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ALBRECHT FRÖHLICH

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GALOIS MODULE STRUCTURE AND ARTIN L-FUNCTIONS

by

Albrecht FRÖHLICH

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1) Let L and K be number fields, always of finite degree over \mathbb{Q} , and L normal over K with Galois group $\text{Gal}(L/K) = \Gamma$. Let χ be a character of Γ . We shall write $\Lambda(s, \chi)$ for the enlarged Artin L -function, to include gamma and exponential factors. It satisfies a functional equation

$$W(\chi) \Lambda(s, \chi) = \Lambda(1-s, \bar{\chi}) ,$$

where $\bar{\chi}$ is the complex conjugate of χ and $W(\chi)$, the Artin root number, is of absolute value 1. We write

$$W(L/K, \chi) = W(\chi) = \tau(\chi) W_{\infty}(\chi) / Nf(\chi)^{1/2} ,$$

where $W_{\infty}(\chi)$ is a power of $i = \sqrt{-1}$, depending on ramification at infinity, $Nf(\chi)^{1/2}$ is the positive square root of the absolute norm of the conductor $f(\chi)$, and $\tau(\chi)$ is the "Galois Gauss-sum" (Hasse).

If χ is real valued then $W(\chi) = \pm 1$. If actually χ comes from a real

representation then $W(\chi) = 1$.

Example : Let $\Gamma = H_{4\ell^r}$ be the generalised quaternion group of order $4\ell^r$, where ℓ is an odd prime and $r \geq 1$. For each $j = 1, \dots, r$ there are faithful irreducible representations (of degree 2) of the quotient $H_{4\ell^j}$ of $H_{4\ell^r}$, which have real valued characters ψ_j , but which cannot be realised over the fields of real numbers. If L/K is tame then the root numbers of the distinct characters $\psi = \psi_j$, with the same j , all coincide (cf. [1] theorem 5). We shall write

$$W(L/K, \psi_j) = W_j(L/K), \quad (j = 1, \dots, r).$$

2) Let again L/K be normal and tame, with $\text{Gal}(L/K) = \Gamma$, and let $\Gamma : \Gamma \rightarrow \text{GL}(n, E)$ be a representation of Γ over some number field E with character χ . Let $a \in L$, define the resolvent

$$(a | \chi) = \det \left(\sum_Y a^Y T(\gamma)^{-1} \right).$$

Let $K(\chi)$ be the field obtained by adjoining the values of χ to K and let $\mathfrak{o}_{K(\chi)}$ be its ring of algebraic integers. Let \mathfrak{O} be the ring of algebraic integers in L . Then the $(a | \chi)$, with $a \in \mathfrak{O}$, generate a finitely generated rank one module over $\mathfrak{o}_{K(\chi)}$, contained in $L(\chi)$, which we denote by $(\mathfrak{O} | \chi)$. This module is basic for the determination of the Galois module structure of \mathfrak{O} .

Let from now on

$$K = \mathbb{Q}.$$

As a special case of a general theorem one has

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THEOREM 1. - $(\mathfrak{O} | \chi) = \mathfrak{o}_{\mathbb{Q}(\chi)} \tau(\chi)$ (cf. [3]).

Let (\mathfrak{O}) be the class in the class-group $\text{Cl}(\mathbb{Z}(\Gamma))$ of the integral group-ring $\mathbb{Z}(\Gamma)$, given by the $\mathbb{Z}(\Gamma)$ -module \mathfrak{O} . Let $\text{D}(\mathbb{Z}(\Gamma))$ be the kernel group, i. e. the kernel of $\text{Cl}(\mathbb{Z}(\Gamma)) \rightarrow \text{Cl}(\mathbb{M})$, \mathbb{M} being a maximal order of $\mathbb{Q}(\Gamma)$. From Theorem 1, one deduces (cf. [3]):

THEOREM 2. - $(\mathfrak{O}) \in \text{D}(\mathbb{Z}(\Gamma))$. (Martinet's conjecture).

3) With L/\mathbb{Q} as above, let again $\Gamma = H_{4\ell^r}$, with each $j = 1, \dots, r$ (or rather with the corresponding class of characters $\psi = \psi_j$) there is associated a surjection :

$$\theta_j : \text{D}(\mathbb{Z}(H_{4\ell^r})) \rightarrow \pm 1,$$

and these are independent (cf. [2]). Write

$$\theta_j((\mathfrak{O})) = U_j(L).$$

Then one has (cf. [2]) :

THEOREM 3. - (i) If $\ell \equiv -1 \pmod{4}$ then

$$U_j(L) = W_j(L/\mathbb{Q}) \quad (j = 1, \dots, r),$$

(ii) If $\ell \equiv 1 \pmod{4}$ then

$$U_j(L) = 1 \quad (j = 1, \dots, r).$$

One can show that given

$$f : [1, \dots, r] \rightarrow [\pm 1]$$

there are infinitely many fields L with $W_j(L) = f(j)$.

4) Case (ii) of Theorem 3 shows that the Galois module structure of \mathcal{O} will not suffice to determine the root numbers. One needs additional structure, and has to consider in the general situation of tame extensions \mathcal{O} as a "Hermitian Galois module". There is indeed a Hermitian form of L over $\mathbb{Q}(\Gamma)$, induced by the trace.

In the new situation one has to construct a Hermitian Class-group $HCl(Z(\Gamma))$, which classifies locally free rank one $Z(\Gamma)$ -modules M with a non-singular Hermitian form on the $\mathbb{Q}(\Gamma)$ -module spanned by M . The element corresponding to \mathcal{O} will be denoted by $[\mathcal{O}]$.

Now return to the case $\Gamma = H_{4\ell^r}$. Then $[\mathcal{O}]$ belong to a certain subgroup G_r of $HCl(Z(H_{4\ell^r}))$ which under the map $HCl(Z(H_{4\ell^r})) \rightarrow Cl(Z(H_{4\ell^r}))$ falls into $D(Z(H_{4\ell^r}))$. Let \mathbb{F}_ℓ be the field of ℓ element, \mathbb{F}_ℓ^* its multiplicative group. There are then canonical surjections $\omega_j : G_r \rightarrow \mathbb{F}_\ell^*$, so that the diagram

$$(D) \quad \begin{array}{ccc} G_r & \xrightarrow{\omega_j} & \mathbb{F}_\ell^* \\ \downarrow & & \downarrow \\ D(Z(H_{4\ell^r})) & \xrightarrow{\theta_j} & \pm 1 \end{array} \quad [x \rightarrow x^{(\ell-1)/2} = \left(\frac{x}{\ell}\right)_2]$$

commutes. Write $\omega_j(\mathcal{O}) = V_j(L)$.

In place of Theorem 3 one now gets the better

THEOREM 4. - $V_j(L) = W_j(L/\mathbb{Q}) \quad (j = 1, \dots, r)$.

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Note that the distinctions of cases in Theorem 3 now becomes obvious from diagram (D).

For other surveys of this general topic see Martinet's Bourbaki report (cf. [4]) and the report of my address to the Vancouver congress (cf. [5]).

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Albrecht FRÖHLICH
University of London
King's College
Département of Mathematics
Strand London WC 2R 2LS