(2)}^{(2)} (x,\xi)$$

We can state now

Theorem 3.-

Let

(5.3) $$q(x,\xi) \sim - i \, \pi_{j=1}^{3} (\xi - \Sigma_{k=M(j)}^{\infty} \beta_{j,k} x^{k/q})$$

be the formal Puiseux expansion of q, labelled so that $\{M(j)\}$ is non-increasing. Then the derivatives of the determining factors of L are given by

(5.4) $$\frac{dQ_j}{dx} = i \quad \Sigma_{k=M(j)}^{-q-1} \beta_{j,k} x^{k/q} \; , \; j = r + 1, \ldots, 3.$$

Proof :

Choose complex numbers c_j and d_j such that $c_j + d_j = -(j+1)/2$, $c_j d_j = (j+1)(j+2)/12$. Repeated application of Lemma 3.1 yields that the operator $q(x,D)$, whose j-th coefficient is

$$b_j(x) - (j+1)b'_{j+1}(x)/2 + (j+2)(j+1)b''_{j+2}(x)/12 =$$

$$= \left[b_j(x) + c_j b'_{j+1}(x)\right] + d_j \left[b_{j+1}(x) + c_j \, b'_{j+2}(x)\right]$$

has the same characteristic index $3 - r$ as L (here we have put $b_4(x) = b_5(x) = 0$). Hence the use of the index r in (5.4) is justified. By lemma 2.1, it suffices to consider the characteristic index of the operator $Mv = e^{-T(x)}L(e^{T(x)}v(x))$, where

(5.5) $$\frac{dT}{dx} = i \quad \Sigma_{k=M(j)}^{\infty} \beta_{j,k} x^{k/q}$$

for some j, $r < j \leq 3$. Set $Mv = \Sigma_{j=0}^{m} c_j(x)d^jv/dx^j$. Then

(5.6) $\quad c_3(x) = 1$

(5.7) $\quad c_2(x) = b_2(x) + 3T'(x)$

(5.8) $\quad c_1(x) = 3[T'(x)]^2 + 2b_2(x)T'(x) + b_1(x) + 3T''(x)$

(5.9) $$c_0(x) = [T'(x)]^3 + b_2(x)[T'(x)]^2 + b_1(x)T'(x) + b_0(x) + \\ + 3T'(x)T''(x) + b_2(x)T''(x) + T^{(3)}(x)$$

Inserting (5.2) into (5.3) (using also (5.3) and (5.5)), we see that

(5.10) $\quad c_0(x) = 3T'T'' + T^{(3)} + b_2T'' + b_2'T' + B_1'/2 - b_2''/6$

Differentiating (5.7) and (5.8), we get :

$$\frac{1}{2}\ \frac{d}{dx}\, c_1(x) = 3T'T'' + \frac{3}{2}\, T^{(3)} + b_2'T' + b_2T'' + \frac{1}{2}\, b_1'$$

$$\frac{1}{6}\,\frac{d^2}{dx^2}\ c_2(x) = \frac{1}{6}\, b_2''(x) + \frac{T^{(3)}(x)}{2}$$

Hence

(5.11) $\quad c_0(x) = \frac{1}{2}\ \frac{d}{dx}\, c_1(x) - \frac{1}{6}\,\frac{d^2}{dx^2}\ c_2(x)$

and

$$o(c_0) \geq \min\,(o(c_1'), o(c_2'')) \geq \min\,(o(c_1) - 1,\ o(c_2) - 2)$$

so that the characteristic index of M is strictly less than 3. Note that if $r = 0$ and the right hand side of (5.4) is independent of j, $j = 1,2,3$, then $o(c_2(x)) = o(q^{(2)}(x, - iT'(x))) \geq -1$ which proves that the characteristic index of M is 0 and $Q_1(x)$ is a triple determining factor. If the right hand sides of (5.4) are the same for exactly two values of j, say for $j = j_1$ and $j = j_2$, then it is easily seen that

$$o(q^{(1)}(x, - iT'(x))) \geq o(q^{(2)}(x, - iT'(x))) - 1$$

where $- iT'(x)$ is given by (5.5) for $j = j_1$. But

$c_2(x) = - q^{(2)}(x, - iT'(x))/2$ and

$c_1(x) = - i\, q^{(1)}(x, - iT'(x)) + c_2'(x)$.

It follows that $o(c_1) \geq o(c_2) - 1$ and the characteristic index of M is 1.

Hence formula (5.4) gives also the correct multiplicities of the determining factors.

We can now characterize hypoelliptic ordinary differential operators of the third order.

<u>Theorem 4</u> :

L is hypoelliptic near the origin if and only if (i) $n_r = 0$ and (ii) whenever $\eta_j(x)$ is an unbounded branch of zeros of $q(x,\xi)$ where $q(x,\xi)$ is given by (5.2), then $|x|\ |\mathrm{Im}\ \eta_j(x)| \to \infty$ as $x \to 0$.

Theorem 4 is an immediate corollary of theorem 1 of [4] (stated also in section 4 of the present paper), Theorem 3, and the fact that the Puiseux expansion of $\eta_j(x)$ is actually asymptotic expansion (compare Lemma 6.2 in [4]).

Condition (i) of Theorem 4 can be replaced by a condition on the zeros sets of $q(x,\xi)$ and of $a_3(x)p(x,\xi)$. We leave the details to the reader.

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