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INEQUALITIES FOR DECOMPOSABLE FORMS

by

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1.- Results. We shall indicate some inequalities for resultants of polynomials with integer coefficients and for decomposable forms in n variables with coefficients in $\mathbb{Q}$. A form $f(\underline{x}) = f(x_1,\dots,x_n)$ is called decomposable if it is a product of linear forms with algebraic coefficients.

In the sequel let M be an arbitrary but fixed finite set of valuations of $\mathbb{Q}$ containing the archimedean one. A point $\underline{x} \in \mathbb{Z}^n$ is called primitive for M if the greatest common divisor of the components of $\underline{x}$ is not divisible by the primes corresponding to the non-archimedean valuations in M.

THEOREM 1.1. *Let $f(\underline{x}) = f(x_1,\dots,x_n)$ be a decomposable form of degree d with rational coefficients. Let k be an integer with $d > k \geq n > 1$ and suppose that f is not divisible by a rational form of degree less than k and that moreover in the factorization $f(\underline{x}) = l_1(\underline{x})\dots l_d(\underline{x})$ of f in a product of linear forms with algebraic coefficients any k factors $l_i(\underline{x})$ have rank n. Then for every $\varepsilon > 0$ there are only finitely many integer points $\underline{x} = (x_1,\dots,x_n)$, which are primitive for M, such that the inequality*

$$(1.1) \qquad \prod_{p \in M} |f(\underline{x})|_p < \|\underline{x}\|^{d-2(k-1)-\varepsilon}$$

is satisfied, where $\|\underline{x}\| = \max\{|x_1|,\dots,|x_n|\}$.

As a consequence of theorem 1.1 we can prove

THEOREM 1.2. Let $R(X) \in \mathbb{Z}[X]$ be of degree d and without multiple roots. Suppose $0 < t < d$, and $R(X)$ has no factor of degree $< t$ in $\mathbb{Z}[X]$. Let $\{p_1,\dots,p_s\}$ be a finite set of primes and denote the archimedean prime by p_o . For polynomials P and Q write $r(P,Q)$ for their resultant. Then for every $\varepsilon > 0$ there are only finitely many polynomials $Q(X) \in \mathbb{Z}[X]$ of degree $\le t$, whose coefficients have no common factor, divisible by one of the p_i $(1 \le i \le s)$ and with

$$(1.2) \qquad \prod_{i=o}^{s} |r(R,Q)|_{p_i} < H(Q)^{d-2t-\varepsilon} ,$$

where $H(Q)$ means the height of Q , i.e. the maximum of the absolute values of its coefficients.

COROLLARY. Let the hypotheses be the same as in theorem 1.1, but suppose moreover $d > 2t$ and that we are considering only polynomials Q with respectively coprime coefficients. Write $P(Q)$ for the greatest prime factor of $r(R,Q)$. Then

$$(1.3) \qquad P(Q) \to \infty \text{ as } H(Q) \to \infty .$$

The following theorem gives an estimate for decomposable, irreducible forms. Let $\alpha_1,\dots,\alpha_n$ be elements of a number field K and write $\mathfrak{N}$ for the norm from K to $\mathbb{Q}$.

THEOREM 1.3. Suppose $N(\underline{x}) = \mathfrak{N}(\alpha_1 x_1 + \dots + \alpha_n x_n)$ is an irreducible norm form of degree $d > n > 1$, whose coefficients $\alpha_1,\dots,\alpha_n$ are linearly independent over $\mathbb{Q}$ and ly in a number field K which has no nontrivial subfields. Then for every $\varepsilon > 0$ there are only finitely many integer points $\underline{x}$, which are primitive for M , such that

(1.4) $$\prod_{p \in M} |N(\underline{x})|_p \leq \|\underline{x}\|^{d-d'-\varepsilon} ,$$

<u>where</u>

(1.5) $$d' = \frac{n}{n-1} \, 2^{(n-2)/2} \, d^{(n-2)/(n-1)} .$$

All these results have been proved by W.M. SCHMIDT [2] in the archimedean case.

2.- <u>On the proofs</u>. The method of proof is that by SCHMIDT [2]. The main work consists in reducing the assertions in such a way that the following criterion for the estimation of products of norm forms is applicable (this is a special version of theorem 1.1 [1]) :

THEOREM 2.1. <u>Suppose</u> $f(\underline{x}) = f(x_1,\dots,x_n)$ <u>is a form with rational coefficients, which is decomposable in a product of</u> d <u>linear forms</u> $l_i(\underline{x})$ $(1 \leq i \leq d)$ <u>with algebraic coefficients. If the inequality</u>

(2.1) $$\prod_{p \in M} |f(\underline{x})|_p < c \, \|\underline{x}\|^{d-\eta}$$

<u>has infinitely many solutions</u> $\underline{x}$, <u>which are primitive for</u> M , <u>with positive constants</u> c <u>and</u> η , <u>then there is a rational subspace</u> S^t <u>of dimension</u> t <u>with</u> $1 \leq t \leq n$ <u>and there are</u> m <u>forms</u> $l_{i_1},\dots,l_{i_m}$ <u>among the forms</u> $l_1,\dots,l_d$ <u>with</u> $1 \leq m \leq d$ <u>and</u> $i_1 < \dots < i_m$, <u>whose restrictions to</u> S^t <u>have rank</u> r <u>satisfying</u>

(2.2) $$r < t \quad \underline{\text{and}} \quad r \leq \frac{tm}{\eta} .$$

Applying (2.2) of theorem 2.1 for the proof of theorem 1.1 one sees that it suffices to show that

(2.3) $$0 < r < t \quad \text{implies} \quad \frac{2(k-1)}{t} \geq \frac{m}{r} ,$$

since by the hypotheses of theorem 1.1 a linear factor l_i of f has $l_i(\underline{x}) = 0$

only for $\underline{x} = 0$.

We take an arbitrary but fixed rational subspace S^t of dimension $t \geq 2$ and we write $m(r)$ for the maximum number of linear forms $l_{i_1}, \dots, l_{i_m}$ of rank r on S^t . Putting $q(r) = \frac{m(r)}{r}$ and $q_o = \max_{0<r<t} q(r)$ one shows that the hypotheses of theorem 1.1 imply $q_o \leq \frac{2(k-1)}{t}$, which proves by (2.3) the theorem (for more details cf. [2], pp. 242-245).

Theorem 1.2 is proved by using theorem 1.1 . Let $\alpha_1, \dots, \alpha_d$ be the roots of R . Then

$$r(R,Q) = a_d^q \, Q(\alpha_1) \dots Q(\alpha_d)$$

where $q = \deg Q$ and a_d is the leading coefficient of R .

Since $\deg Q \leq t$, Q may be written as $Q(X) = b_t X^t + \dots + b_1 X + b_o$ with $(b_t, \dots, b_1, b_o) \in \mathbb{Z}^{t+1}$. Therefore as Q runs through the polynomials in $\mathbb{Z}[X]$ of degree $\leq t$, $Q(\alpha_i)$ runs through the values of the linear form $l_i(y) = y_t \alpha_i^t + \dots + y_1 \alpha_i + y_o$ at points $y \in \mathbb{Z}^{t+1}$. The hypotheses of theorem 1.2 imply those of theorem 1.1 with $t+1 = n = k$; thus by (1.1) the assertion follows. (It should be noted that the condition that R has no multiple roots guarantees that any k of the linear forms $l_i(y)$ $(1 \leq i \leq d)$ have rank n .)

Theorem 1.2 is also obtained as an application of theorem 2.1. But the necessary considerations to obtain the shape of d' as in (1.5) are more intricated. For the precise details we refer the reader to [2], pp. 245-253.

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