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EXISTENCE AND UNIQUENESS OF NONLINEAR REALIZATIONS

by

Bronislaw JAKUBCZYK

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1. - INTRODUCTION.

The realization problem can be formulated as follows. Given an input-output system ("black box") described by an input-output map, find an internal description of the system (called realization of the system) and show that a "minimal internal description" is, in a sense, unique. In the case of discrete time and a general (set theoretical) input-output map, this is a problem of the automata theory (cf. [6]) and is solved by introducing the concept of "state space" as a "minimal memory" of the system.

We are concerned with the case of continuous-time systems with the output having finite number of real valued components. The problem has a satisfactory solution for the case of linear systems (cf. KALMAN [6] and the bibliography cited there) and bilinear systems (cf. e.g. [3]). In the case of the input-output map given by a finite Volterra series direct constructions of linear-analytic realizations were given by BROCKETT [1] and CROUCH [2] (see these proceedings).

In the general, nonlinear case a basic result was obtained by SUSSMANN [8], [10] (for related topics see SUSSMANN [9], [11] and HERMANN, KERNER [4]), who proved that if an input-output map has a realization which is either analytical or smooth symmetric, then it has a minimal realization which is unique up to a diffeomorphism.

Here, we give general necessary and sufficient conditions for existence of realizations of nonlinear input-output maps. We show that two minimal realizations are diffeomorphic (our definition of minimality is slightly modified with respect to [10]). We outline the construction of a realization in the general case. The

detailed construction and proofs are contained in [5].

2. - INPUT-OUTPUT MAP OF A GIVEN CONTROL SYSTEM.

Consider a control system of the form

$$(1) \quad \begin{aligned} \dot{x} &= f(x, u), \quad x(0) = x_0 \\ y &= h(x), \end{aligned}$$

where $x(t) \in X$ n -dimensional, differentiable manifold, $u(t) \in \Omega$ and $y(t) \in \mathbb{R}^r$. Let U be a class of admissible control functions u defined on finite subintervals $[0, t_u)$ of $\mathbb{R}_+ = [0, \infty)$. We assume that for any $u \in U$ the equation $\dot{x} = f(x, u)$ has a well defined unique solution on $[0, t_u]$. Let Φ_u^f denote the diffeomorphism $\Phi_u^f : X \longrightarrow X$ which maps initial points of the trajectories into their finite points. For a given quadruple $\Sigma = (X, f, h, x_0)$ we define the input-output map $p_\Sigma : U \longrightarrow \mathbb{R}^r$ of system (1) by

$$(2) \quad p_\Sigma(u) = h(\Phi_u^f(x_0)).$$

3. - CONTROL SEMIGROUP AND CONTROL GROUP.

For the sake of simplicity, we shall consider here the class of piecewise constant controls only (see [5] for a general class of controls). Let Ω denote a set of admissible values of controls (its elements will be denoted by α, β).

Denote by

$$(3) \quad a = (t_k \alpha_k) \dots (t_2 \alpha_2) (t_1 \alpha_1)$$

the piecewise constant function $[0, \sigma_k) \longrightarrow \Omega$, $a(\tau) = \alpha_i$ for $\tau \in [\sigma_{i-1}, \sigma_i)$, $\sigma_i = \sum_{j=1}^i t_j$ ($\sigma_0 = 0$), where $t_i \in \mathbb{R}_+ = [0, \infty)$ and $k \geq 0$. The set of all such functions will be denoted by S and its elements by a, b, c . There is a natural semigroup structure in S with multiplication defined by concatenation

$$(4) \quad ba = (\tau_m \beta_m) \dots (\tau_1 \beta_1) (t_k \alpha_k) \dots (t_1 \alpha_1)$$

where $b = (\tau_m \beta_m) \dots (\tau_1 \beta_1)$. The identity in S is the empty sequence (3).

There is a natural action of R_+ on S

$$(5) \quad ta = ((t t_k) \alpha_k) \dots ((t t_1) \alpha_1)$$

(expansion). We identify $\alpha = (1\alpha)$.

The semigroup S can be extended to a group G called control group (see LOBRY [7]). The elements of G are formal sequences of the form (3) with $t_i \in R$ and multiplication defined by (4), where we identify $(t_1\alpha)(t_2\alpha) = (t_1+t_2)\alpha$ and $(0\alpha) = e$. The element ta is defined by (5) for $t > 0$ and by $ta = ((t t_1)\alpha_1) \dots ((t t_k)\alpha_k)$ for $t < 0$.

4. - INPUT-OUTPUT SYSTEMS.

Assume that R^r is our output space. Any mapping $p: S \longrightarrow R^r$ will be called an input-output map. By an input-output system we shall mean the triple (S, p, R^r) . To have existence of realizations we shall impose two basic assumptions on the input-output map p (they have parallel versions if p is defined on the group G).

Denote $\underline{b} = (b_1, \dots, b_m)$, $b_i \in S$ ($b_i \in G$), $m \geq 1$, $\underline{a} = (a_1, \dots, a_q)$ $a_i \in S$ ($a_i \in G$), $q \geq 1$, $\underline{t} = (t_1, \dots, t_q)$, $t_i \in R_+$ ($t_i \in R$) and define the functions :

$$\psi_{\underline{a}}^{\underline{b}} : R_+^q \longrightarrow R^{rm} \quad (R^q \longrightarrow R^{rm}) \quad \text{by} \quad \psi_{\underline{a}}^{\underline{b}} = (\psi_{\underline{a}}^{b_1}, \dots, \psi_{\underline{a}}^{b_m}), \text{ where}$$

$$\psi_{\underline{a}}^{b_i}(\underline{t}) = p(b_i(t_q a_q) \dots (t_1 a_1)).$$

It may be useful to imagine $(t_q a_q) \dots (t_1 a_1)$ as a basic control and b_i , $i = 1, \dots, m$, as measure experiments.

Let $k = 2, 3, \dots, \infty, \omega$. The regularity assumption on p takes the form

(A.1) The functions $\psi_{\underline{a}}^{\underline{b}}$ belong to the class C^k for any $\underline{a}, \underline{b}, m \geq 1, q \geq 1$.
In the case of $k = \omega$ and p defined on the semigroup S we shall also need a stronger version of (A.1).

(A.1)' The functions have real analytic extensions onto R^q for any $\underline{a}, \underline{b}, m \geq 1, q \geq 1$.

Define

$$\text{rank } p = \sup_{\underline{a}, \underline{b}, \underline{t}} \text{rank } D\psi_{\underline{a}}^{\underline{b}}(\underline{t})$$

where $D\psi$ denotes the differential of ψ . We shall also assume that

$$(A.2) \quad \text{rank } p < \infty .$$

In the nonanalytical case the following additional assumption will be used

$$(A.3) \quad \forall a \in \Omega \quad \exists \alpha \in \Omega \quad \forall a, b \in S \quad \forall t > 0 \quad p(b(t\beta)(t\alpha)a) = p(ba) = p(b(t\alpha)(t\beta)a) .$$

5. - REALIZATIONS.

Now we shall precise what we mean by realizations of the input-output system (S, p, R^r) . The quadruple $\Sigma = (X, f, h, x_0)$ will be called a C^k realization of the input-output system (S, p, R^r) , $k = 2, 3, \dots, \infty, \omega$, if

- (i) X is a C^k manifold (Hausdorff, without boundary) and $x_0 \in X$,
- (ii) $f : X \times \Omega \longrightarrow TX$ is a function such that the vector fields $f(\cdot, \alpha)$ are complete and generate C^k flows $\Phi_{(t\alpha)}^f$,
- (iii) $h : X \longrightarrow R^r$ is a function of the class C^k ,
- (iv) the input-output map p_Σ is equal to p i.e.

$$p(a) = h(\Phi_a^f(x_0)), \quad a \in S .$$

The realization is called reachable (weakly reachable) if $\forall x \in X \quad \exists a \in S \quad (a \in G) \quad \Phi_a^f(x_0) = x$ (for $a = (t_k \alpha_k) \dots (t_1 \alpha_1) \in G$ we define $\Phi_a^f = \Phi_{(t_k \alpha_k)}^f \circ \dots \circ \Phi_{(t_1 \alpha_1)}^f$).

It is called observable if $\forall x_1, x_2 \in X, \quad x_1 \neq x_2 \quad \exists b \in S \quad h(\Phi_b^f(x_1)) \neq h(\Phi_b^f(x_2))$. A reachable and observable realization is called minimal. Weakly reachable and observable realization is called C^ω -minimal (minimal in the class C^ω). The realization is called symmetric if $\forall \alpha \in \Omega \quad \exists \beta \in \Omega \quad \forall x \in X \quad f(x, \alpha) = -f(x, \beta)$.

Two C^k realizations Σ and Σ' are said to be C^k -diffeomorphic if there is a C^k diffeomorphism $\chi : X \longrightarrow X'$ which carries Σ to Σ' i.e.

$$(D\chi f) \circ \chi^{-1} = f', \quad h \circ \chi^{-1} = h', \quad \chi(x_0) = x'_0 .$$

6. - THE MAIN RESULT.

The following theorem gives general conditions for existence and uniqueness of realizations of the system (S, p, R^r) .

Theorem 1. - Let $k = 2, 3, \dots, \infty, \omega$. The input-output system (S, p, R^r) has a C^k realization if and only if the map p can be extended to a map $\bar{p} : G \rightarrow R^r$ which satisfies (A.1) and (A.2).

Any two C^k , $k = 2, \dots, \infty$, minimal (C^ω -minimal) realizations of (S, p, R^r) are C^k (C^ω) diffeomorphic.

The existence criterion of the above theorem is somewhat implicit. However the criterion can be transformed to an explicit form for two important classes of realizations.

Theorem 2. - a) $k = 2, \dots, \infty$. Any input-output system, which satisfies (A.1), (A.2) and (A.3) has a minimal, symmetric, C^k realization Σ such that $\dim X = \text{rank } p$.

b) $k = \omega$. Any input-output system which satisfies (A.1)' and (A.2) has a C^ω -minimal realization Σ such that $\dim X = \text{rank } p$.

Theorems 1, 2 are reformulations of the results of [5] (extended version). Namely, the existence part of Theorem 1 is contained in Theorem 4 of [5] and the uniqueness part of Theorem 1 in [5]. The full proofs are contained in [5]. Below we shall outline the proof of the existence part of Theorem 1.

7. - NECESSITY.

If there exists a realization Σ , then the extension $\bar{p} : G \rightarrow R^r$ can be defined by

$$\bar{p}(a) = h(\Phi_a^f(x_0)), \quad a \in G.$$

We define the following maps $\psi_a : R^q \rightarrow X$, $\psi_b : X \rightarrow R^{rm}$,

$$\psi_a(\underline{t}) = \Phi_{(t_q a_q) \dots (t_1 a_1)}^f(x_0)$$

$$\psi_b(x) = (h(\Phi_{b_1}^f(x)), \dots, h(\Phi_{b_m}^f(x))).$$

We have that $\psi_{\underline{a}}^{\underline{b}} = \psi_{\underline{a}}^{\underline{b}} \circ \psi_{\underline{a}}$, thus it is easy to see that (A.1) and (A.2) are satisfied and $\text{rank} \psi_{\underline{a}}^{\underline{b}} \leq \dim X$.

8. - CONSTRUCTION OF A REALIZATION (OUTLINE).

For a given map $\bar{p} : G \rightarrow R^r$ we introduce an equivalence relation in G

$$a \sim b \Leftrightarrow \forall c \bar{p}(ca) = \bar{p}(cb).$$

We define

$$X = G/\sim$$

and $[a]$ denotes the equivalence class of a . Define the maps $\bar{\psi}_{\underline{a}} : X \rightarrow X$, $a \in G$, and $h : X \rightarrow R^r$ by

$$\bar{\psi}_{\underline{a}}([b]) = [ab], h([b]) = \bar{p}(b)$$

and let $x_0 = [e]$.

The topology in X is defined as the strongest topology such that the maps $\psi_{\underline{a}} : R^q \rightarrow X$ are continuous for all $\underline{a} = (a_1, \dots, a_q)$, $q \geq 1$, where

$$\psi_{\underline{a}}(\underline{t}) = \bar{\psi}_{(t_q a_q)} \dots \bar{\psi}_{(t_1 a_1)}(x_0).$$

The C^k differential structure on X is introduced by defining the class of real valued functions of the class C^k on X :

$$\varphi \in C^k(X, R) \Leftrightarrow \varphi \circ \psi_{\underline{a}} \in C^k(R^q, R) \quad \forall \underline{a}.$$

Using (A.1) and (A.2) it can be proved that X is C^k , finite dimensional manifold and $\bar{\psi}_{\underline{a}}, h$ are functions of the class C^k . The vector fields $f(\cdot, \alpha)$ are defined as infinitesimal vector fields of the flows $\bar{\psi}_{(t\alpha)}$.

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