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## Invariant Functionals and Polynomial Growth

by

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Recall that a locally compact group  $G$  is said to be amenable if there is an element  $p \in L^\infty(G)^*$  such that i)  $p \geq 0$ , ii)  $\langle p, \delta_g * \psi \rangle = \langle p, \psi \rangle$  for all  $g \in G$  and  $\psi \in L^\infty(G)$  (here  $\delta_g$  is the unit mass at  $g$  and  $*$  denotes convolution), and iii)  $\langle p, \chi_G \rangle = 1$ , where  $\chi_A$  denotes the characteristic function of a subset  $A \subseteq G$ . In this note we consider the situation in which the "normalization condition" iii) is replaced by iii)'  $\langle p, \theta \rangle = 1$ , where  $\theta$  is a given non zero, non negative element of  $L^\infty(G)$ . The first results on this problem were obtain by Rosenblatt [6].

We begin with the following observation : if  $G$  is not compact and  $K$  is a compact subset of  $G$  then there is an infinite sequence of elements  $\{g_i\} \subseteq G$  such that for any positive integer  $N$ ,

$$\chi_G \geq \sum_{i=1}^N \delta_{g_i} * \chi_K .$$

If  $p$  is an element of  $L^\infty(G)^*$  that satisfies i), ii), iii)' for  $\theta = \chi_K$ , then for all positive integers  $N$ ,

$$\langle p, \chi_G \rangle \geq \sum_{i=1}^N \langle p, \delta_{g_i} * \chi_K \rangle = N .$$

This contradiction shows that, in general, given a  $\theta$ , the desired functional can exist only on a proper subspace of  $L^\infty(G)$ . Hence we begin by considering the smallest translation invariant subspace of  $L^\infty(G)$  containing  $\theta$ .

Definition 1.- A locally compact group is said to be type A if for any non zero, non negative  $\theta \in L^\infty(G)$ , there is a linear functional  $p$

defined on

$$S_\theta = \{ \sum \alpha(g) \delta_g * \theta \mid \text{supp } \alpha \text{ finite} \}$$

such that i)  $p \geq 0$  , ii)  $\langle p, \delta_g * \psi \rangle = \langle p, \psi \rangle$  for  $g \in G$  ,  
 $\psi \in S_\theta$  , iii)  $\langle p, \theta \rangle = 1$  .

In order to state our first theorem we need to recall the notion of polynomial growth. Fix a left Haar measure on  $G$  and denote the measure of a subset  $A$  of  $G$  by  $|A|$  . For any positive integer  $n$  , let  $A^n = \{a_1 a_2 \dots a_n \mid a_i \in A\}$  .  $G$  is said to have polynomial growth if for every compact neighborhood of  $e$  in  $G$  ,  $U$  , there is a polynomial  $p$  such that  $|U^n| \leq p(n)$  for  $n = 1, 2, 3, \dots$  . The following theorem was first proved in [6] for discrete groups and in [3] for arbitrary groups.

Theorem 2.- A locally compact group with polynomial growth is type A .

Proof. We define a linear functional  $p$  on  $S_\theta$  by setting  $\langle p, \sum \alpha(g) \delta_g * \theta \rangle = \sum \alpha(g)$  . It is obvious that  $p$  satisfies conditions ii) and iii) of definition 1. It only remains to show that  $p$  is positive, i.e. if  $\sum \alpha(g) \delta_g * \theta \geq 0$  then  $\sum \alpha(g) \geq 0$  . For this purpose, let  $U = U^{-1}$  be a compact neighborhood of  $e$  in  $G$  containing the support of  $\alpha$  and such that  $\langle \chi_U, \theta \rangle > 0$  . Set

$$A_n = \int_{U^n} \theta(s) \, ds \quad ,$$

and note that  $0 < A_n \leq \|\theta\|_\infty |U^n|$  . Hence  $\lim(A_n)^{1/n} = 1$  . Thus there is a subsequence  $\{n_k\}$  such that  $A_{n_k-1} / A_{n_k+1} \rightarrow 1$  . Now if  $g \in \text{supp } \alpha$  then  $U^{n-1} \subseteq gU^n \subseteq U^{n+1}$  , and hence, for such  $g$  ,

$$A_{n-1} \leq \int_{gU^n} \theta(s) \, ds \leq A_{n+1} \quad .$$

Recalling that  $\sum \alpha(g) \delta_g * \theta \geq 0$  , we have

$$0 \leq \int_U n_k \sum \alpha(g) \delta_g * \theta(s) ds = \sum \alpha(g) \int_U n_k \theta(s) ds .$$

Dividing the inequality by  $A_{n_k}$  and taking the limit on  $k$  we have, since  $\text{supp } \alpha$  is finite,  $\sum \alpha(g) \geq 0$  .

There is a partial converse to theorem 2.

Theorem 3.- Suppose that  $G$  is a finitely generated, discrete, solvable group or that  $G$  is a connected group, and that  $G$  is type A . Then  $G$  has polynomial growth.

Proof. Suppose that  $G$  a finitely generated, discrete, solvable group or that  $G$  is a connected group, and that  $G$  does not have polynomial growth. Then by [4] and [6] , there exist elements  $a, b \in G$  and a compact neighborhood  $U$  of  $e$  in  $G$  such that if  $[a, b]$  denotes the semi group generated by  $a$  and  $b$  , we have i)  $a[a, b] \cap b[a, b] = \emptyset$  and ii)  $sU \cap tU = \emptyset$  if  $s, t \in [a, b]$  ,  $s \neq t$  . Let  $V = [a, b] U$  . Then

$$\begin{aligned} \text{supp } \delta_a * \chi_V \cap \text{supp } \delta_b * \chi_V &\subseteq aV \cap bV \\ &= (a[a, b] \cap b[a, b]) U = \emptyset . \end{aligned}$$

Thus  $\chi_V \geq \delta_a * \chi_V + \delta_b * \chi_V$  . Hence there can not be a functional  $p$  on  $S_{\chi_V}$  satisfying the conditions of definition 1.

$S_\theta$  is the smallest subspace of  $L^\infty(G)$  on which we might hope to find a translation invariant positive functional normalized with respect to  $\theta$  . By contrast, the largest subspace is

$$L_\theta = \{ \psi \in L^\infty(G) \mid |\psi| \leq \mu * \theta , \mu \in M(G) \} ,$$

where  $M(G)$  denotes the algebra of all bounded, regular Borel measures on  $G$  . Further, instead of asking only for invariance with respect to convolution by point masses, one could require that  $\langle p, \mu * \psi \rangle = \mu(G) \langle p, \psi \rangle$  for all  $\psi \in L_\theta$  and all  $\mu \in M(G)$  . We will construct such functionals  $p$  , inductively, obtaining at each stage a

functional invariant with respect to a larger subalgebra of  $M(G)$ . This construction also requires continuity of the functionals with respect to the following norm,  $\|\cdot\|_\theta$ , on  $L_\theta$ .

$$\|\psi\|_\theta = \inf \{ \|\mu\| \mid |\psi| \leq \mu * \theta, \mu \in M(G) \} .$$

Definition 4.- Let  $A$  be a subalgebra of  $M(G)$ .  $G$  is said to have  $A$ -invariance if for all  $0 \neq \theta \in L^\infty(G)$  for which  $0 \leq \theta \leq f * \theta$  for some  $f \in L^1(G)$ , there is a  $p \in L_\theta^*$  such that

- i)  $p \geq 0$ ,
- ii)  $\langle p, \mu * \psi \rangle = \mu(G) \langle p, \psi \rangle$ ,  $\mu \in A$ ,  $\psi \in L_\theta$ ,
- iii)  $\langle p, \mu * \psi \rangle = \int \langle p, \delta_X * \psi \rangle d\mu(X)$ ,  $\mu \in M(G)$ ,  $\psi \in L_\theta$ ,
- iv)  $\langle p, \theta \rangle = 1$ .

The following theorem and proof were inspired by Ludwig [5].

Theorem 5.- Suppose  $G$  is a connected group with polynomial growth or that  $G$  is a compact extension of a nilpotent group. Then  $G$  has  $M(G)$ -invariance.

The proof of this theorem requires the following.

Lemma 6.- Suppose  $G$  has polynomial growth and that  $G$  contains normal subgroups  $H$  and  $K$  with  $K \subseteq H$ . Suppose further that each element of  $H/K$  is contained in a compact neighborhood that is invariant under the inner-automorphisms from  $G/K$ . Then, if  $G$  has  $M(K)$ -invariance it has  $M(H)$ -invariance.

Proof. Let  $\theta$  be a non zero, non negative element of  $L^\infty(G)$  and let  $p \in L_\theta^*$  satisfying i) - iv) of definition 4 with respect to  $M(K)$ .

Let  $\dot{U}$  be a compact neighborhood of  $e$  in  $H/K$  that is invariant under inner-automorphism from  $G/K$ . Given  $\varepsilon > 0$ , we define  $f_\varepsilon, \dot{U}$  on  $H/K$  by  $f_\varepsilon, \dot{U}(\dot{x}) = (1 + \varepsilon)^{-1}$  if  $\dot{x} \in \dot{U}$ ,  $f_\varepsilon, \dot{U}(\dot{x}) = (1 + \varepsilon)^{-n}$  if  $\dot{x} \in \dot{U}^n - \dot{U}^{n-1}$ ,  $n \geq 2$ , and  $f_\varepsilon, \dot{U}(\dot{x}) = 0$  if  $\dot{x} \notin \langle \dot{U} \rangle$ , the closed

(and open) subgroup of  $H/K$  generated by  $\dot{U}$ . Since  $H/K$  has polynomial growth

$$\|f_{\varepsilon, \dot{U}}\|_1 = (1 + \varepsilon)^{-1} |\dot{U}| + \sum_{n=2}^{\infty} (1 + \varepsilon)^{-n} |\dot{U}^n - \dot{U}^{n-1}| < \infty .$$

Furthermore, for  $\dot{y} \in \dot{U}$  and  $\dot{x} \in \langle \dot{U} \rangle$

$$|f_{\varepsilon, \dot{U}}(\dot{x} \dot{y}) - f_{\varepsilon, \dot{U}}(\dot{x})| \leq \varepsilon f_{\varepsilon, \dot{U}}(\dot{x}) .$$

Since  $\theta \leq f_0 * \theta$  for some  $f_0 \in L^1(G)$ ,

$$1 = \langle p, \theta \rangle \leq \langle p, f_0 * \theta \rangle = \int_G f_0(x) \langle p, \delta_x * \theta \rangle dx .$$

Thus,  $\langle p, \delta_x * \theta \rangle > 0$  on some open subset of  $G$ . Hence, for some  $a \in G$ ,  $\langle p, \delta_{ax} * \theta \rangle > 0$  for  $x$  is some open subset of  $\langle \dot{U} \rangle$ .

Since  $\langle p, \delta_k * \psi \rangle = \langle p, \psi \rangle$  for all  $k \in K$ , we can define  $\langle p, \delta_{\dot{x}} * \psi \rangle$  for  $\dot{x} \in H/K$  by setting  $\langle p, \delta_{\dot{x}} * \psi \rangle = \langle p, \delta_x * \psi \rangle$ . We define  $p'_{\varepsilon, \dot{U}}$  on  $L_{\theta}$  by

$$\langle p'_{\varepsilon, \dot{U}}, \psi \rangle = \int_{H/K} f_{\varepsilon, \dot{U}}(\dot{x}) \langle p, \delta_{\dot{x}} * \psi \rangle d\dot{x} .$$

Fix  $\dot{b} \in H/K$  so that

$$(1 - \varepsilon)^{-1} \langle p'_{\varepsilon, \dot{U}}, \delta_{\dot{b}} * \theta \rangle \geq \sup_{\dot{y}} \langle p'_{\varepsilon, \dot{U}}, \delta_{\dot{y}} * \theta \rangle := \alpha ,$$

and define  $p_{\varepsilon, \dot{U}}$  on  $L_{\theta}$  by setting

$$\langle p_{\varepsilon, \dot{U}}, \psi \rangle = \alpha^{-1} \langle p'_{\varepsilon, \dot{U}}, \delta_{\dot{b}} * \psi \rangle ,$$

for all  $\psi \in L_{\theta}$ .

We first show that  $\|p_{\varepsilon, \dot{U}}\|_{\theta} \leq 1$ . For this, suppose that  $\psi \in L_{\theta}$  and that  $|\psi| \leq \mu * \theta$ . Then

$$\begin{aligned}
 | \langle p_\varepsilon, \dot{U}, \psi \rangle | &\leq \langle p_\varepsilon, \dot{U}, |\psi| \rangle = \alpha^{-1} \langle p'_\varepsilon, \dot{U}, \delta_b * |\psi| \rangle \\
 &= \alpha^{-1} \int_{H/K} f_\varepsilon, \dot{U}(\dot{x}) \langle p, \delta_{axb} * |\psi| \rangle d\dot{x} \\
 &\leq \alpha^{-1} \int_{H/K} f_\varepsilon, \dot{U}(\dot{x}) \langle p, \delta_{axb} * \mu * \theta \rangle d\dot{x} \\
 &= \alpha^{-1} \int_G \int_{H/K} f_\varepsilon, \dot{U}(\dot{x}) \langle p, \delta_{axbz} * \theta \rangle d\dot{x} d\mu(z) \\
 &= \alpha^{-1} \int_G \langle p'_\varepsilon, \dot{U}, \delta_{bz} * \theta \rangle d\mu(z) \\
 &\leq \int_G d\mu(z) \leq \| \mu \| .
 \end{aligned}$$

Also note that if  $\dot{y} \in \dot{U}$ ,  $\psi \in L_\theta$ , then

$$\begin{aligned}
 | \langle p_\varepsilon, \dot{U}, \delta_y * \psi \rangle - \langle p_\varepsilon, \dot{U}, \psi \rangle | \\
 &\leq \alpha^{-1} \left| \int f_\varepsilon, \dot{U}(\dot{x}) \langle p, \delta_{axby} * \psi - \delta_{axb} * \psi \rangle d\dot{x} \right| \\
 &= \alpha^{-1} \int | f_\varepsilon, \dot{U}(\dot{x}b\dot{y}^{-1}b^{-1}) - f_\varepsilon, \dot{U}(\dot{x}) | \langle p, \delta_{axb} * \psi \rangle d\dot{x} \\
 &\leq \varepsilon \alpha^{-1} \int f_\varepsilon, \dot{U}(x) \langle p, \delta_{axb} * \psi \rangle d\dot{x} \\
 &= \varepsilon \langle p_\varepsilon, \dot{U}, \psi \rangle .
 \end{aligned}$$

Choose a sequence of compact neighborhoods of  $e$  in  $H/K$ ,  $\dot{U}_n$ , that are invariant under the inner-automorphisms from  $G/K$  and such that  $\bigcup_{n=1}^{\infty} \dot{U}_n = H/K$ . For each positive integer  $n$ , let  $p_n = p_{1/n}, \dot{U}_n$ . By  $\omega^*$ -compactness of the unit ball in  $L_\theta^*$ , the sequence  $\{p_n\}$  has a cluster point  $p_\infty$ . Clearly  $p_\infty \geq 0$  and, since

$$\begin{aligned}
 1 &\geq \langle p_n, \theta \rangle = \langle p_{1/n}, \dot{U}_n, \theta \rangle \\
 &= \alpha^{-1} \langle p'_{1/n}, \dot{U}_n, \delta_{a_n} * \theta \rangle \\
 &\geq 1 - 1/n ,
 \end{aligned}$$

$\langle p_\infty, \theta \rangle = 1$  . It is also clear that for  $k \in K$  and  $\psi \in L_\theta$  ,  
 $\langle p_\infty, \delta_k * \psi \rangle = \langle p_\infty, \psi \rangle$  and hence that  $\langle p_\infty, \delta_{\hat{y}} * \psi \rangle = \langle p_\infty, \delta_y * \psi \rangle$   
 $= \langle p_\infty, \psi \rangle$  for  $\hat{y} \in H/K$  .

If  $\psi \in L_\theta$  and  $\mu \in M(G)$  , then

$$\begin{aligned} \langle p_\infty, \mu * \psi \rangle &= \lim \langle p_{1/n_\alpha}, \dot{U}_{n_\alpha}, \mu * \psi \rangle \\ &= \lim \int_G \langle p_{1/n_\alpha}, \dot{U}_{n_\alpha}, \delta_x * \psi \rangle d\mu(x) . \end{aligned}$$

Since

$$|\langle p_{1/n}, \dot{U}_n, \delta_x * \psi \rangle| \leq \|\psi\|_\theta ,$$

the dominated convergence theorem gives

$$\langle p_\infty, \mu * \psi \rangle = \int_G \langle p_\infty, \delta_x * \psi \rangle d\mu(x) .$$

Finally, note that if  $\mu \in M(H)$  ,  $\psi \in L_\theta$  ,

$$\begin{aligned} \langle p_\infty, \mu * \psi \rangle &= \int_H \langle p_\infty, \delta_x * \psi \rangle d\mu(x) \\ &= \mu(G) \langle p_\infty, \psi \rangle . \end{aligned}$$

Therefore  $p_\infty$  satisfies i) - iv) of definition 4.

Proof of theorem 5. If  $N$  is a normal, nilpotent subgroup of  $G$  such that  $G/N$  is compact and if  $N = N_1 \supset \dots \supset N_k = \{e\}$  is the lower center series for  $N$  , then for each  $i = 1, \dots, k-1$  ,  $N_i/N_{i+1}$  has large compact neighborhoods invariant under the inner-automorphisms form  $G/N_{i+1}$  . If  $G$  is a connected Lie group with polynomial growth then the eigenvalues of  $\text{Ad}_s$  are of modulus 1 for all  $s \in G$  (see [4] or [1] , where the following fact was first pointed out). Let  $S$  be the solu-radical of  $G$  ,  $LS$  its Lie algebra and  $LS_{\mathbb{C}}$  , the complexification of  $LS$  . By Lie's theorem, there is an ordered basis for  $LS_{\mathbb{C}}$  ,  $\{X_1, \dots, X_m\}$  so that the matrix



representation for Ads with respect to this basis is upper triangular for all  $s \in S$ . Let  $V_j$  be the subspace spanned by  $\{X_1, \dots, X_j\}$ . Then the action of Ads on  $V_j/V_{j-1}$  is multiplication by  $\alpha_j(s)$  where  $|\alpha_j(s)| = 1$ , i.e. Ads acts by rotation on each space  $V_j/V_{j-1}$ . Thus we can find subspaces  $\{0\} = W_0 \subset W_1 \subset \dots \subset W_n$  of LS, each invariant under Ads with  $\dim(V_j / V_{j-1}) \leq 2$ , and with Ads acting by rotation on each space  $W_j / W_{j-1}$ ,  $j = 1, 2, \dots, n$ . Thus  $W_j$  is an ideal in LS and if  $S_j$  is the corresponding closed normal subgroup of  $S$ , each element of  $S_j / S_{j-1}$  has large compact neighborhoods invariant under the inner-automorphisms from  $S / S_{j-1}$ , and also  $G/S$ , since,  $G/S$  is compact.

If  $G$  is a connected group with polynomial growth, then there is a compact, normal subgroup  $K$  of  $G$  such that  $G/K$  is a Lie group with polynomial growth. The above argument applied to  $G/K$  produced the desired series of subgroups.

In order to show that polynomial growth is not sufficient to imply  $M(G)$ -invariance, as is the case with being type A, we prove the following.

Theorem 7.- Let  $A$  be a subalgebra of  $M(G)$  that contains the point masses and is closed with respect to involution. If  $G$  has  $A$ -invariance then for each  $f \in A \cap L^1(G)$ ,  $-1 \notin \text{sp}(f * f^*)$ .

Proof. Assume there is an  $f \in A \cap L^1(G)$  such that  $-1 \in \text{sp}(f * f^*)$ . Then for all  $g \in L^1(G)$ ,  $\|g * f * f^* + g + f * f^*\| \geq 1$ , for other wise,

$$[(g-1)(-f * f^* - 1)]^{-1} = \{1 + [(g-1)(-f * f^* - 1) - 1]\}^{-1}$$

exists in  $\mathbb{C} 1 \oplus L^1(G)$ , contradicting the assumption that  $-1 \in \text{sp}(f * f^*)$ .

Hence, there is a  $\varphi \in L^\infty(G)$  such that  $\langle g * f * f^* + g, \varphi \rangle = 0$  for all  $g \in L^1(G)$  and  $\langle f * f^*, \varphi \rangle = 1$ . Thus  $f * f^* * \varphi = -\varphi$ .

Let  $\theta = \|\varphi\|^2$ . Note that for  $h \in L^1(G)$

$$\begin{aligned}
 |h * \varphi|^2(s) &= \left| \int h(t) \varphi(t^{-1}s) dt \right|^2 \\
 &\leq \left[ \int |h(t)|^{1/2} |h(t)|^{1/2} |\varphi(t^{-1}s)| dt \right]^2 \\
 &\leq \int |h(t)| dt \int |h(t)| |\varphi(t^{-1}s)|^2 dt \\
 &= \|h\|_1 |h| * |\varphi|^2 .
 \end{aligned}$$

Hence,  $\theta = |\varphi|^2 = |f * f^* * \varphi|^2 \leq \|f * f^*\| |f * f^*| * \theta$  .

Thus, there is a functional  $p$  on  $h_\theta$  satisfying i) - iv) of definition 4 with respect to  $A$  . Now, if  $g, h \in L^1(G)$  ,

$$(g * \varphi) \overline{(h * \varphi)} \leq \frac{1}{2} \{ |g * \varphi|^2 + |h * \varphi|^2 \} \in L_\theta .$$

We define a bilinear form  $B$  on  $A \cap L^1(G)$  by

$$B(g, h) = \langle p, (g * \varphi) \overline{(h * \varphi)} \rangle .$$

Clearly  $B(g, g) \geq 0$  for  $g \in A \cap L^1(G)$  and  $B(f * f^*, f * f^*) = \langle p, |\varphi|^2 \rangle = 1$  . Also, for  $g \in A \cap L^1(G)$  ,

$$B(g, g) = \langle p, |g * \varphi|^2 \rangle \leq \|g\|_1 \langle p, |g| * \theta \rangle = \|g\|_1^2 .$$

Thus  $B$  is bounded, and if  $g, h \in A \cap L^1(G)$  ,  $s \in G$  ,

$$B(\delta_s * g, \delta_s * h) = \langle p, \delta_s * [(g * \varphi) \overline{(h * \varphi)}] \rangle = B(g, h) .$$

Hence, for  $g, h, k \in A \cap L^1(G)$

$$B(g * h, k) = B(h, g^* * k) .$$

Recalling that  $f * f^* * \varphi = -\varphi$  , we have

$$0 \leq B(f^*, f^*) = -B(f^* * f * f^*, f^*) = -B(f * f^*, f * f^*) = -1 .$$

This contradiction implies that  $-1 \notin \text{sp}(f * f^*)$  .

A Banach  $*$ -algebra  $A$  is said to be symmetric if  $-1 \notin \text{sp}(aa^*)$  for all  $a \in A$ . Theorems 5 et 7 show that  $L^1(G)$  is symmetric if  $G$  is a connected group with polynomial growth or if  $G$  is a compact extension of a nilpotent group. This was first proved by Ludwig [5].

Hulanicki [2], considered the following group: let  $D$  be a countable direct sum of  $\mathbb{Z}_2$ , and let

$$\underline{D} = \prod_{d \in D} (D)_d = \{(X_d)_{d \in D} \mid X_d \in D\}$$

be the direct product of  $D$  copies of  $D$ . Define  $\tau: D \rightarrow \text{Aut}(\underline{D})$  by  $\tau(c)(X_d)_{d \in D} = (X_{d+c})_{d \in D}$ . Then  $D \rtimes_{\tau} \underline{D}$  is a solvable group and for each  $s \in D \rtimes_{\tau} \underline{D}$ ,  $s^4 = e$ . Hulanicki showed that  $\ell^1(D \rtimes_{\tau} \underline{D})$  is not symmetric, hence  $D \rtimes_{\tau} \underline{D}$  is a solvable group with polynomial growth that does not have  $M(D \rtimes_{\tau} \underline{D})$ -invariance. Hence even for solvable groups being type A is a weaker condition than having  $M(G)$ -invariance.

Let  $F(G)$  denote the subalgebra of  $M(G)$  consisting of all measures with finite support.

Using the Krein extension theorem, one can easily see that  $G$  is type A if and only if  $G$  has  $F(G)$ -invariance. Hence polynomial growth implies  $F(G)$ -invariance. Using the same techniques as in the proof of theorem 2, one can show that polynomial growth implies  $K(G)$ -invariance where  $K(G)$  denotes the measures with compact support, or even invariance with respect to algebras of measures that vanish sufficiently rapidly at infinity. (In Hulanicki's example, for instance, one selects a sequence of subsets  $U_n \subset U_{n+1}$  such that  $\cup U_n = D \rtimes_{\tau} \underline{D}$  and requires the measure  $\mu$  to satisfy

$$\lim_n |\langle u_n, \mu \rangle| = 0 \quad (D \rtimes_{\tau} \underline{D} \curvearrowright \langle u_n \rangle = 0).$$

It is not known how large a subalgebra will give invariance for arbitrary groups with polynomial growth.

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