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UNIQUE ERGODICITY FOR FOLIATIONS

Rufus Bowen

Let G be a connected Lie group acting in a locally free way on a compact manifold M. Then $\{Gx: x \in M\}$ is a foliation of M. This orbit foliation in some sense represents the geometry of the group action; for example, two flows are often considered equivalent if there is a homeomorphism of their phase spaces which matches up their orbit foliations.

One can ask how much of the ergodic theory of G-invariant probability measures on M is geometrical in that it depends only on the orbit foliation. There is a notion of invariant measure for a foliation $\mathcal{F} = \{\mathcal{F}_x: x \in M\}$ of M; it is a family

$$\mu = \{\mu_K: K \text{ compact transversal of } \mathcal{F}\}$$

satisfying

(a) μ_K is a finite nonnegative Borel measure on K,

(b) $\mu_K(K) > 0$ for some K, and

(c) if $A_i \subset K_i$ $(i=1,2)$ are Borel and $g: A_1 \to A_2$ a Borel isomorphism with $g(x) \in \mathcal{F}_x \; \forall x \in A_1$, then $\mu_{K_1}(A_1) = \mu_{K_2}(A_2)$.

Two such invariant measures μ and μ' are considered equal if there is a $\lambda > 0$ with $\mu'_K = \lambda\mu_K$ for every K. Notice that μ is not a measure on M unless $\mathcal{F}_x = \{x\}$ for all x. If G is unimodular, then there is a natural bijection between the G-invariant probability measures of a locally free continuous G-action and the invariant measures for the orbit foliation $\{Gx: x \in M\}$ (this works just like the well-known case of $G = R$). For actions of a unimodular G one sees that the ergodicity of a particular measure, existence of an invariant measure and unique ergodicity (unique invariant measure) are geometrical properties.

How about mixing? It does not seem to have any geometrical meaning for continuous flows [10] , but it may for certain smooth flows ([2] , [9]). For foliations $\mathcal{F}$ of dimension $n \geq 2$ one could look for an ergodic theorem (with some condition on $\mathcal{F}$ analogous to amenability for G [14]), an Ambrose-Kakutani theorem, and an analogue of "positive entropy". These general questions are not my motivation for mentioning $\mathcal{F}$-invariant measures; rather it is to state a theorem about Anosov systems.

A diffeomorphism $f: M \to M$ is <u>Anosov</u> [1] , if there is a Df-invariant splitting of the tangent bundle $TM = E^s \oplus E^u$ and constants $c > 0$, $\lambda \in (0,1)$ so that

$$\|Df^n(v)\| \leq c\lambda^n \|v\| \quad \text{for} \quad v \in E^s , \ n \geq 0$$

and

$$\|Df^{-n}(v)\| \leq c\lambda^n \|v\| \quad \text{for} \quad v \in E^u , \ n \geq 0 .$$

Then there are stable and unstable foliations $\{W^s(x)\}_{x \in M}$ and $\{W^u(x)\}_{x \in M}$ defined by

$$W^s(x) = \{y \in M: d(f^n y, f^n x) \to 0 \quad \text{as} \quad n \to \infty\}$$

$$W^u(x) = \{y \in M: d(f^{-n} y, f^{-n} x) \to 0 \quad \text{as} \quad n \to \infty\} .$$

A flow f^t is Anosov if there is a Df^t-invariant splitting $TM = E \oplus E^s \oplus E^u$ with E the one-dimensional bundle tangent to the flow and E^s, E^u satisfying the same conditions as before. There are stable and unstable foliations exactly as before.

An important unsolved problem is the classification of Anosov diffeomorphisms. All known examples lie on quotients of nilpotent

Lie groups and have their stable and unstable foliations given by group actions [11]. These group actions are known to be uniquely ergodic [3]. The geodesic flow on a surface of constant negative curvature is the standard example of an Anosov flow; here the unstable foliation is the family of horocycles. Furstenberg [7] proved that the horocycle flow is uniquely ergodic. Marcus [12], [13] generalized this to any one-dimensional orientable unstable foliation of an Anosov diffeomorphism or flow (these hypotheses allow one to define an "unstable flow"). These results led Marcus and myself to the

<u>Theorem [4].</u>
The stable and unstable foliations are uniquely ergodic for a topologically mixing Anosov diffeomorphism or flow.

The invariant measure has been studied in [17], [18], and [19]. The mixing condition (for U,V nonempty and open, $\exists T$ with $f^tU \cap V \neq \emptyset$ $t \geq T$) excludes certain degenerate cases. One is hopeful that the foliations in question are given by group actions (and in a nice way); our lack of knowledge here is what necessitates the definition of invariant measure for a foliation.

We won't give here any details of the proof of the theorem, but it uses symbolic dynamics and various technical generalizations of the following.

<u>Lemma</u>. Let $X = \prod_0^\infty \{0,1\}$ and σ be the shift map on X. There is a unique Borel probability measure μ on X for which $\mu(A) = \mu(B)$ whenever $\sigma^n|A$, $\sigma^n|B$ are one-to-one and $\sigma^n(A) = \sigma^n(B)$ for some $n > 0$.

This reminds one of the various mistakes possible when first learning the definition of invariant measure for a non-invertible map. It is amusing that, by taking an incorrect definition of invariant measure, the 2-shift becomes uniquely ergodic and that this phony unique ergodicity is related to the real unique ergodicity of the horocycle flow.

Veech [16] generalized Furstenberg's theorem in the direction of certain group actions on quotients of semi-simple Lie groups. His result does not fit into the theorem stated above, but ideas of [4] did lead [5] to a result containing his. Let G be a unimodular Lie group, Γ a cocompact discrete subgroup and μ the G-invariant measure on G/Γ. For $a \in G$, define T_a on G/Γ by $T_a(g\Gamma) = ag\Gamma$, and S_a on G by $S_a(g) = aga^{-1}$. The tangent space T_eG splits into DS_a-invariant subspaces $T_eG = V^s \oplus V^c \oplus V^u$ corresponding to eigenvalues $|\lambda| < 1$, $|\lambda| = 1$, $|\lambda| > 1$.

<u>Theorem [5]</u>.

Assume T_a *is weak mixing on* $(G/\Gamma,\mu)$ *and* DS_a *is semi-simple on* V^c. *Then the left action of* $G^u(a) = \exp V^u$ *on* G/Γ *is uniquely ergodic*.

It would be good to get a version of this theorem for Anosov actions [15] or partially hyperbolic systems [6], i.e. to remove the symmetry assumptions. This question is raised by the generalized horocycle foliations in [8]. The general underlying problem is to find conditions under which minimality implies unique ergodicity [7].

<u>Example</u>. Let G be $SL(3,\mathbb{R})$. Then the subgroup

$$N = \left\{ \begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix} : x,y,z \in \mathbb{R} \right\}$$ is uniquely ergodic on G/Γ [16].

In fact the subgroup of elements of N with $x = 0$ is uniquely ergodic; use $a = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1/4 \end{pmatrix}$ in the above theorem (T_a is mixing by [20]). It would be interesting to know which elements of N are uniquely ergodic on G/Γ for all . Because of conjugacies one need only consider $\begin{pmatrix} 110 \\ 011 \\ 001 \end{pmatrix}$ and $\begin{pmatrix} 100 \\ 011 \\ 001 \end{pmatrix}$.

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