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RELEVANCE OF COMPUTER EXPERIMENTS TO THE PURE MATHEMATICS OF INTEGRABLE AND ERGODIC DYNAMICAL SYSTEMS *

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The computer can no longer be regarded solely as an amusing toy for the mathematically lazy or inept. In this paper, we describe four, simple, computer experiments which have been, or promise to be, of major significance to purely analytic studies of integrable and ergodic dynamical systems.

I. INTRODUCTION

Given a specified dynamical'system whose integrable or ergodic character is in question, a computer can be used to provide an almost definitive answer. Certainly, computer studies cannot yield a rigorous proof, but they can exhibit overwhelming circumstantial evidence. To use an analogy from everyday life, we easily recognize a duck by its unique vocal call, its web feet, its oily feathers, its broad bill, its characteristic tail, and its ability to swim; seldom do we need to anatomically "prove" the given fowl is a duck by dissecting it. Similarly, a computer can be programmed to "see" the distinctive characteristics of integrable or ergodic systems. Granted the computer can be "fooled" but not frequently, provided the programmer is careful. In the following brief paper, we describe several specific cases in which the computer has been used to investigate interesting problems which had stoutly and successfully resisted analytic attack, at least until the computer results were known.

II. THE TODA LATTICE

The Toda lattice[1] is a one-dimensional, conservative, classical dynamical system governed by the Hamiltonian

$$H = \sum_{n=1}^{N} \left[(p_n{}^2/2) + e^{-(q_{n+1} - q_n)} - 1 \right] , \tag{1}$$

where we here assume periodic boundary conditions $q_{n+N} \equiv q_n$ and where we subtract unity from each exponential in order that $H = 0$ when all the q's and p's equal zero. Without entering into the details here, we remark only that this system has been shown to be of both mathematical [2] and physical[3] interest, but until recently, despite the known existance of special, particular solutions, the generic character of the general solution (integrable or ergodic) was unknown. As a consequence, the present author in collaboration with S. D. Stoddard and J. S. Turner investigated this problem numerically.[4] The first non-trivial case is $N = 3$; and it is of particular interest since for it, after neglecting the pure translation degree of freedom, Hamiltonian (1) can be reduced to the two degrees of freedom Hamiltonian

$$H = (1/2)\,(p_1{}^2 + p_2{}^2) + (1/24)\left\{ e^{(2q_2 + 2\sqrt{3}\,q_1)} + e^{(2q_2 - 2\sqrt{3}\,q_1)} + e^{-4q_2} \right\} - (1/8) \quad , \tag{2}$$

which latter can be studied via the well-known Poincare surface of

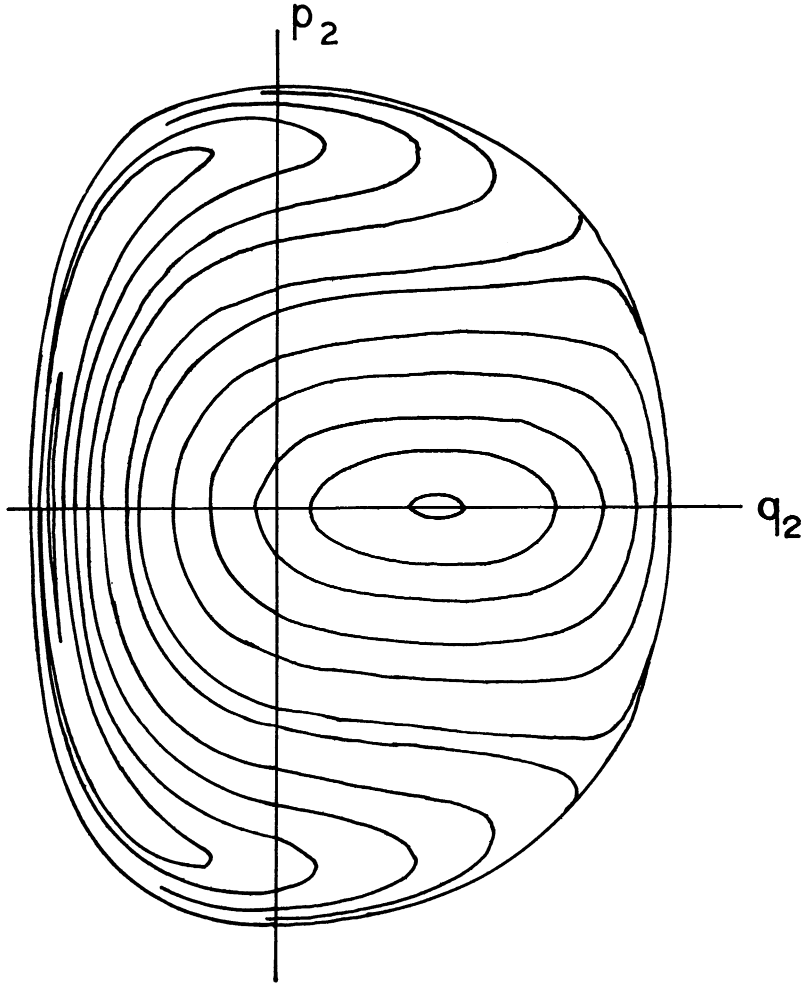

Fig. 1. A surface of section for Hamiltonian (2) at E = 1.

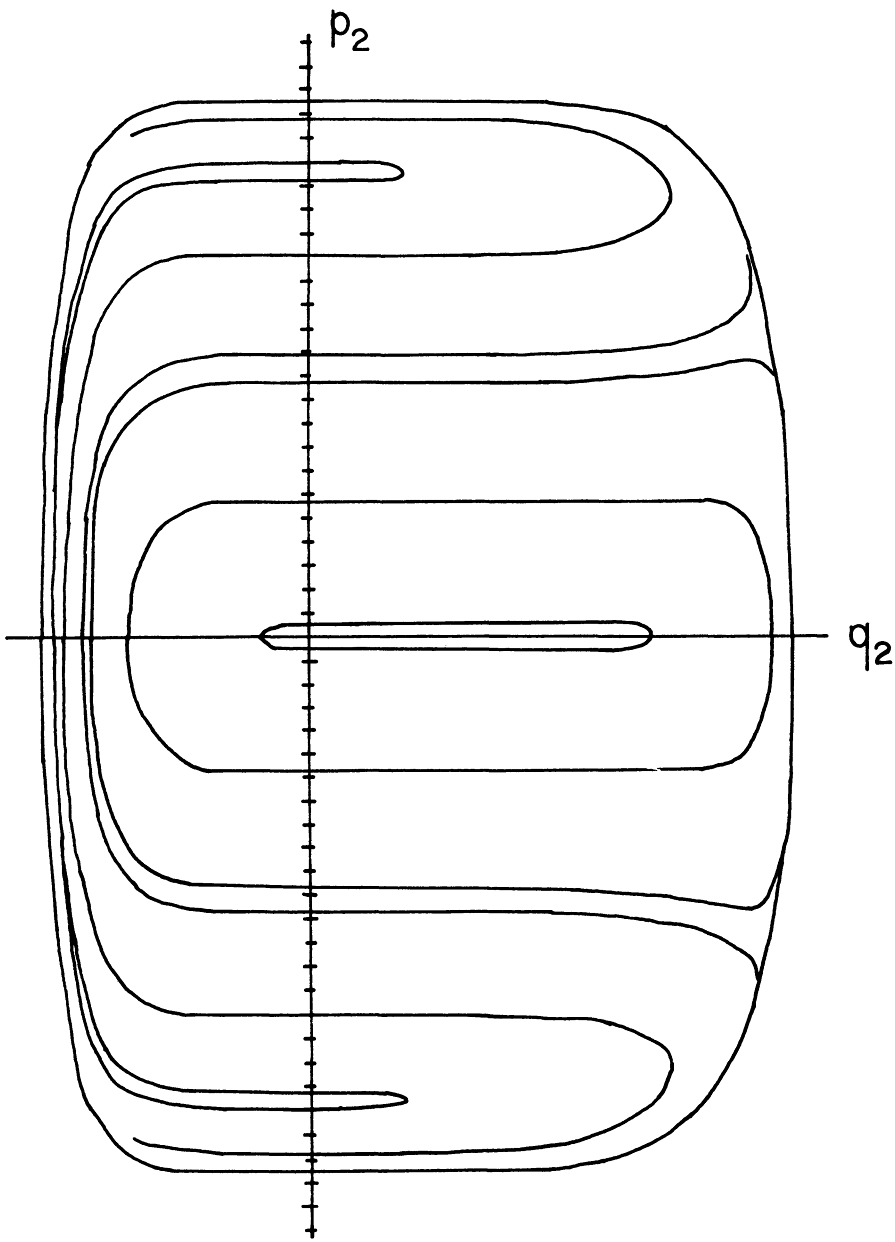

Fig. 2. A surface of section for Hamiltonian (2) at E = 256.

section technique.[5] Since Siegal[6] has shown that generically Hamiltonian systems are not integrable, we anticipated non-integrability for Hamiltonian (2).

We were thus quite surprised when the computer revealed the numerically integrated surface of section shown in Fig. 1 for energy E = 1, and we were shocked by the surface of section shown in Fig. 2 at energy E = 256. Our incredulity only increased when similar surfaces of section were obtained at energies E = 1024 and 56,000. All these surfaces of section, which exhibited curves everywhere even at extremely large nonlinearity, clearly indicate that Hamiltonian (2) is integrable. As an additional test, we computed a formal, truncated series expression for a constant of the motion independent of Hamiltonian (2) using a well-known modification of the standard Birkhoff method,[7] and we numerically integrated the equations of motion to verify the validity of this formal constant. We then used this constant of the motion (in conjunction with Hamiltonian (2)) to analytically compute the numerically integrated curves of Fig. 1, obtaining excellent agreement. We then sought and found a host of periodic orbits for Hamiltonian (2) and established[8] that they all lay on integral surfaces of periodic orbits, again indicating integrability. Finally, we integrated a host of initially close orbit-pairs and observed that the two members of all orbit-pairs separated from each other linearly with time, a well-known characteristic of integrable systems. Lastly, we investigated the case N = 6 for Hamiltonian (1); in particular, we numerically integrated initially close orbit-pairs again obtaining in every case a separation which grew linearly with time. A typical example is shown in Fig. 3.

As we began an analytic search for Hamiltonian (1) constants of the motion, we simultaneously wrote up our computer results[4] and sent a preprint to M. Henon, among others. By return mail, Henon sent us analytic expressions[9] for N constants of the motion (in involution) thus rigorously proving integrability. Only a few days after the arrival of Henon's letter, H. Flaschka dropped by Georgia Tech, and, upon learning of Henon's results, quickly realized that he had already, without previously noticing it, derived[10] Henon's results via an application of the inverse scattering technique to the Toda lattice. Thus, independent rigorous proofs[11] of integrability for Hamiltonian (1) by Henon and Flaschka were presented in back to back Phys. Rev. B articles[9,10]. Subsequently, Kac and van Moerbeke[11] and, independently, Date and Tanaka devised an analytic technique for obtaining the general solution to the equations of motion generated by Hamiltonian (1). Hence, a computer investigation initiated a chain of remarkable achievements in pure mathematics.

III. THE HENON-HEILES HAMILTONIAN SYSTEM

If we expand the exponentials of the integrable Hamiltonian (2) in a power series and retain terms only through cubic order, we obtain the Hamiltonian

$$H = (1/2)\,(p_1^2 + p_2^2 + q_1^2 + q_2^2) + q_1^2 q_2 - q_2^3/3 \qquad (3)$$

studied much earlier by Henon and Heiles,[12] who sought to determine whether or not a third analytic integral (in addition to the Hamiltonian

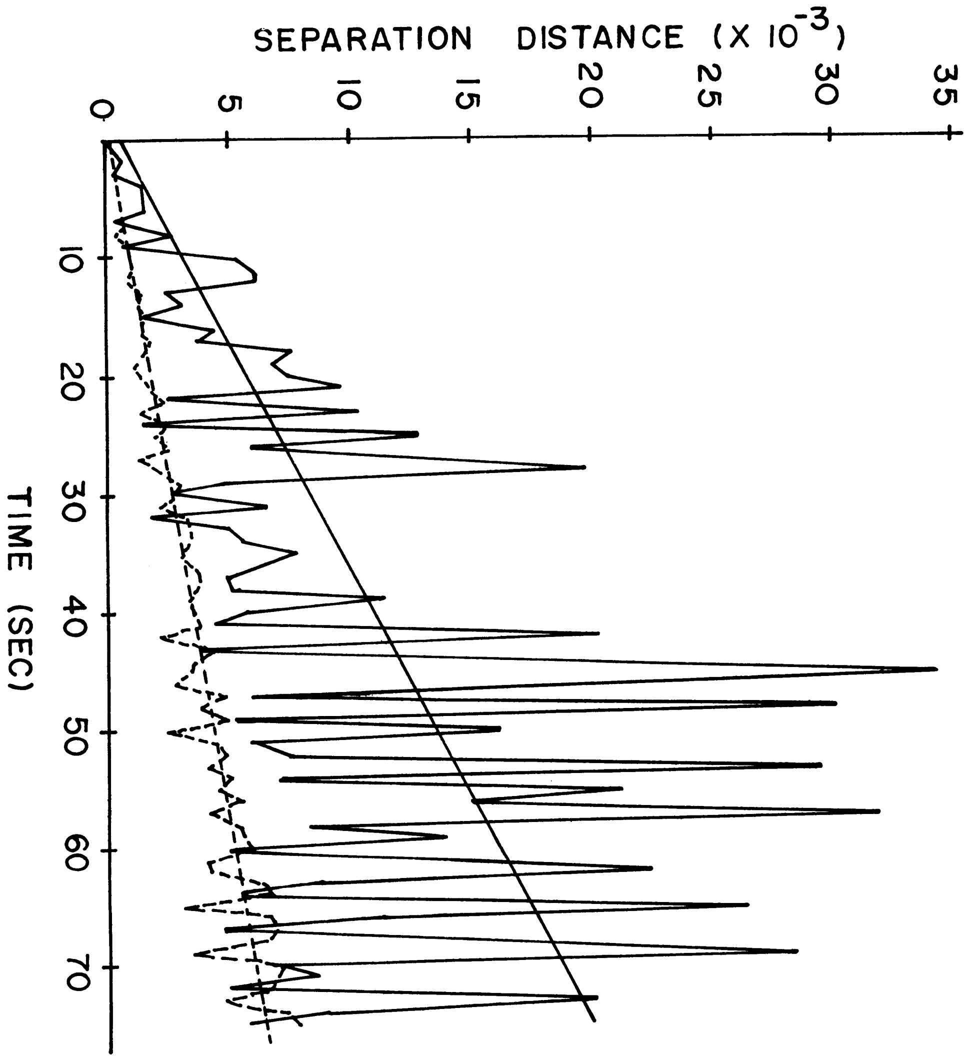

Fig. 3. A graph of separation distance versus time for an orbit-pair of Hamiltonian (1) with N = 6. Here D_q (dashed curve) is position space separation distance between two initially close orbits in phase space. D_p (solid Curve) is momentum space separation distance for the same orbit-pair. Each curve has an average linear growth (least squares straight line fit) with superimposed oscillations. Were this an ergodic, K-system, the orbits would have separated by many powers of 10 during the time interval shown.

and the z-component of angular monentum) exists for a star moving in an axially-symmetric galactic potential. As in the previous section, one here again seeks to determine whether or not Hamiltonian (3) is integrable.

In Fig. 4, we show a (q_2, p_2)- plane, Poincare surface of section for Hamiltonian (3) at energy E = 1/12. To the computer accuracy used here, curves appear to be everywhere dense, indicating integrability. However, in Fig. 5 at E = 1/8, Hamiltonian (3) is seen to have a divided phase space, part of which is still filled with integrable orbits but part contains "ergodic" trajectories. The "random" splatter of dots in Fig. 5 was generated by numerical integration of a single "ergodic" trajectory. In the integrable region, initially close orbits separate linearly with time while in the "ergodic" region they separate exponentially as is characteristic of C-systems[13], known to be ergodic and mixing. Finally in Fig. 6, at energy E = 1/6, one sees that most of phase space is filled with wild and erratic trajectories.

The Henon-Heiles work not only provides a highly unexpected answer to the original astronomical question, but it also is now regarded as a classic example[14] of the transition from integrable to stochastic behavior. It thus serves as a non-rigorous but highly illuminating link between the near-integrable behavior predicted by the rigorous KAM (Kolmogorov-Arnold-Moser) theorem[15] and the rigorous K-systems[13] behavior of Sinai's hard sphere gas.[16] The recent, as yet unpublished, work of Benettin and Strelcyn[17] provides further numerical insight into this link through an investigation of the transition from rigorous integrable behavior

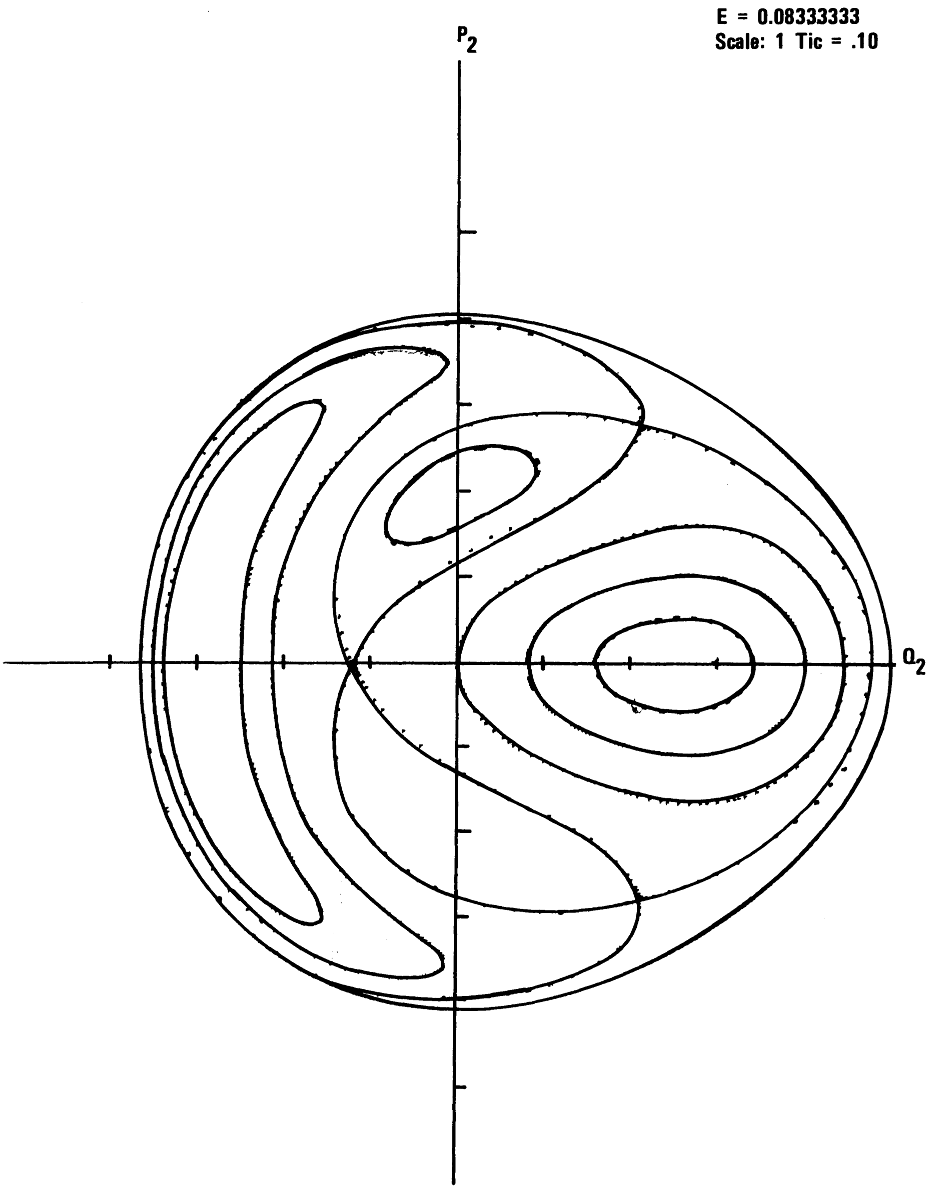

Fig. 4. A surface of section for Hamiltonian (3) at E = 1/12.

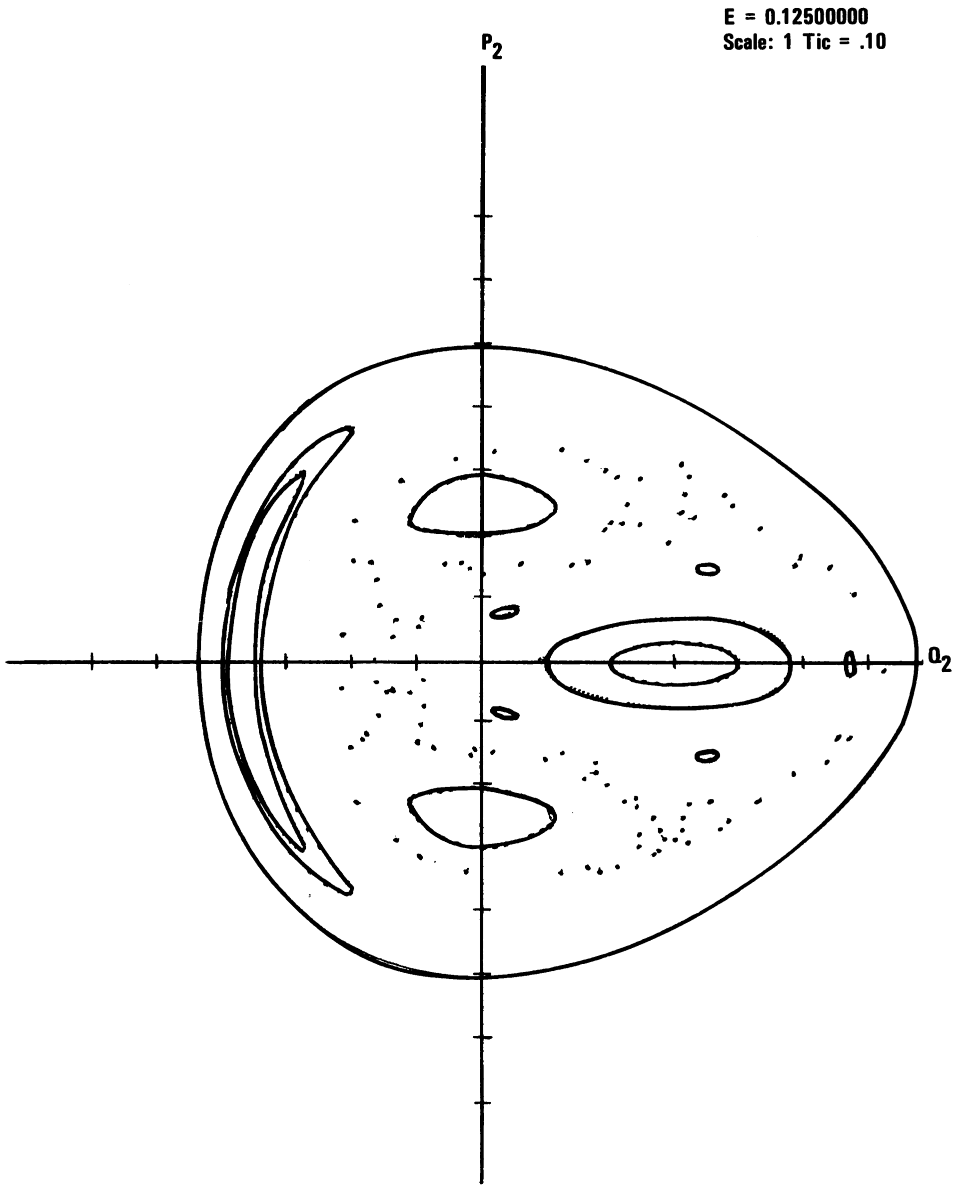

Fig. 5. A surface of section for Hamiltonian (3) at E = 1/8.

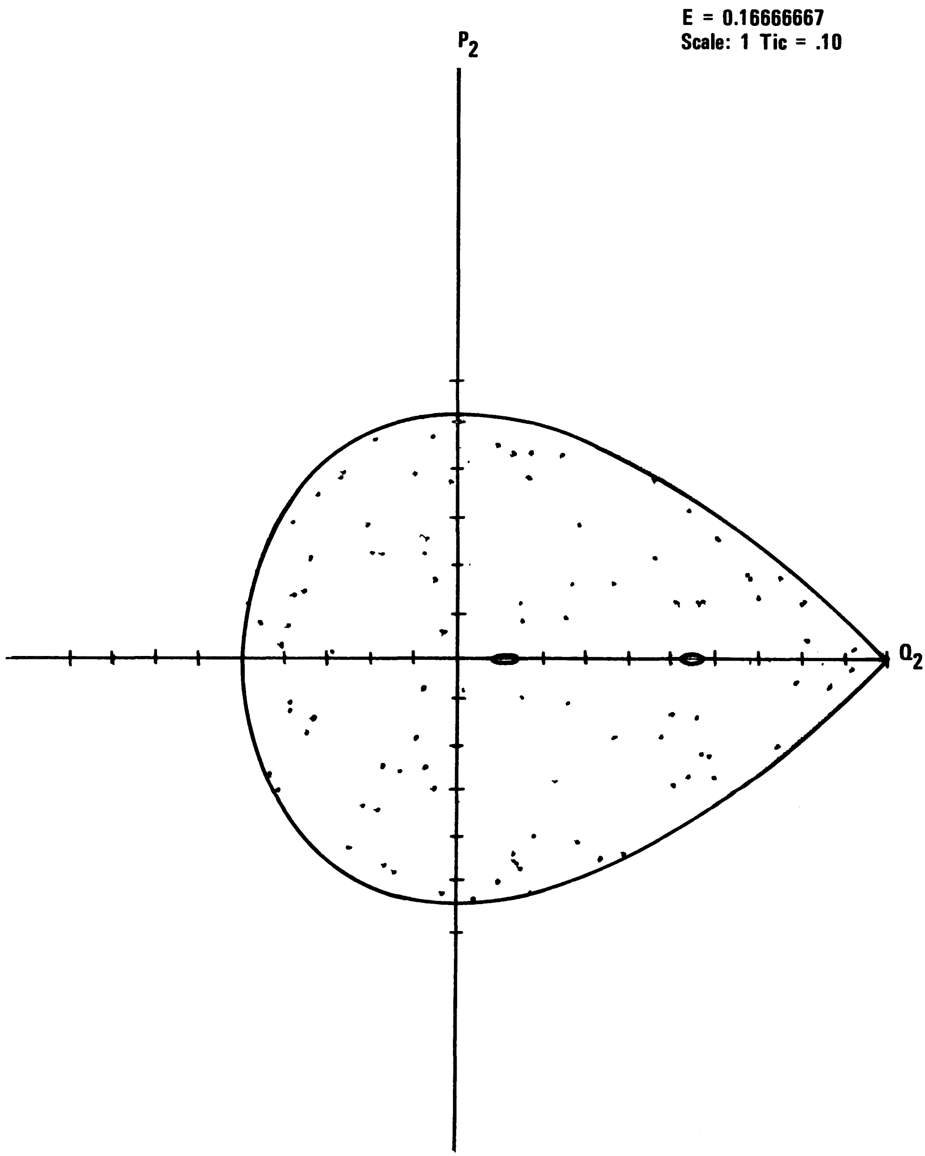

Fig. 6. A surface of section for Hamiltonian (3) at E = 1/6.

to rigorous ergodic, K-system behavior for the motion of a billard confined to a plane, oval region as the boundary changes from a circle to a stadion (two parallel straight line segments joined at their ends by two semi-circles). This paper numerically computes surfaces of section as well as Lyapunov characteristic numbers for the flow. Thence it computes the Kolmogorov-Sinai entropy for any ergodic components which may exist in the flow. In the KAM, near-integrable parameter range (nearly circular boundary) for this flow, many small, disjoint ergodic components are detected. As the boundary approaches the K-system stadion shape, these disjoint components sequentially merge into the single, completely ergodic component of the stadion. This Benettin-Strelcyn work uses a remarkable blend of pure analysis and computer experiments to obtain results of interest for both physics and pure mathematics.

IV. A MAPPING WITH A STRANGE ATTRACTOR

As an example of computer experiments on strictly chaotic systems, we describe the numerical study of a mapping possessing a strange attractor which has been recently published by M. Henon.[18] This work is relevant for weather prediction,[19] turbulence theory,[20] and pure mathematics.[21]

First using "practical" and theoretical arguments, Henon sought and found a non-area-preserving, simple, quadratic mapping which appeared a likely candidate for possessing a strange attractor. The specific mapping he chose to investigate may be written

$$x_{n+1} = y_n + 1 - a\,x_n^2, \tag{4a}$$

$$y_{n+1} = b\,x_n, \tag{4b}$$

where a and b are constants. Henon shows that this mapping is, in fact, the most general quadratic mapping having a constant Jacobian (here equal to (-b)). If this mapping possesses a strange attractor, then one can easily and accurately numerically investigate the "flow" for long "time" intervals and one, hopefully, can develop a purely analytic theory for this model since it would appear to be much simpler to treat than the earlier more complicated models.

For his numerical study, Henon set $a = 1.4$ and $b = 0.3$ for reasons which are set forth in this paper. Figure 7 shows the general appearance of the attractor using 10^4 mapping iterates of Eq. (4a, b) starting from an initial mapping point "on" the attractor. Aside from possible initial transient behavior, one obtains this same figure for any initial mapping point "trapped" by the attractor. The attractor in Fig. 7 appears to consist of a set of parallel "curves" having a certain "thickness". Upon magnification, the small square shown in Fig. 7 becomes Fig. 8, where the number of mapping iterates is now 10^5. Enlarging the square in Fig. 8 yields Fig. 9 which uses 10^6 mapping iterates. These figures strongly suggest that the attractor being graphed here is indeed a strange attractor, i.e., a product of a one-dimensional manifold by a Cantor set. Although these non-rigorous numerical results were initially greeted with skepticism by the relevant mathematical community, recent additional computer results and further (still non-rigorous) mathematical analysis have convinced all but the most devout mathematical skeptics of the significance of Henon's model. His computer study thus provides a significant model for future purely analytic study.

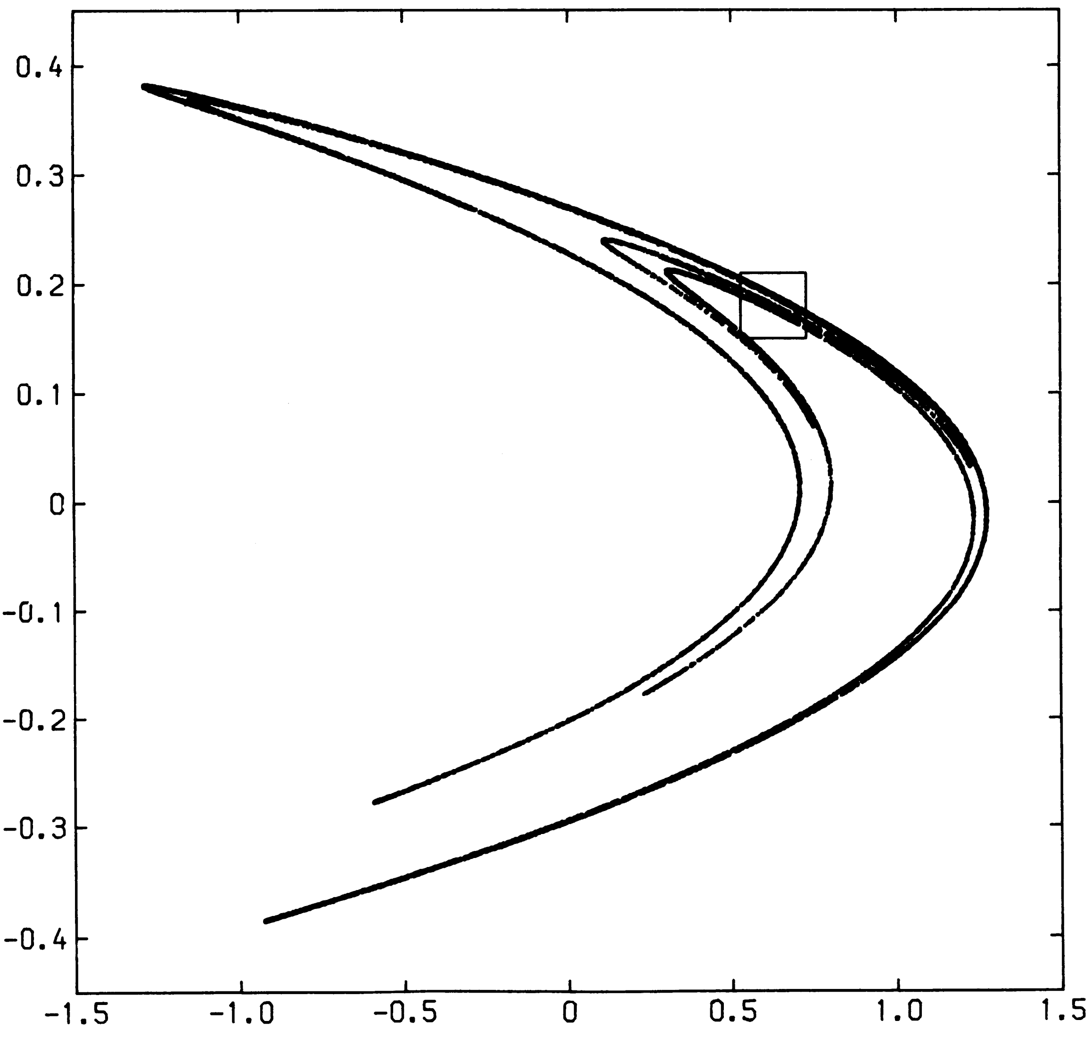

Fig. 7. 10,000 iterates of the mapping in Eq. (4a, b) starting from the point x_o = 0.63135448 and y_o = 0.18940634.

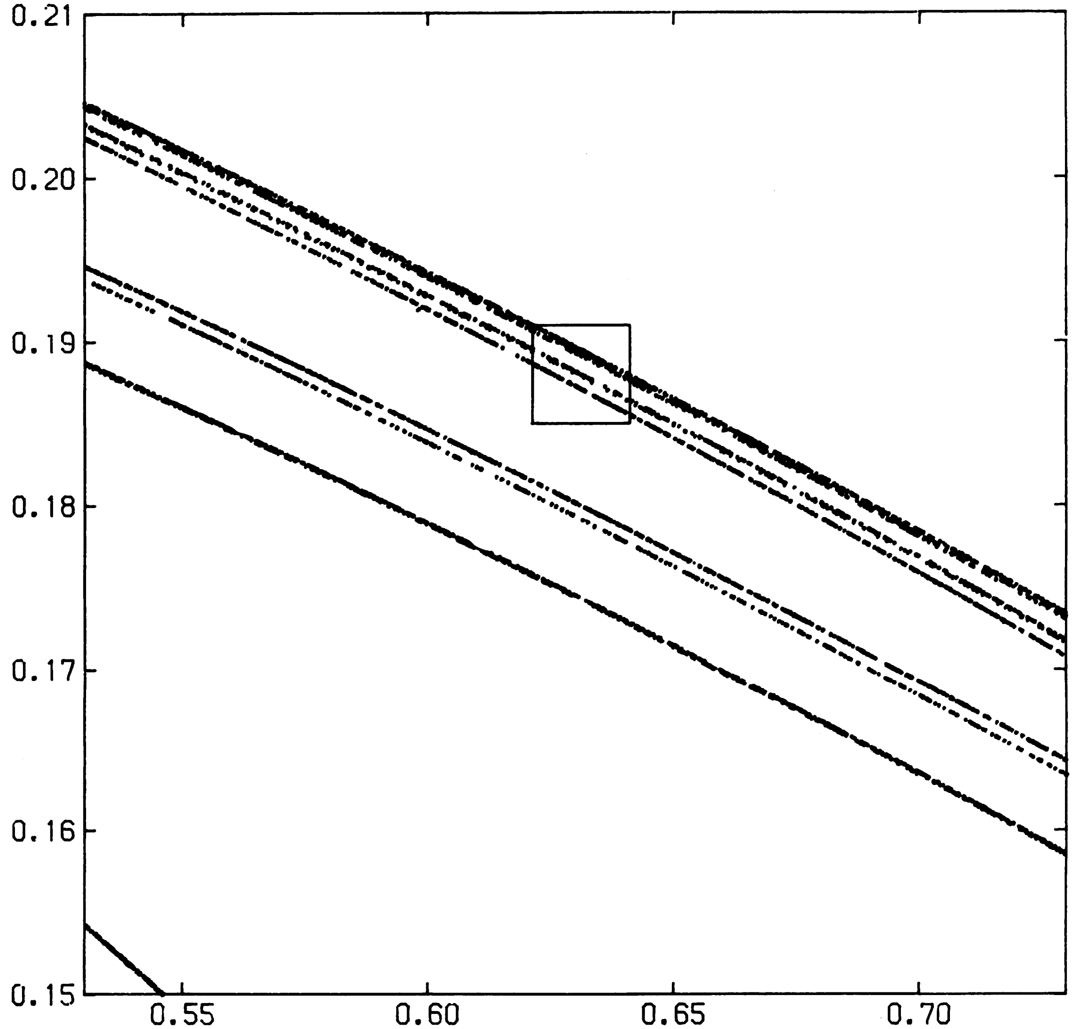

Fig. 8. A magnified view of the small square shown in Fig. 7.

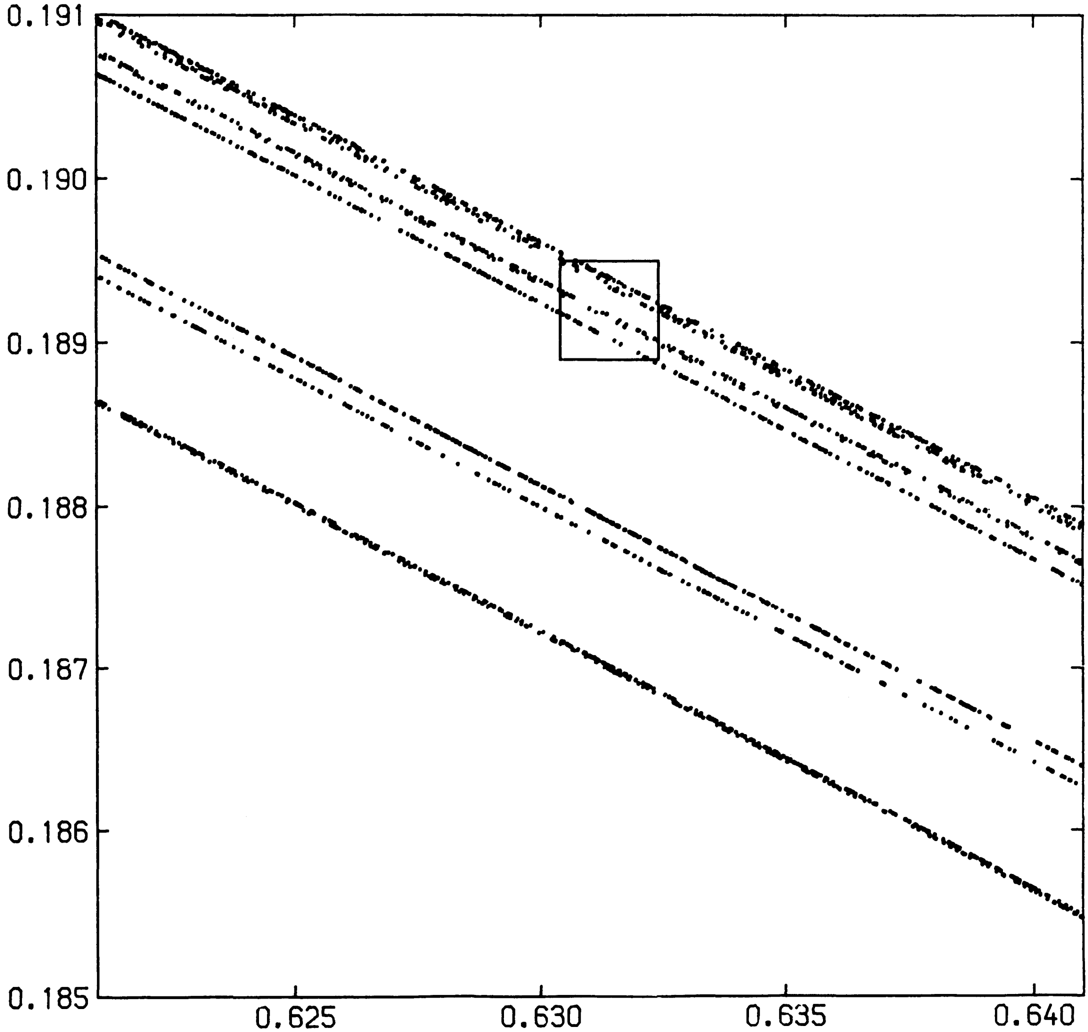

Fig. 9. An enlargement of the square shown in Fig. 8. Note the structural similarity between Figs. 8 and 9. Further enlargements reveal this same structure.

V. CONCLUDING REMARKS

In this paper we have discussed four mathematically significant computer experiments. By presenting this article in the proceedings of a conference dedicated primarily to pure mathematics, we hope to increase the flow of information between physical scientists performing numerical experiments and mathematicians developing rigorous theories. Finally, as a further device for increasing information flow, (to repeat an announcement made verbally at this conference), the present author has initiated a free, nonlinear science abstract service with the first, September, 1977, issue now available on request. This service collects, collates, and distributes a list of preprint and reprint abstracts covering work in all areas of nonlinear science giving titles, author's name and address, abstract, and proposed (or actual) publication journal. Please send your abstracts and/or requests to be placed on the mailing list to the present author at the by-line address.

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