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ASYMPTOTIC EVOLUTION OF A STOCHASTIC CONTROL PROBLEM WHEN THE DISCOUNT VANISHES

by

R. TARRES

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Here we are studying the minimization problem of an integral discounted funtional, on a set of non explosive and non constrained diffusions ; the running cost is "weakly coercive", which leads us, using the dynamic programming method, to characterize the optimal cost among the solutions of the solving equation, with radiative conditions expressing the centripetal aspect of the optimal control. The behaviour of the problem when the discount vanishes is then considered ; a limit problem is defined and similarly studied ; the convergence results are analogous to the ones obtained by J.M. Lasry in the case of a periodical running cost [8]. For details and others results, see [10] and [11].

I.- STATEMENT OF THE PROBLEMS.

Let's consider the stochastic differential equation of Ito type

(1) $$\xi(0) = x \ , \quad d\xi(t) = p(\xi(t))dt + \rho dw(t) \ ,$$

where :

- . w is a normalized brownian motion on the real line $\mathbb{R}$
- . $\rho > 0$ is a constant ($\rho = \sqrt{2}$ in order to simplify formulas)
- . $p \in \Lambda$ (Λ is the control set of our problem), namely p is a function from $\mathbb{R}$ into itself satisfying the following properties
  - . p is weakly growing (this means that there exist positive constants b_p and m_p such that $\forall u \in \mathbb{R} \ |p(u)| \leq b_p(1+|u|^{m_p}))$
  - . p is lipschitzian on every bounded interval
  - . there exists $c_p \in \mathbb{R}_+$ such that $\forall u \in \mathbb{R}^*, \frac{u}{|u|} p(u) \leq c_p(1+|u|)$.

We know that, for each initial state $x \in \mathbb{R}$ and for each control $p \in \Lambda$, this equation (1) has a solution $\xi_{x,p}$ defined on $[0, +\infty[$; this solution is unique (pathwise uniqueness on each interval $[0, T]$), and is a non explosive diffusion process with diffusion coefficient $\rho^2 = 2$ and drift coefficient p (see [1], [4], [5], [9]).

For each constant s $(s > 0)$, $x \in \mathbb{R}$ and $p \in \Lambda$, the relation

$$(2) \qquad J_s(x,p) = E \int_0^{+\infty} e^{-st} [g(\xi_{x,p}(t)) + f(p(\xi_{x,p}(t)))] \, dt$$

defines the discounted cost of our problem ; the functions $f \in C^2(\mathbb{R} ; \mathbb{R}_+)$ and $g \in C^1(\mathbb{R} ; \mathbb{R}_+)$ are given.

We are interested in the following two problems

- the problem $(P_{x,s})$: to minimize $J_s(x,p)$, for $p \in \Lambda$
- what is the behaviour of $(P_{x,s})$ when the discount s vanishes ?

Remarks.- The controls are closed-loop deterministic ones.

- $(P_{x,s})$ is a stationary problem
- Constraint is imposed neither to the controls nor to the trajectories of the controlled diffusion : the controls are only non explosive ones, and the processes $\xi_{x,p}$ evolve on the non bounded set $\mathbb{R}$.
- In 1975, J.M. Lasry (see [8]) studied the problems above and obtained interesting convergence results, in n-dimensional case, without constraints on the controls, but with the restriction that g was a periodic function, or that the diffusion $\xi_{x,p}$ evolves on a bounded set of $\mathbb{R}^n$, with reflection at the boundary. And he thought that, replacing such hypotheses by sufficiently strong coercivity hypotheses on the functions g and f in the running cost, it should be possible to obtain similar convergence results ; the idea of this conjecture is the following one : with such coercivity hypotheses, the optimal control is naturally quite centripetal, that is to say it tends first to bring back the evolution of the process in the region where g takes small values, and secondly, to take in this region values for which f is not too large ; so that the situation is rather similar to the one corresponding to the case where the diffusion

is reflecting at the boundary of a bounded set. This conjecture is the purpose of the present work.

II.- THE PROBLEM $(P_{x,s})$.

It will be solved on the control set Λ_1 defined by $p \in \Lambda_1$ if and only if $p \in \Lambda$ and there exist $c_p \in \mathbb{R}_+$ and $\alpha_p \in [0,1[$ such that $\forall u \in \mathbb{R}^*$, $\frac{u}{|u|}\, p(u) \leqslant c_p(1+|u|^{\alpha_p})$.

The method of solution is the dynamic programming one (see [1], [2], [3], [6], [8], [12]). The solving equation of $(P_{x,s})$ is

$$(R_s) \quad \forall x \in \mathbb{R}\,, \; -y''(x) + sy(x) + h(-y'(x)) = g(x)$$

where $h = f^*$ is the conjugate function of f (according to the convex analysis), defined by $\forall z \in \mathbb{R}$, $h(z) = \sup_{u \in \mathbb{R}} [zu - f(u)]$; we assume that f satisfies the following coercivity hypothesis :

(H_1) there exists $c_o > 0$ such that $\forall u \in \mathbb{R}$, $f''(u) \geqslant c_o$; therefore $h \in C^2(\mathbb{R}\,; [-f(0), +\infty[)$ and $\forall u \in \mathbb{R}$, $0 < h''(u) \leqslant \frac{1}{c_o}$.

Remark : In the problem $(P_{x,s})$, with reflection on the boundary of a bounded set (respectively in the periodical case), we have a limit condition of Neumann type on the boundary of this set (respectively the periodical condition), to characterize the optimal cost among all the solutions of the solving equation ; in our problem, we don't have such conditions, and this characterization is obtained by means of a "radiative condition".

A solution of $(P_{x,s})$ on Λ_1 is summarized in theorem 1 :

THEOREM 1.- We make all the above hypotheses, and assume also that g satisfies the growth and coercivity hypotheses

. g' is weakly growing

(H_2) . there exists $A \in \mathbb{R}_+$ such that $\forall u \in \mathbb{R}^*$, $\frac{u}{|u|}\, g'(u) \geqslant -A$

Then, for each fixed $s > 0$, there exists one and only one solution $y \in C^2(\mathbb{R}\,;\mathbb{R})$ of (R_s) satisfying the growth and radiative conditions :

- y' is weakly growing
- $h'(-y'(.)) \in \Lambda_1$ ("weak radiative condition").

Let y_s denote this solution, and p_s denote the control defined by $\forall u \in \mathbb{R}$, $p_s(u) = h'(-y'_s(u))$; then $p_s \in \Lambda_1$ and $\forall p \in \Lambda_1$, $\forall x \in \mathbb{R}$, $y_s(x) = J_s(x, p_s) \leq J_s(x, p)$; in other words, for each $x \in \mathbb{R}$, $y_s(x)$ is the optimal cost for $(P_{x,s})$ on Λ_1 and p_s is an optimal control (independant of x) for this problem.

Remark : Let's denote by Λ_2 the subset of Λ_1 defined by $p \in \Lambda_2$ if and only if $p \in \Lambda$ and there exists $c_p \in \mathbb{R}_+$ such that $\forall u \in \mathbb{R}^*$, $\frac{u}{|u|} p(u) \leq c_p$; then, we have $p_s \in \Lambda_2 \subset \Lambda_1$.

III.- ASYMPTOTIC BEHAVIOUR OF $(P_{x,s})$ WHEN s VAHISHES.

This study leads us to introduce the limiting stationary problem

(Q_x) : to minimize $\mu(x, p)$, for $p \in \Lambda_3$, where $\mu(x, p)$ is defined by

$$\mu(x,p) = \liminf_{T \to +\infty} \frac{1}{T} E \int_0^T [g(\xi_{x,p}(t)) + f(p(\xi_{x,p}(t)))] \, dt \,, \tag{3}$$

for each $x \in \mathbb{R}$ and $p \in \Lambda_3$ and where the control set Λ_3 is the subset of Λ_2 defined by $p \in \Lambda_3$ if and only if $p \in \Lambda$ and there exist two constants $c_p \geq 0$ and $d_p > 0$ such that $\forall u \in \mathbb{R}^*$, $\frac{u}{|u|} p(u) \leq c_p - d_p |u|$.

We shall solve this problem by means of coercivity hypotheses stronger than those above ; a solution of (Q_x) and the convergence properties are summarized in theorem 2.

THEOREM 2.- We make all the hypotheses of theorem 1, and assume also that g and h satisfy the coercivity hypotheses :

(H_3) . there exists $K_1 > 0$ such that $\forall u \in \mathbb{R}$, $h''(u) \geq K_1$

(H_4) . there exist $A \geq 0$ and $B > 0$ such that $\forall u \in \mathbb{R}^*$, $\frac{u}{|u|} g'(u) \geq -A + B|u|$.

Let's consider the solving equation of (Q_x) :

$$(R_o) \quad \forall x \in \mathbb{R}, \quad -v''(x) + \lambda + h(-v'(x)) = g(x)$$

Then, there exists a pair $(\lambda, v) \in \mathbb{R} \times C^2(\mathbb{R};\mathbb{R})$, unique (except for the addition of a constant for v ; that is, if (λ_1, v_1) is another solution of (R_o), then $\lambda_1 = \lambda$ and $v_1 - v$ is a constant) solution of (R_o) satisfying the "strong radiative condition" $h'(-v'(.)) \in \Lambda_3$.

Let (λ_o, v_o) be this solution of (R_o) and p_o the control defined by $\forall u \in \mathbb{R}$, $p_o(u) = h'(-v'_o(u))$; then $p_o \in \Lambda_3$, and $\forall p \in \Lambda_3$, $\forall x \in \mathbb{R}$, $\lambda_o = \mu(x, p_o) \leqslant \mu(x, p)$ in other words, for each $x \in \mathbb{R}$, λ_o is the optimal cost (independant of x) for (Q_x) and p_o is an optimal control (independant of x) for (Q_x).

Let's consider y_s and p_s defined in theorem 1 ; then, for each $s > 0$, $p_s \in \Lambda_3$. Furthermore, when s vanishes, the problem $(P_{x,s})$ converges to the problem (Q_x) in the following sense :

$$\lim_{s \to 0} s y_s = \lambda_o , \quad \lim_{s \to 0} y'_s = v'_o , \quad \lim_{s \to 0} y''_s = v''_o , \quad \lim_{s \to 0} p_s = p_o$$

uniformly on all compact subsets of $\mathbb{R}$.

<u>Remarks</u> : $\lambda_o = \mu(x, p_o) = \lim_{\tau \to +\infty} \frac{1}{\tau} E \int_0^\tau [g(\xi_{x,p}(t)) + f(p(\xi_{x,p}(t)))]\, dt$.

- The hypotheses of theorem 2 can be weakened : it is possible to replace (H_3) and (H_4) by the following hypothesis (with the same conclusions) : there exist constants $\theta \geqslant 1$, $A \geqslant 0$, $B > 0$, $K_1 > 0$, $\lambda_1 > 0$ and $\alpha_1 \in]1, 2]$ such that :

$$\theta \geqslant \frac{3 - \alpha_1}{\alpha_1 - 1}$$

$$\forall u \in \mathbb{R}^*, \ \frac{u}{|u|} g'(u) \geqslant -A + B|u|^\theta$$

$$\forall u \in \mathbb{R}, \ |u| \geqslant \lambda_1 \Rightarrow h''(u) \geqslant K_1 |u|^{\alpha_1 - 2}$$

- Theorems 1 and 2 are generalized to n-dimensional case (to appear).
- For other results concerning the one-dimensional case, see [10].

IV.- <u>PROOF OF THEOREM 1</u>. (For details, see [10]), $s > 0$ is fixed.

4.1.- Let $y_s \in C^2(\mathbb{R};\mathbb{R})$ be a solution of (R_s) and suppose that y'_s is a weakly growing function ; then for each $p \in \Lambda$, $\tau \geqslant 0$ and $x \in \mathbb{R}$, $y_s(x) \leqslant E \int_0^\tau e^{-st}[g(\xi_{x,p}(t)) + f(p(\xi_{x,p}(t)))]dt + e^{-s\tau} E(y_s(\xi_{x,p}(\tau)))$, and this relation becomes an equality if $p = p_s = h'(-y'_s(.)) \in \Lambda$.

This property is a consequence of the Ito formula applied to the process $\alpha_{s,x,p}$ defined by $\alpha_{s,x,p}(t) = e^{-st} y_s(\xi_{x,p}(t))$; then it is sufficient to write the mathematical expectations, using (R_s) and the definition of h .

4.2. <u>Lemma 1.- If</u> $\varphi : \mathbb{R} \to \mathbb{R}$ <u>is a weakly growing and measurable function, and if</u> $p \in \Lambda_1$, <u>then</u> $E(\varphi(\xi_{x,p}(.)))$ <u>is a weakly growing function, for each</u> $x \in \mathbb{R}$.

<u>Proof</u> : It is sufficient to verify the result when $\varphi(u) = u^{2n}$, $n \in \mathbb{N}^*$. Let Γ_p be the differential operator associated with equation (1) : $\Gamma_p = \frac{d^2}{dx^2} + p\frac{d}{dx}$. Since $p \in \Lambda_1$, there exist $\nu \geq 0$, $\eta \geq 0$ and $\beta \in]0,1[$ such that $\forall x \in \mathbb{R}$, $\Gamma_p \varphi(x) \leq G(\varphi(x))$, where $\forall u \in \mathbb{R}_+$, $G(u) = \nu u^\beta + \eta$. It is well known that, if $m(t) = E(\varphi(\xi_{x,p}(t)))$, then m'_d exists on $\mathbb{R}_+$ and $\forall t \in \mathbb{R}_+$, $m'_d(t) = E(\Gamma_p \varphi(\xi_{x,p}(t)))$ (see [5] and also [10] for details : our hypotheses are not exactly those of [5]).

Therefore $m'_d(t) \leq E[G(\varphi(\xi_{x,p}(t)))] \leq G(m(t))$ because of Jensen's inequality and concavity of the function G ; consequently, $m(t) \leq r(t)$ where r is the maximal solution, defined on $\mathbb{R}_+$, of the differential equation $u' = G(u)$, with initial condition $u(0) = x^{2n}$ (for such results concerning differential inequalities, see [7]) ; the verification that r is a weakly growing function completes the proof.

4.3. <u>Lemma 2.- Under the hypotheses of theorem 1,</u> (R_s) <u>has at least one solution</u> $y_s \in C^2(\mathbb{R};\mathbb{R})$ <u>such that</u> y'_s <u>is weakly growing and</u> $h'(-y'_s(.)) \in \Lambda_2$.

<u>Proof</u> : For each $s > 0$ and $T > 0$, let $y_{s,T} \in C^2([-T,T];\mathbb{R})$ be the solution of (R_s) on $[-T,T]$ such that $y'_{s,T}(\pm T) = 0$. We have for $y_{s,T}$ and $y'_{s,T}$ the following estimates :

. There exist $K \in \mathbb{N}^*$ and $B_1 \geq 0$ such that for each $s > 0$, $T > 0$ and $u \in [-T,T]$

$$(E_1) \qquad \frac{1}{s}(\inf g - h(0)) \leq y_{s,T}(u) \leq u^{2K} + \frac{B_1}{s} \ :$$

the function y defined by $\forall u \in \mathbb{R}$, $y(u) = \frac{1}{s}(\inf g - h(0))$ (respectively $y(u) = u^{2K} + \frac{B_1}{s}$, for sufficiently large K and B_1) satisfies the relation :

$\forall u \in \mathbb{R}$, $-y''(u)+sy(u)+h(-y'(u)) \leq g(u)$ (respectively $\geq g(u)$) ; the application of extremality conditions to $y_{s,T}-y$ at a point where the minimum (respectively the maximum) of this function is reached implies the formula (E_1) .

. Let $n_1 \geq 3$ be an integer such that $g'(u)=o(|u|^{n_1})$ when $|u| \to +\infty$; then, for each $s>0$, there exist $c_s>0$ and $c'_s>0$ such that, for each $T>0$ and $u \neq 0$, $u \in [-T,T]$

(E_2) $$-c_s-|u|^{n_1} \leq -\frac{u}{|u|} y'_{s,T}(u) \leq c_s$$

(E_3) $$-c'_s-\frac{1}{c_o}|u|^{n_1} \leq \frac{u}{|u|} h'(-y'_{s,T}(u)) \leq c'_s \quad :$$

the function v defined by $v(u)=-c$ if $u \leq 0$ and $v(u)=-c-u^{n_1}$ if $u \geq 0$ (respectively $v(u)=c+|u|^{n_1}$ if $u \leq 0$ and $v(u)=c$ if $u \geq 0$) for $c \geq 0$ large enough satisfies the relation : $\forall u \in \mathbb{R}$, $v''(u)-sv(u)+(h'(v(u)))\, v'(u) \geq g'(u)$ (respectively $\leq g'(u)$) ; we write for $y'_{s,T}$ the derived equation from (R_s) ; the application of a maximum method (analogous to the one used for (E_1)) leads to formula (E_2) ; (E_3) is a consequence of (E_2) and the hypotheses concerning f or h .

The above estimates are uniform with respect to T ; therefore, $\{y_{s,T/[-U,U]}\ ;\ T \geq U\}$ is relatively compact in $C^2([-U,U];\mathbb{R})$, and it is possible to construct by recurrence y_s , solution of (R_s) on $\mathbb{R}$ and a sequence $(T_n)_{n \in \mathbb{N}}$ (with $T_n \xrightarrow{n \uparrow \infty} +\infty$) such that $\forall U>0$, $y_{s,T_n/[-U,U]} \xrightarrow[T_n \geq U]{n \to +\infty} y_{s/[-U,U]}$ in $C^2([-U,U];\mathbb{R})$ and y_s, y'_s, p_s satisfy also the above estimates.

4.4.- A solution y_s of (R_s) in lemma 2 is also weakly growing and such that $h'(-y'_s(.)) \in \Lambda_1$; hence, because of lemma 1, if $p \in \Lambda_1$ (then if $p=p_s$) $e^{-s\tau} E(y_s(\xi_{x,p}(\tau))) \xrightarrow{\tau \uparrow +\infty} 0$; the relations 4.1 complete the proof.

Remark : The uniqueness property of y_s (which is a consequence of its interpretation) implies that $\forall U>0$, $y_{s,T/[-U,U]} \xrightarrow[T \geq U]{T \uparrow +\infty} y_{s/[-U,U]}$ in $C^2([-U,U];\mathbb{R})$.

V.- PROOF OF THEOREM 2.

5.1. Let $(\lambda_o, v_o) \in \mathbb{R} \times C^2(\mathbb{R};\mathbb{R})$ be a solution of (R_o) such that v'_o is weakly growing ; applying the Ito formula to the process $\alpha_{x,p}$ defined by $\alpha_{x,p}(t) = -\lambda_o t + v_o(\xi_{x,p}(t))$, and taking the expectations, with the help of (R_o) and the definition of h, we get : $\lambda_o \leq \frac{1}{\tau} E\int_0^\tau [g(\xi_{x,p}(t))+f(p(\xi_{x,p}(t)))]\,dt - \frac{v_o(x)}{\tau} + \frac{1}{\tau} E(v_o(\xi_{x,p}(\tau)))$ for each $p \in \Lambda$, $\tau > 0$, and $x \in \mathbb{R}$; and this relation becomes an equality if $p = p_o = h'(-v'_o(.)) \in \Lambda$.

5.2. Lemma 3.- <u>If</u> $\varphi : \mathbb{R} \to \mathbb{R}$ <u>is a weakly growing and measurable function, and if</u> $p \in \Lambda_3$, <u>then for each</u> $x \in \mathbb{R}$, $E(\varphi(\xi_{x,p}(.)))$ <u>is bounded</u>.

<u>Proof</u> : The proof is similar to that of lemma 1 ; using the same notations, since $p \in \Lambda_3$, there exist $\nu_1 > 0$ and $\eta_1 > 0$ such that $\forall x \in \mathbb{R}$, $\Gamma_p \varphi(x) \leq G_1(\varphi(x))$, where $\forall u \in \mathbb{R}_+$, $G_1(u) = -\nu_1 u + \eta_1$; consequently, $m(t) \leq r_1(t)$, where r_1 is the maximal solution, defined on $\mathbb{R}_+$, of the differential equation $u' = G_1(u)$, with initial condition $u(0) = x^{2n}$; the verification that r_1 is bounded completes the proof.

5.3. Lemma 4.- <u>Under the hypotheses of theorem 2,</u> (R_o) <u>has at least one solution</u> $(\lambda_o, v_o) \in \mathbb{R} \times C^2(\mathbb{R};\mathbb{R})$ <u>such that</u> v'_o <u>is weakly growing and</u> $h'(-v'_o(.)) \in \Lambda_3$.

<u>Proof</u> : For each $s > 0$, $T > 0$ and $a \in \mathbb{R}$, let $y^{(a)}_{s,T} \in C^2([-T,T];\mathbb{R})$ be the solution of (R_s) on $[-T,T]$, such that $y^{(a)\prime}_{s,T}(T) = aT$ and $y^{(a)\prime}_{s,T}(-T) = -aT$. We have for $y^{(a)}_{s,T}$ and $y^{(a)\prime}_{s,T}$ the following estimates :

. for each $s_o > 0$, there exist constants $a > 0$, $b > 0$, $a' > 0$, $b' > 0$, such that for each $s \in]0, s_o]$, $T > 0$ and $u \in [-T,T]$, $u \neq 0$

$$(E_4) \qquad -a|u| - b - |u|^{n_1} \leq -\frac{u}{|u|}\, y^{(a)\prime}_{s,T}(u) \leq -a|u| + b$$

$$(E_5) \qquad \frac{u}{|u|} h'(0) - \frac{1}{c_0}(a|u| + b + |u|^{n_1}) \leq \frac{u}{|u|} h'(-y^{(a)\prime}_{s,T}(u)) \leq -a'|u| + b' .$$

. **For such** a constant a, there exist constants $K \in \mathbb{N}^*$, $B_2 > 0$ and $T_o \geq 0$ such that, for each $s \in]0, s_o]$, $T > T_o$ and $u \in [-T,T]$,

$$(E_6) \qquad \inf g - h(0) \leq s y^{(a)}_{s,T}(u) \leq s_o u^{2K} + B_2 .$$

The proofs of (E_4), (E_5) and (E_6) are respectively similar to those of (E_2), (E_3)

and (E_1).

These estimates are uniform with respect to $T > T_o$; consequently, there exist for each $s \in]0, s_o]$ a solution z_s of (R_s) on $\mathbb{R}$ satisfying on $\mathbb{R}$ the above estimates ; it is clear, according to theorem 1, that $z_s = y_s$ for each $s \in]0, s_o]$, and that, for each $U > T_o$,

$$y^{(a)}_{s,T/[-U,U]} \xrightarrow[T \geq U]{T\uparrow +\infty} y_{s/[-U,U]} \quad \text{in} \quad C^2([-U,U] ; \mathbb{R}) .$$

The new estimates obtained for y_s , y'_s and $h'(-y'_s(.))$ are uniform with respect to $s \in]0, s_o]$; then there exists a pair $(\lambda_o, v_o) \in \mathbb{R} \times C^2(\mathbb{R} ; \mathbb{R})$, solution of (R_o) and satisfying the following estimates :

$$\inf g - h(0) \leq \lambda_o \leq B_2 \text{ , and for each } u \in \mathbb{R} ,\ u \neq 0 ,$$

$$-a|u| - b - |u|^{n_1} \leq - \frac{u}{|u|} v'_o(u) \leq -a|u| + b$$

$$\frac{u}{|u|} h'(0) - \frac{1}{c_0}(a|u| + b + |u|^{n_1}) \leq \frac{u}{|u|} h'(-v'_o(u)) \leq -a'|u| + b' :$$

indeed, because of the relative compactness of $\{y'_{s/[-U,U]} ; s \in]0, s_o]\}$ in $C^1([-U,U] ; \mathbb{R})$, one can construct such a solution of (R_o) and a sequence $(s_n)_{n \in \mathbb{N}}$ such that $\lim_{n\to\infty} s_n = 0$, $\lim_{n\to\infty} y'_{s_n} = v'_o$ in $C^1([-U,U];\mathbb{R})$, and $\lim_{n\to\infty} s_n y_{s_n} = \lambda_o$ in $C^0([-U,U] ; \mathbb{R})$.

5.4.- A solution (λ_o, v_o) of (R_o) in lemma 4 is such that v_o is weakly growing and $h'(-v'_o(.)) \in \Lambda_3$; hence, because of lemma 3, if $p \in \Lambda_3$ (then if $p = p_o$) , $\frac{1}{\tau} E(v_o(\xi_{x,p}(\tau))) \xrightarrow{\tau\uparrow\infty} 0$; the relations 5.1. complete the proof of the results concerning (Q_x) . The convergence properties stated in theorem 2 follow from the construction of (λ_o, v_o) and the uniqueness property.

Remark : Let $(\lambda^{(a)}_T, v^{(a)}_T) \in \mathbb{R} \times C^2([-T,T] ; \mathbb{R})$ be the solution of (R_o) such that $v^{(a)\prime}_T(T) = aT$ and $v^{(a)\prime}_T(-T) = -aT$, defined for each $a \in \mathbb{R}$ and $T > 0$ (such a pair is unique, except for the addition of a constant for $v^{(a)}_T$) ; if a is the constant occuring in the proof of lemma 4, then we have the following convergence properties :

$$\lim_{s\to 0} s y^{(a)}_{s,T} = \lambda^{(a)}_T \ , \ \lim_{s\to 0} y^{(a)\prime}_{s,T} = v^{(a)\prime}_T \ , \ \lim_{s\to 0} y^{(a)\prime\prime}_{s,T} = v^{(a)\prime\prime}_T \text{ , uniformely on } [-T,T] ;$$

$\lim_{T\uparrow+\infty} \lambda_T^{(a)} = \lambda_o$ and $\lim_{T\uparrow+\infty} v_T^{(a)\prime} = v'_o$, $\lim_{T\uparrow+\infty} v_T^{(a)\prime\prime} = v''_o$ uniformely on all compact subsets of $\mathbb{R}$.

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