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non constant length**

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MINIMAL SETS GENERATED BY A  
 SUBSTITUTION OF NON CONSTANT LENGTH

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Let  $P$  denote the alphabet  $\{0,1\}$  and let

$$\Omega = P^{\mathbb{Z}} = \{\omega = \dots \omega_{-1} \omega_0 \omega_1 \omega_2 \dots \mid \omega_i \in P\}$$

A substitution  $\theta$  is a map from  $P$  to  $\bigcup_{n \geq 2} P^n$ . If  $\text{card } \theta(0) = \text{card } \theta(1)$  the substitution is said of constant length. If not, it is said of non constant length.

For suitable  $i$  and  $j$  we can generate (1) a bisequence by the following:

$$\omega[-\text{card } \theta^{2n}(i), \text{card } \theta^{2n}(i) + \text{card } \theta^{2n}(j)] = \theta^{2n}(i) \theta^{2n}(j), \forall n \in \mathbb{N}$$

where  $\omega(a,k)$  is the  $k$ -block of  $\omega$  which begin at  $\omega_a$ .

Under a condition of non-degeneracy, the dynamical system  $(O_\omega, T)$  where  $O_\omega$  is the orbit closure of  $\omega$ , and  $T$  the shift, is minimal (1) and strictly ergodic (2).

We can prove that the dynamical system associated with the substitution  $\theta_1$  defined by

$$\begin{cases} \theta_1(0) = 0^{n+1-p} 1 0^p \\ \theta_1(1) = 1 0^n \end{cases} \quad \text{for } 0 \leq p \leq n,$$

has a purely discrete spectrum. ( $i^k$ ,  $i \in P$ ,  $k \in \mathbb{N}$  means  $\underbrace{i i \dots i}_n$  times).

(1) GOTTSCHALK. Substitution minimal sets. Trans. Amer. Math. Soc. 109 (1963)

(2) MICHEL. Stricte ergodicité d'ensembles minimaux de substitution. CRAS Paris, 278, (1974).

For this, we introduce the notion of "coIncidence-value of the substitution  $\theta$ " by the following :

$$\begin{aligned} \text{if } \theta^\infty(0) &= \lim_{n \rightarrow \infty} \theta^{2n}(0) = \omega_1^0 \omega_2^0 \dots \omega_n^0 \dots \\ \theta^\infty(1) &= \lim_{n \rightarrow \infty} \theta^{2n}(1) = \omega_1^1 \omega_2^1 \dots \omega_n^1 \dots \end{aligned}$$

we say that  $n$  is a "coIncidence value of  $\theta$ " if and only if

$$\omega_n^0 = \omega_n^1 .$$

If we denote the density of the coIncidence values of  $\theta$  in  $\mathbb{N}$  by  $\delta_c(\theta)$ , we can prove that  $\delta_c(\theta_1)$  is equal to one.

Conversely if we consider the substitution  $\theta_2$  :

$$\begin{cases} \theta_2(0) = 0 \ 1 \\ \theta_2(1) = 1 \ 1 \ 0 \ 0 \end{cases}$$

we have  $\delta_c(\theta_2) = \frac{1}{3}$  and we can prove that there is a continuous part in the spectrum of  $\theta_2$  .

Moreover, if we consider the two closely related substitutions  $\theta_2'$  and  $\theta_2''$  defined by

$$\theta_2'(0) = \theta_2''(0) = 0 \ 1$$

and

$$\begin{cases} \theta_2'(1) = 1 \ 0 \ 1 \ 0 \\ \theta_2''(1) = 1 \ 0 \ 0 \ 1 \end{cases}$$

we can prove that  $\theta_2'$  and  $\theta_2''$  have purely discrete spectra and that

$$\delta_c(\theta_2') = \delta_c(\theta_2'') = 1 .$$

More generally, we conjecture that the substitution  $\theta_n$  :

$$\begin{aligned} \theta_n(0) &= 0^n \ 1^n = \underbrace{0\dots 0}_{n \text{ times}} \ \underbrace{11\dots 1}_{n \text{ times}} \\ \theta_n(1) &= 1^{2n} \ 0^{2n} = \underbrace{11\dots 1}_{2n \text{ times}} \ \underbrace{00\dots 0}_{2n \text{ times}} \end{aligned}$$

has :

- 1) a purely discrete spectrum if  $n$  is even (then  $\delta_c=1$ )
- 2) a mixed spectrum (partly continuous and partly discrete) if  $n$  is odd (then  $\delta_c=\frac{1}{3}$ ).

At last, if we consider the constant-length case (3), we know that the so-called

- regular Toeplitz substitution generates a dynamical system with discrete spectrum. In this case  $\delta_c = 1$ .
- generalized Morse substitution generates a dynamical system with mixed spectrum and in this case  $\delta_c = 0$ .

Thus we can draw the following table, and we state the question

Is the following true :

$$\delta_c = 1 \implies \text{discrete spectrum}$$

$$\delta_c < 1 \implies \text{mixed spectrum ?}$$

(see the following page)

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(3) COVEN and KEANE. The structure of substitution minimal sets. (1971)

SUBSTITUTION		$\delta_c$	SPECTRUM	
constant length	Regular Toeplitz substitution $(\theta(0) \neq \widetilde{\theta(1)})$ (1)	1	Discrete	
	Generalized Morse substitution $(\theta(0) = \widetilde{\theta(1)})$ (1)	0	Mixed	
non constant length	$\begin{cases} \theta(0) = 0^{n+1-p} 1 0^p \\ \theta(1) = 1 0^n \end{cases}$ $0 \leq p \leq n$	1	Discrete	
	$\begin{cases} \theta(0) = 01 \\ \theta(1) = 1010 \end{cases} \text{ and } \begin{cases} \theta(0) = 01 \\ \theta(1) = 1001 \end{cases}$	1	Discrete	
	$\begin{cases} \theta(0) = 0 1 \\ \theta(1) = 1 1 0 0 \end{cases}$	$\frac{1}{3}$	Mixed	
	$\begin{cases} \theta(0) = 0^n 1^n \\ \theta(1) = 1^{2n} 0^{2n} \end{cases}$	n even	1	Discrete
		n odd	$\frac{1}{3}$	Mixed
other cases		?	?	

(1) with  $\widetilde{\omega}_k = 1 - \omega_k$