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ORBITS OF PATHS UNDER HYPERBOLIC TORAL AUTOMORPHISMS

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In [1], M. Hirsch considers the existence of compact sets invariant under hyperbolic toral automorphisms (h.t.a.'s), and mentions the question :

Can a h.t.a. $f:T^n \to T^n$ have a compact invariant set of dimension 1 ?

J. Franks [2] went some way towards providing a negative answer when he proved that a compact f-invariant set which contains a C^2 arc must contain a coset of an invariant toral subgroup of dimension at least 2. If we impose the condition that the characteristic polynomial of f be irreducible over Z , then there are no proper invariant toral subgroups, so every C^2 arc must have a dense orbit. The following simple result shows that this is usually the case for C^0 arcs, even without the irreducibility assumption :

Proposition :

Let $f:T^n \to T^n$ be a h.t.a. . Then $\{\{\sigma:I \to T^n : O(\sigma)$ is dense$\} = D$ is a Baire set in $C(I,T^n)$.

Proof.

Let $\{U_m\}$ be a countable open base for T^n , and $D_m = \{\sigma:\sigma(I) \cap O(U_m) \neq \phi\}$.Then D_m is open since $O(U_m)$ is open, and

$O(U_m)$ is dense since f is ergodic, so D_m is also dense, and $D = \bigcap D_m$.

Against this we have :

Theorem

Let $f:T^n \to T^n$ be a h.t.a. with $u > 1$, $s > 1$, where u and s denote the respective dimensions of the unstable and stable manifolds of f. Then $\{\sigma:I \to T^n : O(\sigma)$ is not dense$\}$ is dense in $C(I,T^n)$.

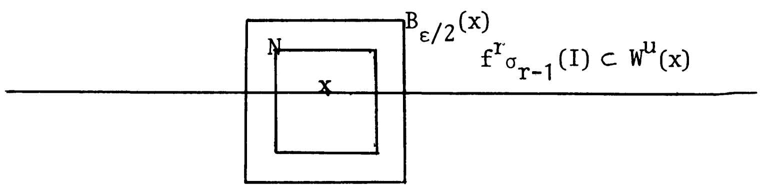
Sketch of proof.

Given $\sigma \in C(I,T^n)$ and $\varepsilon > 0$, we take $x \in T^n$ and a closed neighbourhood N of x of diameter less than ε, and first construct a sequence of paths $\sigma_{-1} = \sigma, \sigma_0, \sigma_1, \sigma_2, \ldots$ such that

1) $f^r\sigma_r(I) \cap N = \phi$ $(r \geq 0)$

2) $f^r\sigma_r(t) \in W^u_\varepsilon(f^r\sigma_{r-1}(t))$ $(t \in I,\ r \geq 0)$

Thus given σ_{r-1}, we obtain $f^r\sigma_r$ by moving each point $f^r\sigma_{r-1}(t)$ by a small amount in its own unstable manifold to a point $f^r\sigma_r(t) \notin N$. The hypothesis $u > 1$ is necessary at this stage, for suppose $W^u(x)$ were 1-dimensional and $f^r\sigma_{r-1}$ passed through N along $W^u(x)$:



It is clearly impossible to move $f^r\sigma_{r-1}$ by at most ε along $W^u(x)$ and obtain a continuous path $f^r\sigma_r$ avoiding N. With the condition $u > 1$, it is always possible to make this construction.

From 2), $d(\sigma_r,\sigma_{r-1}) \leq \alpha^r d(f^r\sigma_r,f^r\sigma_{r-1}) \leq \alpha^r \varepsilon$, where α gives the contraction of the unstable manifolds under f^{-1}, and if α is small enough (as can be ensured by taking a power of f) it follows that the sequence (σ_r) converges uniformly to a path τ , with $d(\sigma,\tau) \leq 2\varepsilon$, whose forward orbit misses a neighbourhood $N' \subset N$ of x . Now using the same method with f replaced by f^{-1} we move the path τ by an even smaller amount, say at most δ , $2\delta <$ diam N' , in the direction of the stable manifolds of f , to get a path ρ with $0^-(\rho) \cap N'' = \phi$ for some neighbourhood N'' of x , $N'' \subset N'$.

Since $\rho(t) \in W^s_\delta(\tau(t))$, the forward orbit of ρ will be within δ of that of τ and so $0^+(\rho) \cap N'' = \phi$. Thus ρ is a path within 3ε of σ and $0(\rho)$ is not dense .

Remarks:

1) If $u > 1$ and $s = 1$, the first half of the proof goes through to give a path τ with a non-dense forward orbit. If the original path σ lies in $W^u_\gamma(p)$ for a fixed point p, then $\tau(I)$ will lie in $W^u_{\gamma+2\varepsilon}(p)$ and the backward iterates of $\tau(I)$ will remain in in this set. We can thus obtain paths with non - dense orbits in this case. In particular, suppose $u = 2$, $s = 1$ and τ is such a path. Then $K = \overline{0(\tau)}$ is a compact invariant set of dimension at least one. By a result of Hirsch and R. Williams, dim $K \leq 1$, so the original question is answered.

For $n > 3$, it would be interesting to know whether the resulting invariant sets can be 1-dimensional.

2) It seems to be possible, using a similar method, to prove the theorem for maps $\sigma: I^m \to T^n$ if $m < \min\{u,s\}$.

3) The results probably go through largely unchanged if f is any Anosov diffeomorphism with $\Omega(f) = M$.

4) Using Markov partitions, we can answer Hirsch's question more directly. For, in the notation of [3], if $\mathcal{C}$ is a Markov partition for an Anosov diffeomorphism $f:M \to M$ with $\Omega(f) = M$ and $\dim M = n$, then $\overline{\mathcal{O}(\partial^s \mathcal{C} \cap \partial^u \mathcal{C})}$ is an invariant set of dimension $n-2$. This follows from Hirsch and Williams'sresult and the product structure of rectangles in $\mathcal{C}$.

References

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