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ON SEQUENTIAL ENTROPY OF
MEASURE PRESERVING TRANSFORMATION

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The aim of this article is to present a simple proof of a slightly more general result than those ones obtained in [2] and [4].

Definition

Let $A = \{t_1, \dots, t_n, \dots\}$ be a subset of the set Z_+ of the set Z_+ of nonnegative integers. For a given positive integer n, k we denote

$$A_{n,k} = \bigcup_{i=1}^n \{t_i, t_{i+1}, \dots, t_i + k-1\}$$

$$S_{n,k}(A) = \text{card}(A_{n,k})$$

$$\overline{S}_k(A) = \overline{\lim}_{n \rightarrow \infty} S_{n,k}(A) \quad \text{and} \quad K(A) = \lim_{k \rightarrow \infty} \overline{S}_k(A)$$

Newton in [2] defined $K(A)$ in another way, but the definitions are equivalent. Let $h_A(T)$ denote the sequential entropy introduced by Kushnirenko [1].

Theorem 1.

The equation

$$h_A(T) = K(A) \cdot h(T)$$

holds for any measure transformation T except of the case $h(T) = 0$ and $K(A) = \infty$. In case $K(A) = 0$, $h_A(T) = 0$.

For the proof we recall some properties of conditional entropy.

Let ξ be a finite measurable partition. We denote

$$\xi^K = \bigvee_{i=0}^{k-1} T^{-i}(\xi) \quad \text{for any positive integer } K.$$

Lemma 1.

$$\begin{aligned} H(\xi^k) &= H(\xi) + \sum_{i=1}^{k-1} H(T^{-i}(\xi) \mid \bigvee_{j=0}^{i-1} T^{-j}(\xi)) = \\ &= H(\xi) + \sum_{i=1}^{k-1} H(\xi \mid \bigvee_{j=1}^i T^{-j}(\xi)) \end{aligned}$$

Lemma 2.

For $k \leq q$ we have

$$\frac{1}{q} \cdot H(\xi^q) \leq \frac{1}{k} \cdot H(\xi^k)$$

Proof.

$$\begin{aligned} H(\xi^q) &= H(\xi) + \sum_{i=1}^{q-1} H(\xi \mid \bigvee_{j=1}^i T^{-j}(\xi)) = H(\xi) + \sum_{i=1}^{k-1} H(\xi \mid \bigvee_{j=1}^i T^{-j}(\xi)) + \\ &+ \sum_{i=k}^{q-1} H(\xi \mid \bigvee_{j=1}^i T^{-j}(\xi)) \leq H(\xi^k) + (q-k) \cdot H(\xi \mid \bigvee_{j=1}^{k-1} T^{-j}(\xi)) \leq \\ &\leq H(\xi^k) + \frac{q-k}{k} \cdot [H(\xi) + \sum_{i=1}^{k-1} H(\xi \mid \bigvee_{j=1}^i T^{-j}(\xi))] = \frac{q}{k} \cdot H(\xi^k) \end{aligned}$$

Proof of the theorem.

The fact that in case $K(A) = 0$ $h_A(T) = 0$ was proved in [3].

Suppose that the product $K(A) \cdot h(T)$ is well defined. For any measurable partition ξ and any positive number k we have

$$H_A(T, \xi) \leq H_A(T, \xi^k)$$

For a given k we can write the set $A_{n,k}$ as a union of its maximal connected subsets.

$$A_{n,k} = \bigvee_{j=1}^m C_j$$

We denote

$$q_j = \text{card}(C_j) \quad , \quad n_j = \min(C_j)$$

From the construction of $A_{n,k}$ we have

$$q_j \geq k \quad , \quad \sum_{j=1}^m q_j = S_{n,k}(A)$$

According the definition of sequential entropy

$$\begin{aligned} H_A(T, \xi) &\leq H_A(T, \xi^k) = \overline{\lim} \frac{1}{n} \cdot H(\bigvee_{i \in A_{n,k}} T^{-i}(\xi)) \leq \\ &\leq \overline{\lim} \frac{1}{n} \cdot \sum_{j=1}^m H(\bigvee_{i \in C_j} T^{-i}(\xi)) = \overline{\lim} \frac{1}{n} \cdot \sum_{j=1}^m H(T^{-n_j}(\xi^{q_j})) \leq \\ &\leq \overline{\lim} \frac{1}{n} \cdot \sum_{j=1}^m \frac{q_j}{k} \cdot H(\xi^k) = \frac{1}{k} \cdot H(\xi^k) \cdot \overline{\lim}_{n \rightarrow \infty} \frac{S_{n,k}(A)}{n} = \\ &= \overline{S}_k(A) \cdot \frac{1}{k} \cdot H(\xi^k) \quad . \end{aligned}$$

Taking limit with respect to k we get

$$H_A(T, \xi) \leq K(A) \cdot H(\xi)$$

Therefore the inequality

$$h_A(T) \leq K(A) \cdot h(T)$$

holds.

The converse inequality was proved in [3].

References

- [1] Kushnirenko, A.G. : O matricěskich invariantach tipa entropii, UMN 22(5), 1967, 57-62

- [2] Newton D. : On sequential entropy I - MST 4, 119-125 (1970)
- [3] Newton D. : On sequential entropy II - MST 4, 126-128 (1970)
- [4] Krug E. , Newton D. : On sequence entropy of Automorphisms of Lebesgue space . Zeits. f. Wahrsch. 24, 211-214 (1972)

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