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§1. Introduction. The main result of this paper is that functions in certain Bergman spaces and in certain generalizations of Bergman spaces can be written as sums of building blocks of an especially simple type. This can be regarded as an extension to the Bergman spaces of the atomic theory of generalized Hardy spaces presented in [3] and of the molecular theory of Hardy spaces which is presented in this volume by Taibleson and Weiss [26]. (See also [31].)

Let D be a bounded symmetric domain in $\mathbb{C}^n$. (That is, D has a transitive group of biholomorphic automorphisms and each point of D is the fixed point of a biholomorphic automorphism of period two.) For example D could be the unit disk in $\mathbb{C}$ or the unit polydisk or the unit ball in $\mathbb{C}^n$. Let $dV(z)$ be Lebesgue measure on D and let $B(z, \zeta)$ be the Bergman kernel for D. For instance, if D is the unit ball in $\mathbb{C}^n$, then

$$B(z, \zeta) = C_n(1 - z.\bar{\zeta})^{-(n+1)}$$

where $z.\bar{\zeta} = \Sigma z_i\bar{\zeta}_i$. For $0 < p < \infty$ the Bergman space $A^p = A^p(D)$ is defined to be the supapace of $L^p(D, dV(z))$ consisting of holomorphic functions.

We will prove the following result.

Theorem I. *Suppose* $0 < p < 2$. *There are points* ζ_i, $i = 1, 2, \ldots$ *in* D *and constants* C_1, C_2 *which depend on* p *and on the points* ζ_i *so that*

(a) *if* F *is in* A^p *then there are numbers* λ_i *such that*

$$F(z) = \sum_{i=1}^{\infty} \lambda_i \left(\frac{B^2(z,\zeta_i)}{B(\zeta_i,\zeta_i)}\right)^{\frac{1}{p}} \qquad (1.1)$$

with $\Sigma|\lambda_i|^p \leq C_1 \int_D |F(z)|^p \, dV(z)$, *and*

(b) *if* λ_i *are numbers such that* $\Sigma|\lambda_i|^p < \infty$ *then the function* $F(z)$

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<u>defined by</u> (1.1) <u>is in</u> A^p <u>and satisfies</u>

$$\int_D |F(z)|^p \, dV(z) \le C_2 \Sigma |\lambda_i|^p .$$

Our main result is actually for Bergman spaces defined on unbounded realizations of such domains. In that context we obtain results for the full range $0 < p < \infty$ and also for weighted Bergman spaces. A similar, but slightly more complicated result is obtained corresponding to the end point $p = \infty$. The end point result involves spaces of functions similar to Bloch functions. Decomposition theorems are also obtained for certain spaces of harmonic functions defined on the unit ball in $\mathbb{R}^n$.

Part (a) of the theorem is derived from, and can be regarded as a discrete analog of, an integral reproducing formula for functions in A^p. Part (b) of the theorem is elementary for $p \le 1$. For $p > 1$, part (b) is derived from and can be regarded as the discrete analog of, the fact that the integral reproducing formula gives a bounded projection of L^p onto A^p.

The proofs of the decomposition theorems for holomorphic functions is given in Section 2. The proof of the analogous result for harmonic functions is in Section 3.

In Section 4, we give various consequences of these decomposition theorems. One of the main themes of these applications is that various computations or estimates which are relatively simple for the individual summands in a decomposition such as (1.1) can then be extended by linearity to the entire space. Using this point of view we obtain inclusion relations between various Bergman and Hardy spaces and description of the behaviour of the Bergman spaces under various integral and differential operators. Other applications include results about zero sets for holomorphic and harmonic functions, results about automorphic forms on the disk, and a nuclearity criteria for Hankel operators.

In proving the decomposition theorems for holomorphic functions we use certain facts about the Bergman kernel function on Siegel domains. Since the proof of these facts has a different character than the rest of the arguments we use, and since the facts themselves may be of independent interest; we present those results in an Appendix.

We acknowledge with thanks the many helpful discussions we had with our colleague, G. Weiss, during this research.

We follow the custom of using the letter "c" to denote many different constants.

§2. Spaces of holomorphic functions.

Our main results for holomorphic functions, Theorems 2 and 2', are proved for Bergman spaces of holomorphic functions on symmetric Siegel domains of type two. The primary reason for working in that setting is that the proofs are based on certain estimates for the Bergman kernel and those estimates, Lemmas 2.1, 2.2, and 2.3 are obtained using, among other things, Gindikin's theory of special functions for such domains [11]. Due to work of Stoll [25] and Forelli and Rudin [9], similar estimates are available for products of complex balls and hence we also obtain Theorem 2 in that context. Certain cases of Theorem 2 are invariant under biholomorphic automorphisms of the domain and hence will be valid for all bounded symmetric domains. This observation produces Theorem 1 from Theorem 2.

The proof we offer makes frequent use of the fact that the domain being considered has a transitive group of automorphisms. In most cases it is clear that this is just a convenience, and it seems reasonable to conjecture that results similar to Theorems 1 and 2 are true for a much more general class of domains. In this direction we note that Theorem 3 (the analog of Theorem 1 for the ball in $\mathbb{R}^n$) does not use (in any obvious way) the homogeneous structure of the real ball.

For any open set D in $\mathbb{C}^n$, we denote by $B(z, \zeta) = B_D(z, \zeta)$ the Bergman kernel function for D. We will denote the Euclidean volume element of D by $dV(z)$ and will denote the volume of a subset B of D by $|B|$. Thus $|B| = \int_B dV$: The spaces of holomorphic functions which we will consider are the weighted Bergman spaces $A^{p,r}(D)$ which are defined as follows. For p and r, $0 < p < \infty$ and r real, $A^{p,r}(D)$ is the space of all holomorphic functions F on

D for which

$$\|F\|_{p,r} = \left(\int_D |F(z)|^p \, B(z,z)^{-r} \, dV(z)\right)^{\frac{1}{p}} < \infty .$$

We will call $\|F\|_{p,r}$ the "norm" of F even in the case when $p < 1$. For the case of D the unit disk of $\mathbb{C}$, many properties of these spaces are presented in [23].

For z and w in D, we denote by $d(z, w)$ the Bergman distance between z and w. This distance is invariant under biholomorphic automorphisms of D and on compact subsets of D is equivalent to the Euclidean distance. A fuller discussion of the Bergman kernel and the associated geometry can be found in the books by Baily [1] and Kobayashi [29] and in the references listed there.

Suppose η is a given positive number. We will call a sequence of points $\{\zeta_i\}$ in D an η-lattice if $d(\zeta_i, \zeta_j) \geq \eta$ whenever $i \neq j$ and if given any z in D there is a ζ_i with $d(z, \zeta_i) \leq \eta$. Hence the balls centered at ζ_i with radius η (in the Bergman metric) cover D and there is control on the amount of overlap in the covering. (We will also use this terminology when the actual inequalities satisfied are $d(\zeta_i, \zeta_j) \geq \eta/a$ and $d(z, \zeta_i) \leq a\eta$ for a large constant a which does not depend on z, i, j, or η. This slightly imprecise usage avoids the need for an additional parameter in the definition.)

We now recall the terminology and notation for Siegel domains. A homogeneous Siegel domain of type two (an "affine-homogeneous Siegel domain of the second kind" in the terminology of [11]) is obtained as follows. We start with V a regular open affine-homogeneous cone in $\mathbb{R}^n$ and a homogeneous V-Hermitian bilinear form F defined on $\mathbb{C}^m \times \mathbb{C}^m$. The domain D is then defined to be the (z_1, z_2) in $\mathbb{C}^{n+m} = \mathbb{C}^n \times \mathbb{C}^m$ with z_1 in $\mathbb{C}^n$, z_2 in $\mathbb{C}^m$ for which

$$\mathrm{Im}(z_1) - F(z_2, z_2) \in V .$$

A detailed discussion of analysis on such domains is presented by Gindikin in [11]. Other good sources of background information on such domains are the article by Vagi [32] and the references there and the book by Kobayashi [29].

A discussion of the Fourier analysis for the simpler special case of domains of tube-type is in [24].

One reason for the significance of this class of domains is the result of Vinberg, Gindikin, and Pyatetzki-Shapiro which states that every bounded homogeneous domain is biholomorphically equivalent to such a domain.

Our results are proved for a subclass of these domains--the symmetric domains. This corresponds to the case where the cone V is self-dual. We suspect the results are valid for the full class of homogeneous domains but we have not been able to provide proofs for the results of the Appendix in that more general case.

The simplest example of such a domain is the upper half plane in $\mathbb{C}$ ($n = 1$, $m = 0$). This example should be kept in mind when reading the proof of Theorem 2. Also, Theorem 2 is presented for spaces $A^{p,r}(D)$. The choice $r = 0$ simplifies the reading of the proof and involves no real conceptual loss.

We now state the main result of this section.

Theorem 2. Let D be the unit ball in $\mathbb{C}^n$ or a symmetric Siegel domain of type two. There is a positive constant ϵ_D which depends on D so that the Bergman space $A^{p,r}(D)$ with $0 < p < \infty$, $-\epsilon_D < r < \infty$ satisfies the following. Given $\theta > (1 + r)\max(-1, p - 2)$ there is an $\eta_0 = \eta_0(D, p, r, \theta)$ so that if the points $\{\zeta_i\}$ are an η-lattice in D, $\eta < \eta_0$ then

(a) If F is in $A^{p,r}$ then there are numbers λ_i so that

$$F(z) = \sum \lambda_i \left(\frac{B^2(z,\zeta_i)}{B(\zeta_i,\zeta_i)}\right)^{\frac{1+r}{p}} \left(\frac{B(z,\zeta_i)}{B(\zeta_i,\zeta_i)}\right)^{\frac{\theta}{p}} \tag{2.1}$$

and $\sum|\lambda_i|^p \leq C\|F\|^p_{p,r}$.

(b) If $\sum|\lambda_i|^p < \infty$ then F defined by (2.1) is in $A^{p,r}$ and $\|F\|^p_{p,r} \leq$ $\leq c\sum|\lambda_i|^p$.

The constant c can be chosen to depend on only D, p, r, θ and η (not on F, the λ's, or the particular η-lattice).

Before starting the proof, several comments should be made:

1. For these domains the Bergman kernel is never zero, (see the proof of Lemma 2.3 in the Appendix) hence the fractional powers in (2.1) cause no difficulty.
2. The sum in (2.1) converges absolutely uniformly on compact subsets of D and converges in $A^{p,r}$ norm.
3. The numbers λ_i in (2.1) (which are called λ_{ij} in the proof) are obtained in a constructive way as continuous linear functionals on $A^{p,r}$.
4. The representation (2.1) (i.e., the choice of λ_i) is not unique.
5. The Bergman kernel of a product of two domains is the product of the Bergman kernels for the domains. Hence, if Theorem 2 is true for two domains, it is true for their product. In particular the theorem is true for the polydisk.
6. ϵ_D is less then 1. Hence $1 + r > 0$ and thus the simplifying choice $\theta = 0$ is always allowed if $0 < p < 2$.
7. For D the ball in C^n, $\epsilon_D = \frac{1}{n+1}$. For this D, the spaces $A^{p,r}$ with $r \leq -\epsilon_D$ contain only the zero function.

The basic idea of the proof of part (a) is the following. We use an integral reproducing formula (Lemma 2.1) to represent F in $A^{p,r}$ by an integral. This integral is then approximated by a Riemann sum using the values of F at the points of an η-lattice. If η is sufficiently small then this produces a good approximation and iteration of the process yields the representation (2.1).

Part (b) of the Theorem is elementary if $0 < p \leq 1$. For $p > 1$ we must show that a certain linear operator from ℓ^p to $A^{p,r}$ is bounded. We do this by using a boundedness criterion of Schur [21], which was used by Forelli and Rudin [9] in a similar context (Lemma 2.7). The estimates which are needed to apply this criterion are estimates on integrals of powers of the Bergman kernel. (Lemma 2.2).

The history of the reproducing formulas which we use, and of the closely related results on the boundedness of certain projection operators, is complicated. We refer to [9] and the discussion there and also to [14] and [22].

The following three Lemmas are estimates we will need for the Bergman kernel.

We suppose D satisfies the hypothesis of Theorem 2.

Lemma 2.1. *There is a constant* $\epsilon_D > 0$ *which depends on the domain* D *such that if* $r > -\epsilon_D$ *then for appropriate constant* c_r, $c_r B(z, \zeta)^{1+r}$ *is the reproducing kernel for* $A^{2,r}$; *that is, for all* F *in* $A^{2,r}$.

$$F(z) = c_r \int_D F(\zeta) B(z, \zeta)^{1+r} B(\zeta, \zeta)^{-r} dV(\zeta)$$

In particular, setting $F(z) = B(z, w)^{1+r}$ and then taking $w = z$ and recalling that $B(z, \zeta) = \overline{B(\zeta, z)}$ we obtain

$$B(z, z)^{1+r} = C_r \int_D |B(z, \zeta)|^{2(1+r)} B(\zeta, \zeta)^{-r} dV(\zeta).$$

This is related to the next result.

Lemma 2.2. *If* $\alpha > r > -\epsilon_D$ *then*

$$\int_D |B(z, \zeta)|^{1+\alpha} B(\zeta, \zeta)^{-r} dV(\zeta) \leq C_{\alpha,r} B(z, z)^{\alpha-r}$$

with $c_{\alpha,r}$ *not depending on* z.

Lemma 2.3. *if* z, ζ, ζ_0 *are in* D *and* $d(\zeta, \zeta_0) < 10$ *then*

$$\left|\frac{B(z,\zeta)}{B(z,\zeta_0)} - 1\right| \leq c_D\, d(\zeta, \zeta_0).$$

with c_D *not depending on* z, ζ, *or* ζ_0.

These results are proved for symmetric Siegel domains of type two in the Appendix. In that case we have equality in Lemma 2.2. For bounded symmetric domains, Lemma 2.1 is due to Stoll [25]. Lemmas 2.1 and 2.2 for the ball in $\mathbb{C}^n$ are due to Forelli and Rudin [9]. In that case $\epsilon_D = \frac{1}{n+1}$. If D is the ball in $\mathbb{C}^n$ then Lemma 2.3 is straightforward.

We now describe a decomposition of D which we will associate with any given η-lattice (with η small). Informally, and slightly imprecisely, the construction is as follows. We cover D with sets B_i, balls of radius one which are almost disjoint. Each B_i is covered by sets B_{ij} which are balls of radius η, are almost disjoint and each B_{ij} contains exactly one point of the η-lattice.

More precisely, let z_0 be some fixed base point of D. Pick z_1 in D with $d(z_1, z_0) \geq 1$ and $d(z_1, z_0)$ as small as possible subject to that condition. (The fact that the choice is not unique does not matter.) Proceed inductively picking z_j so that $d(z_j , z_i) \geq 1$ for $i = 0 , \ldots , j - 1$ and $d(z_j , z_0)$ is as small as possible subject to this constraint. It is clear that the balls $\tilde{B}_i = \{z ; d(z , z_i) < 1/2\}$ are disjoint and it is clear after a moment's thought that the balls $B_i = \{z ; d(z , z_i) < 1\}$ form a cover of D. (If z were in no B_i then the sequence of numbers $d(z_0 , z_i)$ would be bounded. However, the balls $\tilde{B}_i$ are disjoint and all have the same invariant volume, hence they can not all fit in a set of the form $\{z; d(z , z_0) < M\}$.) The set $\{z_i\}$ is a 1-lattice. (A similar construction could be used to obtain an η-lattice for any η and hence there are η-lattices on any such D.)

Let $\{\zeta_i\}$ be the points of the given η-lattice. We rename the ζ_i's as follows. Let $\zeta_{0,j}$, $j = 1 , 2 , \ldots, N_0$ be those ζ_i which are in B_0 but not in $\tilde{B}_i$ for $i = 1 , 2 , \ldots.$ Let $\zeta_{1,j}$ for $j = 1 , 2 , \ldots , N_1$ be those ζ_i which have not been previously selected and which are in B_1 but not in $\tilde{B}_i$ for $i = 2 , 3 , \ldots.$ Proceeding in this way we end up with ζ_i being renumbered as ζ_{ij} in such a way that all the ζ_{ij} are in B_i and every point of the η-lattice which is in $\tilde{B}_i$ is among the ζ_{ij}.

We now construct a covering of D associated with this η-lattice. Let B_{ij} and $\tilde{B}_{ij}$ be the balls of radius η and $\eta/2$ respectively, centered at ζ_{ij}. Let D_{ij} be the B_{ij} made disjoint. That is, let

$$D_{01} = B_{01}\setminus[\bigcup_{(i,j)\neq(0,1)} \tilde{B}_{ij}]$$

$$D_{02} = B_{02}\setminus[D_{01} \cup (\bigcup_{(i,j)\neq(0,2)} \tilde{B}_{ij})]$$

$$D_{03} = B_{03}\setminus[D_{01} \cup D_{02} \cup (\bigcup_{(i,j)\neq(0,3)} \tilde{B}_{ij})]$$

Continue this way through the finite number of B_{0k} and then set

$$D_{11} = B_{11}\setminus[\bigcup_k D_{0k} \cup (\bigcup_{(i,j)\neq(1,1)} \tilde{B}_{ij})]$$

etc. Let $D_i = \bigcup_j D_{ij}$.

These quantities will be fixed for the rest of this proof. The following is a summary of the properties of this construction.

Lemma 2.4. (a) *the* ζ_{ij} *are an* η *-lattice. The* D_{ij} *are a disjoint cover of* D , i.e., $\bigcup_{i,j} D_{ij} = D$.

$$\zeta_{ij} \in \tilde{B}_{ij} \subset D_{ij} \subset B_{ij}$$

(b) *The points* z_i *are a* 1 *-lattice and, denoting by* $\overline{B}_i$ *the ball centered at* z_i *and of radius* 2 , *and* $\overline{\overline{B}}_i$ *the ball of radius* 4

$$z_i \in \tilde{B}_i \subset D_i \subset \overline{B}_i \subset \overline{\overline{B}}_i .$$

(c) *There is a constant* M *(which depends on* D *and* η *but not on the* z_i *or* ζ_{ij}*) such that no point of* D *belongs to more than* M *of the* B_{ij} *nor to more than* M *of the* $\overline{B}_i$ *nor of the* $\overline{\overline{B}}_i$.

We will make frequent use of the following fundamental invariance property of the Bergman kernel. Let g be any biholomorphic map of D to itself. For z , ζ in D

$$B(gz , g\zeta) \det g'(z) \overline{\det g'(\zeta)} = B(z , \zeta)$$

(Here $g'(z)$ denotes the complex differential of g at z .) Now suppose z_0 is a fixed base point of D . When we set $z = \zeta = g(z_0)$ in the previous equation we obtain

$$B(z , z) = \frac{1}{|\det g'|^2} B(z_0 , z_0) \tag{2.2}$$

Lemma 2.5. *If* f *is holomorphic on* D *then*

$$\int_{\overline{B}_i} |f(z)| dV(z) \leq c\left(\int_{\overline{\overline{B}}_i} |f(z)|^p B^{-r}(z , z) dV(z)\right)^{\frac{1}{p}} B^{\frac{1+r}{p}-1}(z_i , z_i) .$$

Proof. $\overline{B_i} = g\overline{B_v}$ for some transformation g . Hence, by a change of variables

$$\int_{\overline{B}_i} |f(z)| \, dV(z) = \int_{\overline{B}_0} f(g\zeta) \, |\det g'|^2 \, dV(\zeta) .$$

$|f(g(\zeta))|^p$ is subharmonic for any positive p , hence, for any small Euclidean ball Δ centered at ζ ,

$$|f(g(\zeta))| \leq \left(\frac{1}{|\Delta|} \int_\Delta |f(g\zeta)|^p \, dV(\zeta)\right)^{\frac{1}{p}} .$$

Since we can require that Δ be inside $\overline{\overline{B_0}}$ and that $|\Delta|^{-1}$ be bounded, this yields

$$\int_{\overline{B_i}} |f(z)|dV(z) \leq c\ |\det g'|^2\ (\int_{\overline{B_0}} |f(g\zeta)|^p\ dV(\zeta))^{\frac{1}{p}}$$

$$= c\ |\det g'|^{2(1-\frac{1}{p})}\ (\int_{\overline{\overline{B_i}}} |f(z)|^p\ dV))^{\frac{1}{p}}$$

By Lemma 2.3, $B(z_i\ ,\ z_i)/B(z\ ,\ z)$ is bounded below for z in $\overline{\overline{B_i}}$. Hence

$$\int_{\overline{B_i}} |f(z)|\ dV(z) \leq c\,B(z_i\ ,\ z_i)^{\frac{r}{p}}\ |\det g'|^{2(1-\frac{1}{p})}\ \int_{\overline{\overline{B_i}}} |f(z)|^p\ B(z\ ,\ z)^{-r}\ dV(z))^{\frac{1}{p}}$$

An application of equality (2.2) yields the desired result.

Lemma 2.6. *If f is holomorphic on D then*

$$\int_{D_{ij}} |f(\zeta) - f(\zeta_{ij})|\ dV(\zeta) \leq c\eta^{2(n+m)+1} \int_{\overline{B_i}} |f(\zeta)|\ dV(\zeta)$$

(*recall that* $n+m$ *is the complex dimension of* D.)

Proof. $D_{ij} \subseteq gB_{01}$ for some g transforming D to itself with $gz_0 = \zeta_{ij}$. Hence,

$$I = \int_{D_{ij}} |f(\zeta) - f(\zeta_{ij})|\ dV(\zeta) \leq |\det g'|^2\ (\int_{B_{01}} |f(gs) - f(gz_0)|dV(\zeta))\ .$$

Now $|f(gs) - f(gz_0)| \leq (\sup_{\zeta\in B_{01}} |\nabla(f(gs))|)\ |\zeta - z_0|$. We dominate $|\zeta - z_0|$ by $c\eta$ and dominate $\sup_{B_{01}}$ by c times the volume integral over the much bigger ball B_0 and obtain

$$|f(gs) - f(g\zeta_0)| \leq c\eta \int_{B_0} |f(gz)|\ dV(z)\ .$$

Hence,

$$I \leq c\eta\ |\det g'|^2\ (\int_{B_0} |f(gz)|\ dV(z)) \int_{B_{01}} dV(\zeta)$$

$$\leq c\eta^{2(n+m)-1}\ (\int_{B_0} |f(gz)|\ |\det g'|^2\ dV(z))\ .$$

Reversing the original change of variables yields the desired estimate.

The next lemma is the extension used by Rudin and Forelli [9] of the boundedness criterion of Schur [21].

Lemma 2.7. Let (X, ν) be a measure space. Suppose $1 < p < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$ and that $Q(x, y)$ is a non-negative measurable function on $X \times X$. If there is a $g(x)$ which is non-negative and measurable and a constant c such that

$$\int_X Q(x, y) g(y)^q \, dV(y) \le c g(x)^q \quad \text{a.e.} \tag{2.3}$$

and

$$\int_X Q(x, y) g^p(x) \, dV(x) \le c g(y)^p \quad \text{a.e.}, \tag{2.4}$$

then the operator which sends f into Tf given by

$$(Tf)(x) = \int Q(x, y) f(y) \, dV(y)$$

is a bounded map of $L^p(X, \nu)$ to itself.

Lemma 2.8. Suppose $p > 1$, $r > -\epsilon_D$, $\theta > (p-2)(r+1)$ and $\alpha = 2r + 2 + \theta$. Let T be the linear map which sends the function f defined on D to the new function Tf defined on D by

$$(Tf)(z) = \int_D \frac{|B(z,\zeta)|^{\frac{\alpha}{p}}}{B(\zeta,\zeta)^{\frac{\alpha}{p}-r-1}} f(\zeta) B^{-r}(\zeta,\zeta) dV(\zeta) .$$

T is a bounded map of $L^p(D, B^{-r}(\zeta,\zeta)dV(\zeta))$ into itself.

Proof. We apply Lemma 2.7 on the measure space $(D, B(\zeta,\zeta)^{-r} dV(\zeta))$. The required function $g(z)$ is $B(z, z)^{\delta}$. The estimates (2.3) and (2.4) follow from Lemma 2.2 for sufficiently small positive δ. More specifically, to obtain (2.3) replace α in Lemma 2.2 by $\alpha/p-1$ and replace r by $\alpha/p - 1 - \delta q$. To obtain (2.4), replace α in Lemma 2.2 by $\alpha/p-1$ and r by $-\delta p + r$. If δ is sufficiently small and positive, then all the required inequalities are satisfied.

We now prove the theorem.

Proof of Theorem II. We begin with part b of the theorem. We wish to show that given a sequence λ_{ij} with $\Sigma|\lambda_{ij}|^p < \infty$ then

$$\int |\Sigma\lambda_{ij}k_{ij}(z)|^p B(z, z)^{-r} dV(z) \le c \, \Sigma|\lambda_{ij}|^p$$

where

$$k_{ij}(z) = \left(\frac{B^2(z,\zeta_{ij})}{B(\zeta_{ij},\zeta_{ij})}\right)^{\frac{1+r}{p}} \left(\frac{B(z,\zeta_{ij})}{B(\zeta_{ij},\zeta_{ij})}\right)^{\frac{\theta}{p}}$$

and ζ_{ij} are the points of an η -lattice.

By Lemma 2.2, $\|k_{ij}(z)\|^p_{p,r} \le c$, for some constant c which does not depend on i , j . If $p \le 1$ then

$$\|\Sigma \lambda_{ij} k_{ij}\|^p_{p,r} \le \Sigma \|\lambda_{ij} k_{ij}\|^p_{p,r} \le c\, \Sigma |\lambda_{ij}|^p$$

and we are done. We now suppose $p \ge 1$.

Let

$$|D_{ij}|_r = \int_{D_{ij}} B^{-r} (\zeta , \zeta) dV(\zeta) .$$

We estimate $|D_{ij}|_r$ by first using Lemma 2.3 to replace $B^{-r} (\zeta , \zeta)$ in the integral by $B^{-r} (\zeta_{ij} , \zeta_{ij})$ and then evaluating the resulting integral by making the change of variables which sends ζ_{ij} to ζ_{01} . This yields

$$|D_{ij}|_r \sim \eta^{2(n+m)} B^{-(r+1)} (\zeta_{ij} , \zeta_{ij}) . \tag{2.5}$$

Let $\chi_{D_{ij}}$ be the characteristic function of D . Set

$$H(z) = \Sigma |\lambda_{ij}|\, |D_{ij}|_r^{-\frac{1}{p}} \chi_{D_{ij}}(z) .$$

Clearly $\|H\|^p_{p,r} = \Sigma |\lambda_{ij}|^p$.

We now claim that

$$|\Sigma \lambda_{ij} k_{ij}(z)| \le c\, \eta^{2(n+m)(1-\frac{1}{p})} (TH)(z) \tag{2.6}$$

(where T is defined in Lemma 2.8). As in Lemma 2.8, let $\alpha = 2r + 2 + \theta$, then

$$(TH)(z) = \Sigma |\lambda_{ij}|\, |D_{ij}|_r^{-\frac{1}{p}} \int_{D_{ij}} \frac{|B(z,\zeta)|^{\frac{\alpha}{p}}}{B(\zeta,\zeta)^{\frac{\alpha}{p}-r-1}} B(\zeta , \zeta)^{-r} dV(\zeta) .$$

Hence, by Lemma 2.3, $(TH)(z)$ is of size comparable to

$$\Sigma |\lambda_{ij}| \frac{|B(z,\zeta_{ij})|^{\frac{\alpha}{p}}}{B(\zeta_{ij},\zeta_{ij})^{\frac{\alpha}{p}-r-1}} B(\zeta , \zeta)^{-r} dV(\zeta) .$$

Now, using (2.5), we obtain

$$(TH)(z) \sim \Sigma|\lambda_{ij}|\eta^{2(n+m)(1-\frac{1}{p})} B(\zeta_{ij}, \zeta_{ij})^{-(r+1)(1-\frac{1}{p})-\frac{\alpha}{p}+r+1} |B(z, \zeta_{ij})|^{\frac{\alpha}{p}}$$

$$= \Sigma|\lambda_{ij}|\eta^{2(n+m)(1-\frac{1}{p})} |k_{ij}(z)|$$

which establishes (2.6). An appeal to Lemma 2.8 completes the proof of part b of the theorem for $p > 1$.

We now prove part (a) of the theorem. If suffices to give the proof for functions in a dense class with the appropriate norm estimates. Hence we can assume that the integrals in the proof are absolutely convergent. We start with a function $F(z)$ which we may (by the comment just made) assume to be in $A^{2,\alpha/p-1}$. By Lemma 2.1, we may write

$$F(z) = c \int_D \frac{B(z,\zeta)^{\frac{\alpha}{p}}}{B(\zeta,\zeta)^{\frac{\alpha}{p}-1}} F(\zeta)dV(\zeta) .$$

We will approximate $F(z)$ by a Riemann sum for the integral based on the decomposition $D = \cup D_{ij}$. That is, letting c denote the same constant as in the previous equation, set

$$G(z) = c \,\Sigma \frac{B(z,\zeta_{ij})^{\frac{\alpha}{p}}}{B(\zeta_{ij},\zeta_{ij})^{\frac{\alpha}{p}-1}} F(\zeta_{ij}) \,|D_{ij}| .$$

Thus, for any z in D

$$(2.7) \quad |F(z) - G(z)| \le c \,\Sigma \int_{D_{ij}} \left|F(\zeta) \frac{B(z,\zeta)^{\frac{\alpha}{p}}}{B(\zeta,\zeta)^{\frac{\alpha}{p}-1}} - F(\zeta_{ij}) \frac{B(z,\zeta_{ij})^{\frac{\alpha}{p}}}{B(\zeta_{ij},\zeta_{ij})^{\frac{\alpha}{p}-1}}\right| dV(\zeta)$$

$$\le c \,\Sigma \int_{D_{ij}} |F(\zeta) - F(\zeta_{ij})| \frac{|B(z,\zeta_{ij})|^{\frac{\alpha}{p}}}{B(\zeta_{ij},\zeta_{ij})^{\frac{\alpha}{p}-1}} dV(\zeta)$$

$$+ c \,\Sigma \int_{D_{ij}} |F(\zeta)| \left|\frac{B(z,\zeta)^{\frac{\alpha}{p}}}{B(\zeta,\zeta)^{\frac{\alpha}{p}-1}} - \frac{B(z,\zeta_{ij})^{\frac{\alpha}{p}}}{B(\zeta_{ij},\zeta_{ij})^{\frac{\alpha}{p}-1}}\right| dV(\zeta) .$$

By Lemma 2.6, the first sum can be estimated by

$$\Sigma_I \le c \sum_{ij} \eta^{2(n+m)+1} \frac{|B(z,\zeta_{ij})|^{\frac{\alpha}{p}}}{B(\zeta_{ij},\zeta_{ij})^{\frac{\alpha}{p}-1}} \int_{\overline{B}_i} |F(\zeta)|\,dV(\zeta)$$

By Lemma 2.3, this yields

$$\Sigma_I \le \sum_{ij} c\, \eta^{2(n+m)+1} \frac{|B(z,\zeta_i)|^{\frac{\alpha}{p}}}{B(\zeta_i,\zeta_i)^{\frac{\alpha}{p}-1}} \int_{\overline{B}_i} |F(\zeta)|\,dV(\zeta)\ .$$

The terms summed do not depend on j. Using elementary volume estimates, we see that for fixed i there are $O(\eta^{-2(n+m)})$ terms in the sum. Hence

$$\Sigma_I \le c\,\eta \sum_i \left(\int_{\overline{B}_i} |F(\zeta)|\,dV(\zeta)\right) \frac{|B(z,\zeta_i)|^{\frac{\alpha}{p}}}{B(\zeta_i,\zeta_i)^{\frac{\alpha}{p}-1}}\ .$$

We obtain the same majorant for the second sum on the right hand side of (2.7) as follows. Use Lemma 2.3 to obtain the estimate

$$\Sigma_{II} \le c\,\eta \sum_{ij} \left(\int_{D_{ij}} |F(\zeta)|\,dV(\zeta)\right) \frac{|B(z,\zeta_{ij})|^{\frac{\alpha}{p}}}{B(\zeta_{ij},\zeta_{ij})^{\frac{\alpha}{p}-1}}\ .$$

Then, using Lemma 2.3 again to replace ζ_{ij} by ζ_i, note that (by Lemma 2.4) $\bigcup_j D_{ij} \subset \overline{B}_i$.

Combining the estimates for Σ_I and Σ_{II}, and then using Lemma 2.5, yields

$$|F(z) - G(z)| \le c\,\eta \sum_i \left(\int_{\overline{\overline{B}}_i} |F(\zeta)|^p\, B^{-r}(\zeta\,,\,\zeta)\,dV(\zeta)\right)^{\frac{1}{p}} |B(z\,,\,\zeta_i)|^{\frac{\alpha}{p}}\, B(\zeta_i\,,\,\zeta_i)^{\frac{(1+r-\alpha)}{p}}$$

Define $k_i(z)$ by

$$k_i(z) = |B(z\,,\,z_i)|^{\frac{\alpha}{p}}\, B(z_i\,,\,z_i)^{\frac{(1+r-\alpha)}{p}}\ .$$

The previous estimate can be rewritten

$$|F(z) - G(z)| \leq c\,\eta \sum_i \Big(\int_{\overline{\overline{B_i}}} |F(\zeta)|^p\, B^{-r}(\zeta, \zeta)\,dV(\zeta)\Big)^{\frac{1}{p}} k_i(z)\ .$$

Since the $k_i(z)$ are the functions of the type used in the proof of part b of the theorem (only now for the 1 -lattice $\{z_i\}$), we conclude

$$\|F - G\|^p_{p,r} \leq c\,\eta^p \sum_i \Big(\int_{\overline{\overline{B_i}}} |F(\zeta)|^p\, B^{-r}(\zeta, \zeta)\,dV(\zeta)\Big)\ .$$

Since the $\overline{\overline{B_i}}$ are almost disjoint (Lemma 2.4(c)), this implies

$$\|F - G\|^p_{p,r} \leq c\,\eta^p\, \|F\|^p_{p,r}\ . \tag{2.8}$$

Since the size of c is independent of η, we may choose η so small that $c\,\eta^p \leq (\frac{1}{2})^p$. Thus, denoting by A the linear operator which sends F to the approximating sum G, we have the operator norm estimate

$$\|I - A\| \leq \frac{1}{2}\ .$$

Hence the operator $A = I - (I - A)$ is invertible. Thus F can be written as $F = AH$ for some H with norm comparable to that of F. That is the required conclusion. (Note that H can be obtained constructively as $H = \sum_{k=0}^{\infty} (I-A)^k F$.)

The proof of the theorem is complete.

Since every bounded symmetric domain D is biholomorphically equivalent to a domain U to which Theorem 2 applies and since the $\theta = 0$ version of Theorem 2 is invariant under biholomorphic changes of variables, Theorem 1 follows from Theorem 2. In more detail, the relation between the Bergman kernels for the two domains is that for any z, ζ in U and g a biholomorphic map of U to D

$$B_D(g(z), g(\zeta))\, \det g'(z)\, \overline{\det g'(\zeta)} = B_U(z, \zeta)\ . \tag{2.9}$$

Suppose f is in $A^p(D)$. Using g to change variables one verifies directly that $F(\zeta) = f(g(\zeta))\,(\det g'(\zeta))^{2/p}$ is in $A^p(U)$ and has the same norm as f. Theorem 2 can be used to give a decomposition of F. Using (2.9) this is seen to be equivalent to having a decomposition of f of the type described in Theorem 1.

The same considerations also yield the analog of Theorem 1 for the spaces $A^{p,r}, r > -\epsilon_D$. We know of no reason to doubt that the full analog of Theorem 2 (that is, including the case $\theta \neq 0$) is true for this class of domains.

The limiting case of these results as p becomes large is a description of the space $(A^{1,r}(D))^*$, the dual space of $A^{1,r}$. We will denote the norm of a linear functional L on $A^{1,r}$ by $\|L\|_{*,r} = \|L\|_*$. Our description of the dual space will be with respect to the weighted pairing. Set

$$\langle f, g\rangle_r = \int_D f(\zeta)\overline{g(\zeta)}B(\zeta, \zeta)^{-r}\, dV(\zeta) .$$

Theorem 2'. *Let D be a symmetric Siegel domain of type two and ε_D the positive constant associated with D by Theorem 2. Suppose $r > -\varepsilon_D$. There is an $\eta_0 = \eta_0(D, r)$ such that if the points $\{\zeta_i\}$ are an η-lattice, $\eta < \eta_0$, then the space $(A^{1,r}(D))^*$ has the following description:*

(a) *Every L in $(A^{1,r})^*$ can be realized as a weak $*$ convergent series*

$$L = \sum_{i=1}^{\infty} \lambda_i \left(\frac{B(z,\zeta_i)}{B(\zeta_i,\zeta_i)}\right)^{1+r} . \tag{2.9}$$

That is, $L(f) = \lim_n \langle f, \sum_{i=1}^{n} \lambda_i \left(\frac{B(z,\zeta_i)}{B(\zeta_i,\zeta_i)}\right)^{1+r} \rangle_r$.

The constants λ_i may be chosen so that $\sup |\lambda_i| \leq c\|L\|_{*,r}$.

(b) *If $\sup |\lambda_i| < \infty$ then (2.9) is a weak $*$ convergent series and L defined by (2.9) has* $\|L\|_{*,r} \leq c \sup |\lambda_i|$.

(c) *If L is represented by (2.9) and α is positive then $K_\alpha L$ defined by*

$$K_\alpha L = \sum_{i=1} \lambda_i \frac{B(z,\zeta_i)^{1+r+\alpha}}{B(\zeta_i, \zeta_i)^{1+r} B(z,z)^{\alpha}} \tag{2.10}$$

is a bounded function with $\|K_\alpha L\|_\infty \leq c\|L\|_{*,r}$ which represents L in the following sense. There is a constant $d = d(\alpha, r)$ such that

$$L(f) = d\langle f, K_\alpha L\rangle_r \tag{2.11}$$

(d) *If the numbers λ_i are a given bounded sequence and $K_\alpha L$ and L are defined by (2.10) and (2.11) then*

$$\|L\|_{*,r} \leq c\|K_\alpha L\|_\infty \leq c^2 \sup |\lambda_i|$$

The constant c can be chosen to depend only on D, r, η and α.

Informally, (a) and (b) of the theorem are the limiting case, $\alpha = 0$, of (c) and (d). However, the actual situation is a bit more complicated. The sum (2.9) need not converge pointwise to a function $L(z)$ and need not converge in the norm of $(A^{1,r})^*$. These issues and others are discussed after the proof.

The proof is very similar to that of Theorem 2, hence we only give an outline.

Proof outline. We start with an L in $(A^{1,r})^*$. By the Hahn-Banach theorem there is a bounded function b such that

(2.12) $$L(F) \doteq \langle F , b\rangle_r$$

for all F in $A^{1,r}$. For any fixed positive α , F can be represented by the reproducing formula

$$F(\zeta) = c \int \frac{B(\zeta,z)^{1+r+\alpha}}{B(z,z)^\alpha} F(z)B(z , z)^{-r} \, dV(z) .$$

Hence $L(F) = \langle F , L_\alpha\rangle$ with L_α defined by

$$L_\alpha(z) = \int \frac{B(z,\zeta)^{1+r+\alpha}}{B(z,z)^\alpha B(\zeta,\zeta)^r} b(\zeta)dV(\zeta) .$$

Now note that $B_z(\zeta) = \dfrac{B(\zeta,z)^{1+r+\alpha}}{B(z,z)^\alpha}$ is uniformly (with respect to z) in $A^{1,r}$ (as a function of ζ) . Hence $L(B_z)$ is uniformly bounded. However $L(B_z) = \overline{L_\alpha(z)}$. Thus $|L_\alpha(z)| \le c\|L\|_*$. Conversely, using Theorem 2, $|L(F)| \le c\|L_\alpha\|_\infty\|F\|_{1,r}$. Hence $\|L_\alpha\|_\infty \sim \|L\|_*$. We now go back to (2.12) and replace b by L_{α_0} for some fixed positive α_0 . This gives a choice of b which is not holomorphic but which is a function whose modulus of continuity can be estimated.

We would now like to replace the integral representation for $L_\alpha(z)$ with a sum of the type

(2.13) $$L_\alpha(z) = \sum \lambda_i \frac{B(z,\zeta_i)^{\alpha+r}}{B(z,z)^\alpha B(\zeta_i,\zeta_i)^r} ,$$

with ζ_i points of a given η-lattice and λ_i a bounded sequence of scalars. To do this, we start with the integral formula defining L_α. Let $D = \cup D_i$ be the covering of D associated with the η-lattice (analogous to the D_{ij} in the proof of Theorem 2).

$$B(z\,,\,z)^{\alpha}L_{\alpha}(z) = \int \frac{B(z,\zeta)^{1+\alpha+r}}{B(\zeta,\zeta)^{r}} L_{\alpha_0}(\zeta)\, dV(\zeta)$$

$$= \sum_i \frac{B(z,\zeta_i)^{1+\alpha+r}}{B(\zeta_i,\zeta_i)^{r}} L_{\alpha_0}(\zeta_i)\,|D_i|$$

$$+ \sum_i \int_{D_i} \frac{B(z,\zeta)^{1+\alpha+r}}{B(\zeta,\zeta)^{r}} [L_{\alpha_0}(\zeta) - L_{\alpha_0}(\zeta_i)]\, dV(\zeta)$$

$$+ \sum_i \int_{D_i} \left[\frac{B(z,\zeta)^{1+\alpha+r}}{B(\zeta,\zeta)^{r}} - \frac{B(z,\zeta_i)^{1+\alpha+r}}{B(\zeta_i,\zeta_i)^{r}}\right] L_{\alpha_0}(\zeta_i)\, dV(\zeta)$$

$$= \Sigma_1 + \Sigma_2 + \Sigma_3\,.$$

To estimate the terms in Σ_2, note that

$$|L_{\alpha_0}(\zeta) - L_{\alpha_0}(\zeta_i)| \le c \int \left|\frac{B(z,\zeta)^{1+\alpha_0+r}}{B(\zeta,\zeta)^{\alpha_0}} - \frac{B(z,\zeta_i)^{1+\alpha_0+r}}{B(\zeta_i,\zeta_i)^{\alpha_0}}\right| B(z\,,\,z)^{-r}\, dV(z)$$

By Lemma 2.3, this is dominated.

$$\le c\,\eta \int \frac{|B(z,\zeta_i)|^{1+\alpha_0+r}}{B(\zeta_i,\zeta_i)^{\alpha_0}} B(z\,,\,z)^{-r}\, dV(z)$$

which, by Lemma 2.2, is dominated by $c\,\eta$. Hence $B(z\,,\,z)^{-\alpha}\,\Sigma_2$ is dominated by

$$c\,\eta\, \|L\|_{*} \int_D |B(z\,,\,\zeta)|^{1+\alpha+r}\, B(\zeta\,,\,\zeta)^{-r}\, B(z\,,\,z)^{-\alpha}\, dV(\zeta)\,.$$

By Lemma 2.2, this quantity is dominated by $c\,\eta\,\|L\|_{*}$. Application of Lemmas 2.2 and 2.3 yield a similar estimate for $B(z\,,\,z)^{-\alpha}\,\Sigma_3$. Hence

$$\|L_{\alpha_0}(z) - B(z\,,\,z)^{-\alpha}\,\Sigma_1\|_{\infty} \le c\,\eta\,\|L\|_{*}$$

Now note that the bounded function $h(z) = L_{\alpha}(z) - B(z\,,\,z)^{-\alpha}\,\Sigma_1$ has the following property. If we start with the linear functional given by $L_1(F) = \langle F\,,\,h\rangle$ and form the associated bounded function $L_{1,\alpha}$ then we recapture the function h. This fact is a direct consequence of the various definitions and of Lemma 2.2'. Since h is its own representing function we can iterate the previous construction. If $\eta < \eta_0$ and η_0 is sufficiently small, then iteration of this process will converge and will give a representation of $L_{\alpha}(z)$ as a sum. Since

$|D_i| \sim B^{-1}(\zeta_i , \zeta_i)$ the sum is of the form (2.13). Also, by construction, if F is any function in $A^{1,r}$ then

$$L(F) = \langle F , L_\alpha \rangle$$

$$= \Sigma \lambda_i \left\langle F , \frac{B(z,\zeta_i)^{\alpha+r}}{B(z,z)^\alpha B(\zeta_i,\zeta_i)^r} \right\rangle_r$$

To complete the proof we note that representations of functionals given by the function in (2.10) are equivalent to those given by (2.9). The reason for this is the observation that there is a constant $d = d(\alpha , r , D)$ such that

$$d \left\langle F , \frac{B(z,\zeta_i)^{\alpha+r}}{B(z,z)^\alpha B(\zeta_i,\zeta_i)^r} \right\rangle_r = \left\langle F , \frac{B(z,\zeta_i)^r}{B(\zeta_i,\zeta_i)^r} \right\rangle_r$$

for all F and ζ_i. This equality is an immediate consequence of Lemma 2.2' and of the representation of F in $A^{1,r}$ given in Theorem 2.

The proof outline is complete.

Using Riemann-Liouville fractional integrals, the previous result can be reformulated in a way which emphasizes the analogy between these dual spaces and the space of Bloch functions on the unit disk. Let $\mathfrak{D}^\alpha = \mathfrak{R}^{(2d-q)\alpha}$ where $\mathfrak{R}$ is the Riemann-Liouville fractional integral operator for the domain D. This operator is described in a bit more detail in the Appendix and in full in [11]. The fact of interest to us is

$$(\mathfrak{D}^\alpha B^\beta (\cdot , \zeta))(z) = c_{\alpha\beta} B^{\alpha+\beta} (z , \zeta) .$$

Hence the operator $\mathfrak{D}^\epsilon$ for various positive ϵ is the operator which establishes the equivalence of the various decompositions in Theorem 2'. Also, formally, L is in $A^{1,r}$ if and only if L has the representation (2.9) and

(2.14) $$\|B(z , z)^{-\alpha} \mathfrak{D}^\alpha L\|_\infty \leq c_\alpha \|L\|_* .$$

However, L is not really a function. Consider the case of D the upper half plane and $r = 0$. The sum (2.9) is the Riemann sum for the integral

$$L(z) = \int b(\zeta) \frac{1}{(z-\bar{\zeta})^2} dV(\zeta)$$

for some bounded b. However, if we choose $b(\zeta) = (i - \zeta)^2 \, |i - \zeta|^{-2}$ then the integral defining $L(i)$ diverges. However, for any z, the integral for $L(z) - L(i)$ converges absolutely. Hence the integral for $L(z)$ diverges for all z.

Since the functions in $A^{1,0}$ all have mean zero, the "function" $L(z)$ need only be specified up to an addative constant. Hence the absolutely convergent integral for $L(z) - L(i)$ can be used to realize the functional L as a function. The situation for general domains is not clear.

If the domain being considered is the unit ball in $\mathbb{C}^n$ then the very simple formula in Lemma 2.2' is not valid. However, in that case, series such as (2.9) are the discrete analogs of absolutely convergent integrals and results analogues to Theorem 2' can be obtained directly.

If the domain being considered is the disk, then the spaces described by the representations (2.9) for different r are the same; that is, a function has a representation in the form (2.9) for some r if and only if it has a representation for all r. One way to see this is to use the known result that (modulo polynomials) the space of holomorphic functions on the disk for which

$$\sup \left| (1 - |z|^2)^{\alpha} f^{(\alpha)}(z) \right| < \infty$$

doesn't depend on α (even for non-integer α.)

It is an interesting open problem to find the appropriate generalization of these observations.

§3. Spaces of harmonic functions.

The main result of this section is that the natural analog of Theorem 1 is valid for spaces of harmonic functions defined on the unit ball in $\mathbb{R}^n$. Many of the technical details of the proof are quite different from those of the previous section. Instead of working with reproducing kernels obtained from the Bergman kernel, we work with kernels obtained from the Poisson kernel. The automorphism group of the ball is not used; instead local estimates are made directly. However, the basic structure of the proof is the same as the proof of Theorem 2 and at certain points we will only sketch the proof. We also obtain results corresponding to the end point $p = \infty$ which is analogous to Theorem 2'.

Some further remarks on the relation of these results to those of the previous section are presented at the end of this section.

We begin with some notation. Fix $n \geq 2$. Let B^n denote the unit ball in $\mathbb{R}^n$. For x in $\mathbb{R}^n$, let $|x|$ be the length of x, x' the vector of unit length parallel to x; that is, $x' = x/|x|$, and let $\tilde{x}$ denote the inversion of x in the unit sphere, $\tilde{x} = x/|x|^2$. We will use the notation $\varepsilon(x) = 1 - |x|$. Denote Euclidean volume measure by dx and surface measure on the unit sphere $\Sigma_{n-1} = \partial B^n$ by $d\sigma$.

Let $Y_j^k(x)$, $j = 1, 2, \dots, d(k, n)$ be a real orthonormal (on Σ_{n-1}) basis for the spherical harmonics of degree k. Hence any harmonic function $h(x)$ on B^n can be represented as

$$h(x) = \sum_{k,j} a_{k,j} Y_j^k(x) = \sum a_{k,j} |x|^k Y_j^k(x').$$

(See, e.g., Chapter 4 of [24] for this and related points.) If $a_{k,j} = 0$ for $k \leq r$ then we say that $h(x)$ vanished to order r (at the origin).

We will frequently use polar coordinates and will often write $x = Rx'$ and $y = ry'$. With this notation, the Poisson kernel for harmonic functions is given by

$$P(x, y) = c\frac{1-(rR)^2}{(1-2rRx'y'+r^2R^2)^{\frac{n}{2}}} = \sum_{k,r} (Rr)^k Y_r^k(x')Y_r^k(y').$$

We will need reproducing formulas for harmonic functions in the various spaces $L^2(B^n, (1-|x|)^m dx)$. For non-negative integers m let

(3.1)
$$\begin{aligned} b_m(x, y) &= \sum_{k,r} \frac{(2k+n+m)!}{(2k+n-1)!} (Rr)^k Y_r^k(x')Y_r^k(y') \\ &= \left[\rho^{1-n}\left(\frac{\partial}{\partial\rho}\right)^{m+1} \rho^{n+m} P(Rx', \rho^2 y')\right]_{\rho=\sqrt{r}} \end{aligned}$$

<u>Proposition 3.1.</u> <u>If</u> $g(x)$ <u>is harmonic and bounded in</u> B^n <u>then</u>

$$g(x) = c_{n,m} \int_{B^n} g(y)\, b_m(x, y)(1 - |y|)^m\, dy.$$

<u>Proof.</u> It suffices to verify the formula for each Y_r^k. That verification is

immediate when the integration is done in polar coordinates and the orthonormality of the Y_r^k on the sphere is used.

For η with $0 < y \leq 1$, we define an η lattice in B^n to be a set of points. $\{y_i\} \subset B^n$ such that the balls B_i with center at y_i and radius $(1 - |y_i|)\eta/10$ are disjoint and the balls $\overline{B_i}$ with the same center and radius $(1 - |y_i|)\eta$ cover B^n.

For $0 < p < \infty$ and $\alpha > -1$, denote by $A_H^{P,\alpha}(B^n) = A_H^{P,\alpha}$ the intersection of $L^P(B^n, (1-|x|)^\alpha\, dx)$ with the space of harmonic functions. Our main result in this section is the following.

Theorem 3. *Suppose* n, m, α, *and* p *are given with*

n *an integer* $n \geq 2$,

$0 < p < \infty$

$\alpha > -1$ *and* $\alpha < p - 1$ *if* $p > 1$, *and*

m *an integer*, $m > \max(0, n(\frac{1}{p} - 1))$

then there is an η*-lattice* $\{y_i\}$ *in* B^n *so that*

(a) *If* $f(x)$ *is in* $A_H^{P,\alpha}$ *then there are numbers* λ_i *so that*

$$f(x) = \Sigma \lambda_i b_m(x, y_i)(1-|y_i|)^{m+n-(m+\alpha)/p} \tag{3.2}$$

and

$$\Sigma|\lambda_i|^P \leq c\|f\|_{P,\alpha}^P \quad \text{and}$$

(b) *If* $\Sigma|\lambda_i|^P < \infty$ *then the function* $f(x)$ *defined by (3.2) is in* $A_H^{P,\alpha}$ *and satisfies*

$$\|f\|_{P,\alpha}^P \leq c\, \Sigma|\lambda_i|^P$$

The constants c *depend on* p, α, n, m *and* η *but not on* f *or the particular choice of* $\{y_i\}$.

Note. As before, the sum in (3.2) converges uniformly and absolutely on compact subsets of B^n and converges in $A_H^{P,\alpha}$ norm. The λ_i can be chosen in a continuous and linear way and it is not claimed that the representation in (3.2) is unique. Also, it is not clear if all the restrictions on α, m, and p are actually necessary. In particular, it is not clear that $p - 1 > \alpha$ is needed.

Before starting, we need several lemmas. The first collects information about the reproducing kernels.

Lemma 3.2. If $x , y \in B^n$ then

(a) $|b_m(x , y)| \leq c|x - \tilde{y}|^{-n-m}$

(b) $|\nabla_y b_m(x , y)| \leq c|x - \tilde{y}|^{-n-m-1}$

(c) For $m > n(\frac{1}{p} - 1)$,

$$\int_{B^n} |b_m(x , y)(1 - |y|)^{m+n-\frac{n}{p}}|^p \, dx \leq c$$

(d) For $m > n(\frac{1}{p} - 1) - 1$

$$\int_{\Sigma_{n-1}} |b_m(x , y)(1 - |y|)^{m+n-\frac{n-1}{p}}|^p \, d\sigma(x) \leq c .$$

Proof. The results are trivial if $|y| \leq .9$. We now suppose $1 > |y| > .9$. When the differentiation in (3.1) is done explicitly, the dominant term obtained is

$$\tilde{b}_m(x , y) = (\frac{d}{dr})^{m+1} P(Rx' , ry') .$$

We will outline the proof of (a) and (b) for $\tilde{b}_m$, the other terms are treated by similar or simpler arguments. The Poisson kernel can be written as

$$P(x , y) = \frac{1}{r^n|x - \tilde{y}|^n}$$

hence

$$\tilde{b}_m = k_1(x , y)(1 - (rR)^2)|x - \tilde{y}|^{-(n+m+1)}$$

$$+ k_2(x , y)|x - \tilde{y}|^{-(n+m)}$$

for some bounded functions $k_i(x , y)$. Part (a) of the lemma follows from this equation and the estimate

$$|x - \tilde{y}| \geq |\tilde{y}| - |x| = \frac{1}{r} - R = \frac{1}{r}(1 - Rr) .$$

Part (b) of the lemma follows from the same considerations and the fact that the functions $k_i(x , y)$ have bounded derivatives.

Parts (c) and (d) of the lemma are standard consequences of the estimates in part (a). We only outline the proofs. Let $\varepsilon = \varepsilon(y) = 1 - |y|$. By elementary geometry

$$|x - \tilde{y}| \sim |x - y'| + \varepsilon(y) \quad . \tag{3.3}$$

Hence, by (a) of the lemma the integral in part (c) is dominated by

$$\int_{B^n} \frac{\varepsilon^{mp+np-n}}{(|x-y'|+\varepsilon)^{(m+n)p}} \, dx$$

This integral can be estimated as follows. First expand the region of integration to all of $\mathbb{R}^n$. Then make an affine change of variable to eliminate y'. One then uses a dilation of the integration variable to see that the integral does not depend on ε. Finally, by going to polar coordinates, one checks that the resulting integral is finite for the indicated range of m, n, and p.

Part (d) of the lemma is proved by similar argument after first noting that the integral over Σ_{n-1} can be estimated by an analogous integral in $\mathbb{R}^{n-1}$.

The next lemma is the replacement we need for Lemmas 2.7 and 2.8.

<u>Lemma 3.3</u>. <u>Suppose</u> $1 < p < \infty$, $m > 0$, $-1 < \alpha < p - 1$. <u>The operator mapping the function</u> $f(x)$ <u>defined on</u> B^n <u>to the function</u> $(Kf)(x)$ <u>defined by</u>

$$(Kf)(x) = \int_{B^n} |b_m(x, y)|(1 - |y|)^m f(y) \, dy$$

<u>is a bounded linear map of</u> $L^p(B^n, (1 - |x|)^\alpha dx)$ <u>to itself</u>.

<u>Proof</u>. By the previous lemma and (3.3)

$$(Kf)(x) \le c(Sf)(x) = c \int_{B^n} \frac{\varepsilon(y)^m}{(|x-y|+\varepsilon(y))^{n+m}} f(y) \, dy \; .$$

The operator Sf is the adjoint of the operator which sends g to $S^* g$ given by

$$S^* g(y) = c \frac{1}{\varepsilon(y)^n} \int_{B^n} \frac{1}{(1+\frac{|x-y|}{\varepsilon(y)})^{n+m}} \, dx$$

(Here we are realizing the dual space of $L^p(B^n, \varepsilon(x)^\alpha dx)$ as $L^{p'}(B^n, \varepsilon(x)^{-\frac{\alpha p'}{p}} dx)$ with the pairing $\langle f, g\rangle = \int fg \, dx$.) To see that the operator norm of S^* on the dual space is finite, we dominate $S^* g$ pointwise by Mg, the Hardy-Littlewood maximal function of g. Then note that by the theory of weights [2], the Hardy-

Littlewood maximal operator is bounded on $L^{p'}(B^n, \epsilon(x)^{-\frac{ap'}{p}} dx)$ if $\frac{-\alpha p'}{p} > -1$.

We now prove the theorem.

Proof of Theorem 3. First we prove (b). The case $p \leq 1$ follows from Lemma 3.2. Now suppose $p > 1$. Start with an η -lattice $\{y_i\}$. As in section 2, construct $\overline{B_i}$ so that $y_i \in B_i \subseteq \overline{B_i}$, the B_i are disjoint, and $\cup \overline{B}_i = B^n$. Let

$$g(x) = \Sigma|\lambda_i| \|\chi_{B_i}\|_{p,\alpha}^{-1} \chi_{B_i}(x) .$$

Clearly $\|g\|_{p,\alpha}^p = \Sigma|\lambda_i|^p$. Hence, by the previous lemma

$$(Kg)(x) = \Sigma|\lambda_i| \|\chi_{B_i}\|_{p,\alpha}^{-1} \int_{B_i} |b_m(x , y)| (1 - |y|)^m dy$$

is in $L^p(B^n , (1 - |x|)^\alpha dx)$. $\epsilon(y)$ varies by a bounded factor as y varies over $\overline{B_i}$ and $b_m(x , y_i)$ equals its mean value over B_i. Hence the function $f(x)$ of (3.2) is easily seen to satisfy

$$|f(x)| \leq c(Kg)(x) .$$

Thus $f(x)$ is in $A_H^{p,\alpha}$ and satisfies the required norm estimate.

In order to prove part (a) we proceed as we did for Theorem 2. Start with a λ -lattice for fixed small λ. Call it $\{y_i\}$. Now choose an η -lattice $\{w_j\}$ for some η, $\eta << \lambda$. The size of η and λ are to be selected at the end of the proof using the same considerations as those used in selecting η at the end of the proof of Theorem 2. As in the previous section, rename the $\{w_j\}$ as $\{y_{ij}\}$ where $y_{ij} \in \overline{B}_i$. ($\overline{B}_i$ is the disjoint covering associated with the y_i). Let B_{ij} be the disjoint cover of B^n associated with the η -lattice and constructed in a way similar to the construction of the D_{ij} in Section 2. To prove part (a) of the theorem, we start with $F(x)$, and then use Proposition 3.1 to write

$$F(x) = c_{nm} \int_{B^n} F(y) b_m(x , y)(1 - |y|)^m dy .$$

We approximate this by

$$G(x) = c_{nm} \sum_{ij} \int_{B_{ij}} F(y_{ij})b_m(x , y_{ij})(1 - |y_{ij}|)^m dy .$$

In addition to the estimates in Lemma 3.3, we also need the following

Lemma 3.4. *If f is harmonic on B^n then*

$$\sup_{x\in B_{ij}} |\nabla h| \leq (\text{Diameter } B_i)^{-n-1} \int_{B_i} |h(x)|\, dx .$$

Proof. For the unit ball, this type of result follows from, for example, Lemma 3.2(b). The general result then follows by translation and dilation.

Finally we need a substitute for the subharmonicity of $|f(z)|^p$.

Lemma 3.5. *If f is harmonic on B^n and $0 < p < \infty$ then*

$$\frac{1}{|B_i|}\int_{B_i} |h(x)|dx \leq C_{n,p}\left(\frac{1}{|B_i|}\int_{\overline{B_i}} |h(x)|^p\, dx\right)^{\frac{1}{p}}$$

Proof. For $p \geq 1$, this follows from Holder's inequality. For $p < 1$, it is a direct consequence of a result of Kuran [16] or of Fefferman and Stein (Section 9, Lemma 2 of [8]).

Once all these estimates are available, the proof is completed exactly as the proof of Theorem 2.

We now consider the limiting, $p = \infty$, case of these results. We obtain a characterization of the dual space $(A_H^{1,\alpha})^*$ for the case when α is a non-negative integer. We will describe the dual of $A_H^{1,\alpha}$ with respect to the pairing

$$\langle f, g\rangle = \int_B f(x)\overline{g(x)}(1-|x|)^\alpha\, dx . \tag{3.4}$$

That is, we wish to describe the dual space of $A_H^{1,\alpha}$ as a space of harmonic functions g such that

$$|\langle f, g\rangle| \leq c_g \|f\|_{1,\alpha} .$$

We then define $\|g\|_*$ to be the smallest such c_g and wish to find an intrinsic characterization of $\|g\|_*$. (To avoid consideration of integrals which do not converge absolutely, we will obtain an a priori estimate for a dense class of functions f).

Theorem 3'. *Suppose n is an integer, $n \geq 2$ and that α is a non-negative integer. There is an η-lattice $\{y_i\}$ (η only depends on α, n) such that the* following conditions are equivalent for a harmonic function defined on B^n.

(a) $b(x)$ is in $(A_H^{1,\alpha})^*$ with respect to the pairing (3.4).

(b) For some bounded $\ell(y)$,

$$b(x) = \int_{B^n} b_\alpha(x, y)(1-|y|)^\alpha \ell(y)\, dy . \tag{3.5}$$

In this case $\ell(y)$ can be chosen with $\|\ell\|_\infty \sim \|b\|_*$.

(c) $\sup_{x\in B^n} (1-|x|)|\nabla b(x)| = c < \infty$.

In this case $c \sim \|b\|_*$.

(d) For some bounded sequence $\{\lambda_i\}$

$$b(x) = \sum_i \lambda_i b_\alpha(x, y_i)(1-|y_i|)^{n+\alpha} .$$

In this case $\{\lambda_i\}$ can be chosen with $\sup |\lambda_i| \sim \|b\|_*$.

(e) $b(x)$ is in B.M.O.(B^n).

In this case $\|b\|_{BMO} \sim \|b\|_*$.

(f) The harmonic function $B(x)$ such that $b(x) = x \cdot \nabla B(x)$ has boundary values in the class $\Lambda^*(\Sigma_{n-1})$ of Zygmund. In this case $\|b\|_{\Lambda^*} \sim \|b\|_*$.

Furthermore, a harmonic function b which satisfies any of the conditions b, c, d, e, f is in $(A_H^{1,\alpha})^*$ and has $\|b\|_*$ dominated by the appropriate quantity.

Note. Three of the conditions do not involve the index α. Hence the space defined by these conditions does not depend on α. Also, the requirement that α be an integer is forced by the fact that we have only defined b_α for integer α. However, the proof of the theorem actually shows that the space described is also the dual of $A_H^{1,\alpha}$ even if α is not an integer.

First, we recall some definitions. For x in B^n and $r > 0$ let $B(r, x)$ be the intersection of B^n with the ball centered at x with radius r. For f a function defined on B^n, x a point in B^n, and $r > 0$, set

$$M_{r,x}(f) = \frac{1}{|B(r,x)|} \int_{B(r,x)} f(y)\, dy .$$

A function f defined on B^n is said to be in B.M.O.(B^n) (to have bounded mean oscillation on B^n) if

$$\sup_{x\in B^n, r>0} M_{r,x}(|f - M_{r,x}(f)|) = c < \infty .$$

The B.M.O. norm of such a function is given by

$$\|f\|_{B.M.O.} = c + |\textstyle\int f| .$$

We will also be considering the related space of functions of local bounded mean oscillation, $B.M.O._{LOC}(B^n)$ which is defined and normed in the same way but with the additional restruction $r < (1 - |x|)$. A general reference for this type of space is [3]. This type of local B.M.O. space in this context is discussed in [5].

A function $B(x')$ defined on Σ_{n-1} is said to be in the Zygmund class. $\Lambda^*(\Sigma_{n-1})$ if for all rotations ρ of Σ_{n-1} by angle of size at most h

$$|B(\rho x') - 2B(x') + B(\rho^{-1}x')| \leq ch .$$

The infimum of all such constants c (plus the modulus of B at some fixed point) gives the norm $\|B\|_{\Lambda^*}$. These spaces are discussed in detail in [6]. (Although the presentation there is $n = 2$, the ideas generalize in a direct way to higher dimensions.)

<u>Proof of Theorem 3'</u>. Suppose $b(x)$ satisfies (a) . By the Hahn-Banach Theorem, we can find a bounded $\ell(x)$ so that for all f in $A_H^{1,\alpha}$

$$\langle f , b\rangle = \int f(x)\ell(x)(1 - |x|)^\alpha \, dx .$$

By Proposition 3.1, we can write

$$f(x) = c \int f(y)b_\alpha(y , x)(1 - |y|)^\alpha \, dy .$$

Substituting this into the previous equation and interchanging order of integration yields

$$\langle f , b\rangle = c \int f(y) \left(\int \ell(x)b_\alpha(x , y)(1 - |x|^\alpha)dx\right)(1 - |y|)^\alpha \, dy .$$

Hence, $b(y)$ is given in the form (3.5). Thus (a) implies (b).

Suppose b is given by (3.5). We wish to show that b satisfies (c). It suffices to show

$$\sup_{x\in B^n} (1 - |x|) \int |\nabla_x b_\alpha(x , y)|(1 - |y|)^\alpha \, dy < \infty .$$

Let $\varepsilon(x) = 1 - |x|$ and $\varepsilon(y) = 1 - |y|$. Note $|x - \tilde{y}| \sim |x' - y'| + \varepsilon(x) + \varepsilon(y)$. Hence by Lemma 3.2 (b) it suffices to show

$$\sup_x \int_{B^n} \frac{\varepsilon(x)\varepsilon(y)^{\alpha}}{(|x'-y'|+\varepsilon(x)+\varepsilon(y))^{n+\alpha+1}}\, dy < \infty$$

changing to polar coordinates and making trivial estimates for the case $|x| < \frac{1}{2}$, and the part of the integral involving $|y| < \frac{1}{2}$, it suffices to estimate

$$\sup_{0<\varepsilon<\frac{1}{2}} \int_0^{\frac{1}{2}} \int_{\Sigma_{n-1}} \frac{\varepsilon\ \varepsilon(y)^{\alpha}}{(|x'-y'|+\varepsilon+\varepsilon(y))^{n+\alpha+1}}\, d\sigma(x')d\varepsilon(y) .$$

Denoting $\varepsilon(x)$ by t, estimating the integral over Σ_{n-1} by the corresponding integral over $\mathbb{R}^{n-1}$, and changing variables to put x' at the origin of $\mathbb{R}^{n-1}$, we obtain the following expression to be estimated

$$\sup_{0<\varepsilon<\frac{1}{2}} \int_0^{\frac{1}{2}} \int_{\mathbb{R}^{n-1}} \frac{\varepsilon\ t^{\alpha}}{(|y'|+\varepsilon+t)^{n+\alpha+1}}\, dy'\ dt .$$

Now introduce polar coordinates on $\mathbb{R}^{n-1}$. Thus it suffices to estimate

$$\sup_{\varepsilon>0} \int_0^{\infty}\int_0^{\infty} \frac{\varepsilon t^{\alpha}\ r^{n-2}}{(r+\varepsilon+t)^{n+\alpha+1}}\, dr\ dt$$

Making the change of variables which sends t to εt and r to εr shows that the double integral is finite and independent of ε.

We now show (c) implies (a). If b satisfies (c) and f is in $A_H^{1,\alpha}$ then the mapping of f into

$$\ell(f) = \int_{B^n} f(x)(1 - R)\frac{\partial}{\partial R} b(Rx')(1 - |x|)^{\alpha}\, dx$$

is a bounded functional on $A_H^{1,\alpha}$. Write

$$b(x) = \Sigma\, b_{kj}\, R^k\, Y_j^k(x') .$$

Direct computation and then integration by parts yields

$$\ell(Y_j^k(x)) = c_n\, b_{kj}\, k \int_0^1 r^{2k+n-2}\ (1 - r)^{\alpha+1}\, dr$$

$$= c_{n,\alpha}\, b_{k,j}\, \frac{k}{2k+n-1} \int_0^1 r^{2k+n-1} (1-r)^{\alpha}\, dr .$$

Since $\frac{k}{2k+n-1}$ is essentially constant for large k, it follows easily that the linear functional L defined by

$$L(Y_j^k) = c_n\, b_{k,j} \int_0^1 r^{2k+n-1} (1-r)^{\alpha}\, dr$$

is also bounded on $A_H^{1,\alpha}$. But

$$L(f) = \int_{B^n} f(x) b(x) (1-|x|)^{\alpha}\, dx$$

thus (a) holds.

Now that we know $\|b\|_* \leq c \sup (1-|x|)(\nabla b(x))$ we can go back to the computation used in showing that (b) implies (c) and read it as a proof that the map of L^∞ into $(A_H^{1,\alpha})^*$ given by (3.5) is bounded. One that is known, the proof of the equivalence of (d) to the other conditions goes exactly as the proof in Theorem 3.

The equivalence of conditions (c) and (f) in two dimensions is a well known result of Zygmund. The proof given in [6] extends in a straightforward way to $\mathbb{R}^n$.

To obtain (c) from (f), one writes $\nabla b(x)$ as the integral of $b(y) - b(x)$ against the appropriate kernel defined on the shell $\{y ; \frac{1}{2}(1-|x|) < |y-x| < 1-|x|\}$. The integral is then dominated by the L^∞ norm of the kernel times the L^1 norm of $|b(y) - b(x)|$ on the shell. This last quantity is dominated by $\|b\|_{B.M.O.}$.

To show that (c) implies (f) one first notes that a function which satisfies (c) is in local B.M.O. Thus the proof of the theorem is complete as soon as we have the following.

Lemma 3.6. $B.M.O._{LOC}(B^n) = B.M.O.(B^n)$.

This lemma is implicit in recent work by P. Jones [28]. (The proof of Lemma 2.11 of [28] also yields Lemma 3.6.)

The proof that (b) implies (c) in Theorem 3' rests on the fact that a certain linear operator is a bounded map from L^{∞} to $B.M.O.(B^n)$. We could use this fact as the basis of an alternative proof of Lemma 3.3. Namely, the same operator is also a bounded map from a certain weighted L^1 to itself, hence, by the extension to spaces of homogeneous type of the interpolation theorem of Fefferman and Stein (see [18]), we conclude that the operator maps certain weighted L^p to L^p for $1 < p < \infty$. It is not clear if this approach extends to, for example, the complex ball. One difficulty would be proving (or avoiding) the analog of Lemma 3.6. However, it would be interesting to obtain a proof of Lemma 2.8 by such an approach.

§4. Applications.

In this section, we present some applications of the previous results. We emphasize results in familiar contexts. Throughout this section, we denote by B^n the unit ball in $\mathbb{C}^n$, D^n the unit polydisk in $\mathbb{C}^n$, and $\mathbb{B}^n$ the unit ball in $\mathbb{R}^n$.

First we note that many well-known results can be read off quite easily from Theorem 2 by observing that the results are elementary for each summand. For example, suppose f is in $A^{1,r}(D^1)$. By using Theorem 2 with $p = 1$, $\theta = 1$, one can write f as a sum. The sum can be differentiated term by term, and Theorem 2 can be used again (now $p = 1$, $\theta = \frac{1}{2}$) to find that f' is in $A^{1,r+\frac{1}{2}}(D^1)$.

As another example, in [4] it was shown that, for certain r, every function in $A^{1,r}(B^n)$ could be written as a sum of products of functions in $A^{2,r}(B^n)$. This was obtained as a corollary of the corresponding result for the Hardy space $H^1(\partial B^n)$. Theorem 2 together with the observation that the Bergman kernel never vanishes provides an alternative proof and generalization of the result. The difference between the two proofs, one based on Theorem 2 and the other given in [4], provides an instance of the general fact that the Bergman spaces are often easier to work with then the corresponding Hardy spaces. In this case the relative ease of analysis in the Bergman spaces is directly related to the existence of a bounded projection operator mapping the ambient L^1 onto the Bergman space.

4.1. Zero Sets.

In the proof of Theorem 2, and in the proof of Theorem 3, the only thing that was required of the points ζ_i (called y_i in the harmonic case) is that they be an η-lattice for some sufficiently small η. Now note that if the function F being analyzed in the proof of Theorem 2 vanishes at each ζ_i, then the approximation G in the proof of Theorem 2 must vanish identically. Similar observations hold for Theorem 3. Thus we have

Proposition 4.1. *Let D, p, r be such that Theorem 2 applies. There is an $\eta_0 = \eta_0(D, p, r)$ such that if f is in $A^{p,r}(D)$ and f vanishes at each point of an η-lattice for some $\eta < \eta_0$ then f must vanish identically. The same result holds for harmonic functions f in $A_H^{p,\alpha}(\mathbb{R}^n)$.*

For the classical Bergman space (i.e., $D = D^1$), this result on zero sets is less precise than the results of Horowitz [13]. Examination of Horowitz's examples shows that our result is sharp in the following crude sense. Given an η, there is a p sufficiently small so that one can find an f in $A^{p,0}(D^1)$ which vanishes at each point of an η-lattice without vanishing identically. From this we also conclude that the choice of η in Theorem 2 must depend on p. (The discussion in part 5 of this section actually shows that $\eta_0 = \eta_0(D^1, p, 0)$ must satisfy $\eta_0 = O(p^{-1/2})$ as $p \to 0$).

We know of no other results similar to Proposition 4.1 for harmonic functions. It would be interesting to have a more direct proof of the proposition for harmonic functions. Also, it would be interesting to have a local version of Proposition 4.1.

4.2. Inclusion of Hardy and Bergman Spaces in Bergman Spaces.

Consideration of the $\theta = 0$ case of Theorem 2 suggests that the spaces $A^{p,r}(D)$ and $A^{p',r'}(D)$ should be very closely related if $(1 + r)/p = (1 + r')/p'$. Suppose

$$0 < p < p' \leq 1 \ , \ r > r' > -\varepsilon(D) \ , \text{ and } \frac{1+r'}{p'} = \frac{1+r}{p} \ . \tag{4.1}$$

In this case the only difference between the two spaces is that forced by the different homogenieties of their norm functions. To make this precise we recall that for $0 < p \leq 1$, the p-convex hull of a subset X of a vector space is the set of all finite sums $\Sigma\lambda_i x_i$ with $\lambda_i \geq 0$, $\Sigma\lambda_i^p = 1$ and x_i in X ([23]). Since $\ell^p \subseteq \ell^{p'}$ if $p \leq p'$, a consequence of Theorem 2 is

Proposition 4.2. *Suppose (4.1) holds, then*

$$A^{p,r}(D) \subseteq A^{p',r'}(D),$$

the inclusion is continuous, and the closure in $A^{p',r'}$ of the p'-convex hull of the unit ball of $A^{p,r}$ contains a ball of $A^{p',r'}$.

The informal statement of the result is that $A^{p',r'}$ is the "smallest p'-convex space" containing $A^{p,r}$. In particular, when $p' = 1$, $A^{1,(1+r-p)/p}$ is the smallest Banach space containing $A^{p,r}$ (i.e., the Mackey completion of $A^{p,r}$ [23]). Thus

Corollary 4.3. *The spaces $A^{p,r}$ and $A^{1,(1+r-p)/p}$ have the same dual.*

From certain points of view, the Hardy spaces on the boundary of the domain D can be regarded as the limiting case (as r approaches $-\varepsilon_D$) of the Bergman spaces. In particular, let $H^p(T^n)$ denote the Hardy space of the n-torus (see [10] for definitions) and let $H^p(\partial B^n)$ denote the Hardy space associated with the boundary of B^n. (See [12] for definitions.)

Proposition 4.4. *If $p, p' \leq 1$, $r' > -\frac{1}{2}$ and $\frac{1}{2p} = \frac{1+r'}{p'}$ then*

$$H^p(T^n) \subseteq A^{p',r'}(D^n). \tag{4.2}$$

If $p, p' \leq 1$, $r' > -\frac{1}{n+1}$ and $\frac{n}{(n+1)p} = \frac{1+r'}{p'}$ then

$$H^p(\partial B^n) \subseteq A^{p',r'}(B^n) \tag{4.3}$$

In both cases, the inclusion is continuous and the closure in $A^{p',r'}$ of the p'-convex hull of the unit ball of H^p contains a ball of $A^{p',r'}$.

Proof. We first consider the case $n = 1$ (and hence (4.2) is the same as (4.3)). In this case, the inclusion and its continuity is a result of Hardy and Littlewood (page 87 of [6]). The second part of the conclusion follows from Theorem 2 as soon as we show that the family of functions of z given by

$$H_\zeta(z) = \left(\frac{B^2(z,\zeta)}{B(\zeta,\zeta)}\right)^{\frac{1+r'}{p'}}$$

is uniformly (with respect to ζ) in $L^p(d\theta)$. However, if $|z| = 1$, then

$$|H_\zeta(z)|^p = \frac{1-|\zeta|^2}{|1-z\bar{\zeta}|^2}$$

which is the Poisson kernel. Thus $\int_{T^1} |H_s(e^{i\theta})|^p \, d\theta = 1$.

For $n > 1$ the proof follows the same pattern. The result of Hardy and Littlewood has been extended to the polydisk by Frazier [10], and to the ball by Hahn and Mitchell [12]. In both of these cases $|H_\zeta(z)|^p$ is the Poisson kernel.

One immediate consequence of the proposition is that linear operators which map H^p to H^p continuously automatically extend to continuous maps of $A^{p',r'}$ to $A^{p',r'}$.

Taking $p' = 1$ in the proposition and then passing to dual spaces yields the identification of the duals of certain Hardy spaces with the duals of certain Bergman spaces. For $n = 1$ this identification (and some of the other results in this section) are due to Duren, Romberg, and Shields [7]. For the polydisk it is a result of Frazier, for the ball, a result of Hahn and Mitchell.

For $n = 1$, $p' < 1$, this result had been conjectured by J. Shapiro [23]. Modulo the result he gave a proof of the following proposition (extending earlier work by Horowitz, Oberlin, Rudin, Duren and Shields).

Proposition 4.5. Suppose $0 < p \leq \infty$ and $n \geq 2$. The map of holomorphic functions on the polydisk D^n to holomorphic functions on the disk D^1 which sends $f(z_1, z_2, \ldots, z_n)$ to $f(z, z, \ldots, z)$ is a continuous map of $H^p(T^n)$ onto $A^{p,-1+n/2}(D^1)$. (Another proof of this result and some cases of the previous proposition has been given by J. Detraz [27].)

There is an obvious and immediate real variable analog of Proposition 4.2. There is also a natural analog of Proposition 4.4; however, before presenting that result we must introduce the appropriate Hardy space on Σ_{n-1}, the unit sphere in

$\mathbb{R}^n$.

In the discussion which follows, we will identify distributions on Σ_{n-1} with their Poisson integrals defined on $\mathbb{B}^n$.

Suppose $0 < q \leq 1$. We now define $H^q(\Sigma_{n-1})$. Pick an integer r, $r > (n-1)(\frac{1}{q} - 1)$. A function $a(x')$ on Σ_{n-1} is called a q-atom if either

a is a spherical harmonic of degree at most r with $\|a\|_\infty \leq 1$

or

(4.4)

a is supported on a set of diameter ϵ.

$$|a(x')| \leq \epsilon^{-(n-1)/q} \quad \text{and}$$

a is orthogonal to spherical harmonics of degree less that or equal to r.

We define $H^q(\Sigma_{n-1})$ to be the space of distributions which can be written in the form $\sum_i \lambda_i a_i$ with q-atoms a_i and scalars λ_i satisfying $\sum|\lambda_i|^q < \infty$. The H^q norm of this distribution is defined to be $\inf\{(\sum|\tilde{\lambda}_i|^q)^{1/q}\}$, where the infimum is over all sums of the same type which produce the same distribution. We will also denote by $H^q(\Sigma_{n-1})$ the space of harmonic functions on $\mathbb{B}^n$ which are obtained as Poisson integrals of the distributions just described.

This definition of $H^q(\Sigma_{n-1})$ is completely analogous to the description of $H^p(\mathbb{R}^n)$ given by Latter [17]. The fact that the space described does not depend on the integer r is proved by Taibleson and Weiss in [26]. A general discussion of this approach to Hardy spaces is given in [3] (see also [20] and [26]).

<u>Proposition 4.6.</u> <u>If</u> p, $p' \leq 1$, $\alpha > -1$, <u>and</u> $\frac{n+\alpha}{p'} = \frac{n-1}{p}$ <u>then</u>

(4.5) $$H^p(\Sigma_{n-1}) \subseteq A_H^{p',\alpha}(\mathbb{B}^n)$$

<u>The inclusion is continuous and the closure in</u> $A_H^{p',\alpha}$ <u>of the</u> p' <u>convex hull of the unit ball of</u> H^p <u>contains a ball of</u> $A_H^{p',\alpha}$.

<u>Corollary 4.7.</u> H^p <u>and</u> $A^{1,-n+(n-1)/p}$ <u>have the same dual.</u>

<u>Proof of the Proposition.</u> Since $1 \geq p' > p$, to prove (4.5) and to show that the inclusion is continuous, it suffices to show that if a is a p-atom then $A(x)$, the Poisson integral of $a(x')$ given by

$$A(x) = \int a(y')P(x , y')d\sigma(y') ,$$

is in $A_H^{p',\alpha}$ and has $A_H^{p',\alpha}$ norm bounded by a bound which does not depend on a. The case when a is a spherical harmonic is immediate. We now claim that if a satisfies (4.4) and y_0' is any point in the support of $a(x')$ then for all x in $\mathbb{R}^n$

$$|A(x)| \leq c \frac{\epsilon^{n+r-(n+\alpha)/p'}}{(|x-y_0'|+\epsilon)^{n+r}} \tag{4.6}$$

(Here ϵ and r are given by (4.4)). If $|x - y_0'| < \epsilon$ then

$$|A(x)| \leq \|a\|_\infty \|P\|_1 \leq \epsilon^{-(n-1)/p} = \epsilon^{-(n+\alpha)/p'}$$

If $|x - y_0'| > \epsilon$ we must use the moment condition on a and the information on the support of a.

$$A(x) = \int_\Sigma aP = \int_\Sigma a(P - R) = \int_{|y-y_0'|<\epsilon} a(P - R)$$

where R is any sum of spherical harmonics of degree r or less. We need the following

<u>Lemma</u>. <u>Given</u> x , $R = R_x$ <u>can be chosen to that</u>

$$\sup_{|y'-y_0'|<\epsilon} |P(x , y') - R_x(y')| \leq c \frac{\epsilon^{r+1}}{|x-y_0'|^{n+r}}$$

<u>with a constant</u> c <u>independent of</u> ϵ , y_0' , <u>and</u> x (<u>as long as</u> $|x - y_0'| > \epsilon$).

<u>Proof.</u> This is just Taylor's theorem on the sphere. We need to know that any $(r + 1)^{st}$ order partial derivative of $P(x , y')$ (as a function of y') evaluated near y_0' is dominated by $|x - y_0'|^{n+r}$. This follows from the same argument that was used to prove Lemma 3.2.

Using this choice of R we then obtain (4.6) for x far from y_0' by using the estimating

$$|A(x)| \leq (\text{Area of } |y - y_0'| < \epsilon)\|a\|_\infty \|P - R\|_\infty .$$

Once (4.6) is verified, the calculation to show $A(x)$ is in $A^{p',\alpha}$ is the same type of calculation as the calculation done in showing that condition (b) implies condition (c) in the proof of Theorem 3'. The restriction on the exponent is

$r > -n + (n-1)/p$. Since r can be chosen as large as is needed, the inclusion is established.

By Theorem 3, we may finish the proof by showing that the functions of x' given by

$$M_y(x') = b_m(x', y)(1-|y|)^{m+n-(n+\alpha)/p'}$$

are in $H^p(\Sigma_{n-1})$ and have their H^p norm bounded by a constant which does not depend on y . Write $b_m(x', y) = \tilde{b}_m + P$ where P is a linear combination of spherical harmonics of degree at most r and $\tilde{b}_m$ is orthogonal to spherical harmonics of degree less than or equal to r . Let $\tilde{M}_y = \tilde{b}_m(1-|y|)^{m+n-(n+\alpha)/p'}$. We will show that $\tilde{M}_y$ are uniformly in H^p, for $|y|$ near 1 . (The cases of $|y|$ small and the term $P(1-|y|)^{m+n-(n+\alpha)/p'}$ are proved by elementary arguments which we omit.)

We will show that $\tilde{M}_y$ satisfies

$$(4.7)\quad \left(\int_\Sigma |\tilde{M}_y(x')|^2\, d\sigma(x')\right)^p \left(\int_\Sigma |\tilde{M}_y(x')|^2\, |x'-y'|^{2(n-1)/p}\, d\sigma(x')\right)^{2-p} \le c$$

for some c independent of y . A function which satisfies (4.7) and is orthogonal to the harmonics of degree less than or equal to r is called a p -molecule. Such a function is in H^p ([26]). To estimate the left hand side of (4.7) we use part a of Lemma 3.2, which clearly extends to $\tilde{b}_m$, estimate $|x'-y'|$ by $|x'-y'| + \epsilon(y)$, and use the fact that $(n+\alpha)/p' = (n-1)/p$. Thus we must estimate

$$c\left(\int_\Sigma \frac{\epsilon(y)^{2(m+n-(n-1)/p)}}{(|x'-y'|+\epsilon(y))^{2(n+m)}}\, d\sigma(x')\right)^p \left(\int_\Sigma \frac{\epsilon(y)^{2(m+n-(n-1)/p)}}{(|x'-y'|+\epsilon(y))^{2(m+m)}}\, |x'-y'|^{2(n-1)/p}\, d\sigma(x')\right)^{2-p}$$

These integrals are dominated by the corresponding integrals over $\mathbb{R}^{n-1}$. These integrals are computed in polar coordinates centered at y' . Thus we need to show

$$\left(\int_0^\infty \frac{\epsilon^{2(m+n-(n-1)/p)}}{(r+\epsilon)^{2(m+n)}}\, r^{n-2}\, dr\right)^p \left(\int_0^\infty \frac{\epsilon^{2(m+n-(n-1)/p)}}{(r+\epsilon)^{2(m+n)}}\, r^{n-2+2(n-1)/p}\, dr\right)^{2-p}$$

is bounded by a bound independent of ϵ . The substitution of ϵr for r shows that the product is independent of ϵ . To insure that the product is finite for any ϵ , one must select an m that is large enough to insure that both integrals converge at infinity. However, Theorem 3 allows m to be selected as large as is needed. The proof is complete.

As with the case of holomorphic functions, it follows that bounded linear maps of H^p to H^p extend to certain Bergman spaces. For instance we have an extraordinarily roundabout proof of the following.

Corollary 4.8. Suppose $0 < p \leq 1$, $\alpha > -1$. The map which takes a harmonic function U on the disk to its harmonic conjugate $\tilde{U}$, is a bounded map of $A_H^{p,\alpha}(B^2)$ to itself.

Proof. Let $r = p/(2 + \alpha)$. The map of U to $\tilde{U}$ is a bounded map of $H^r(d\theta)$ to itself.

4.3. Inclusion of Bergman Spaces in Hardy Spaces.

In this section we will regard functions f in the various $A_H^{p,0}(\mathbb{B}^n)$ as defined on all of $\mathbb{R}^n$ by setting $f(x) = 0$ if x is in $\mathbb{R}^n \backslash \mathbb{B}^n$. We wish to show that $A_H^{p,0}$ is contained in the Hardy space (in the sense of Stein and Weiss) of the ambient Euclidean space. However, we must first discard some functions in $A_H^{p,0}$. We define $\tilde{A}^p$ by

$$\tilde{A}^p(\mathbb{B}^n) = \tilde{A}^p = \{f \; ; \; f \in A_H^{p,0}(\mathbb{B}^n) \text{ and the expansion of } f \text{ in spherical harmonics contains no terms of degree} \leq n(\tfrac{1}{p} - 1)\} \; .$$

We denote by $H^p(\mathbb{R}^n)$ the Hardy space (in the sense of Stein and Weiss) of $\mathbb{R}^n$. (A discussion of these spaces from the point of view we will use in in [3] or [26].)

Proposition 4.9. Given $n \geq 2$, $0 < p \leq 1$, there is a continuous inclusion

$$\tilde{A}^p(\mathbb{B}^n) \subseteq H^p(\mathbb{R}^n) \; .$$

Proof. The proof is essentially the same as the proof of the last part of Proposition 4.6. By Theorem 3, we may write f in $\tilde{A}^p(\mathbb{B}^n)$ as a sum of terms

$$M_y(x) = b_m(x \; , \; y)(1 - |y|)^{m+n-n/p} \; .$$

Splitting b_m as a sum of spherical harmonics of degree at most r and a remainder $\tilde{b}_m$, f can actually be written as a sum of terms

$$\tilde{M}_y(x) = \tilde{b}_m(x , y)(1 - |y|)^{m+n-n/p}$$

plus a sum of spherical harmonics of degree at most r . This latter sum contains no harmonics of degree $\leq n(\frac{1}{p} - 1)$ (since f does not) and hence is easily seen to be in H^p .

We now show that the $\tilde{M}_y$ are p-molecules and hence, by the results in [26] are in $H^p(\mathbb{R}^n)$ with uniform estimates on their H^p norm. That is, we show that the $\tilde{M}_y$ (now regarded as functions on $\mathbb{R}^n$ which vanish identically off $\mathbb{B}^n$) are orthogonal to the spherical harmonics of degree r (the appropriate r is determined by n and p) and satisfy

$$(\int_{\mathbb{R}^n} M_y^2(x)dx)^p (\int_{\mathbb{R}^n} M_y^2(x)|x - y|^{2n/p} dx)^{2-p} \leq c$$

for some universal constant c .

This condition is verified in the same way that (4.7) was verified.

4.4. The atomic theory of Bergman Spaces.

In this section we let D denote the unit disk in the complex plane.

We will show that the Bergman spaces $A^{p,\alpha}(D)$ are exactly the holomorphic functions in certain "atomic Hardy spaces" associated with certain "spaces of homogeneous type." We must refer to [3] and [26] for motivation, details of the atomic theory of Hardy spaces and definitions of some terms. The spaces we consider are related to those of example 10 in [3]. This is also the point of view taken in Section 5 of [5].

Let p and α be fixed , $0 < p \leq 1$, $\alpha > -\frac{1}{2}$ and consider the measure on D given by $d\nu = (1 - |z|)^{2\alpha} dx\, dy$. For z , w in D we introduce the "measure distance" between z and w =

$$m(z , w) = \inf \{\nu(B \cap D) \; ; \; B \text{ is a Euclidean ball which contains } z \text{ and } w\} .$$

The triple (D, ν, m) is the space of homogeneous type which we will consider.

The atomic Hardy space associated with this triple consists of distributions which can be written as ℓ^p sums of simple functions called "atoms". However, it is proved in [3] and [26] that one obtains the same space if one takes ℓ^p sums of p-molecules. A p-molecule for the triple (D, ν, m) is a function M which is supported on D, has vanishing moments up to order r for some large r and, for some ζ in D, satisfies

$$(4.9) \qquad \left(\int |M(z)|^2 \, d\nu(z)\right)^p \left(\int |M(z)|^2 \, m(z, \zeta)^{2/p} \, d\nu(z)\right)^{2-p} \leq c$$

with a constant c which does not depend on the function M. It is proved in [26] that the space obtained in this way does not depend on r if r is at least as large as some critical r_0 determined by p and α. For example, if $\alpha = 0$, then $r_0 = 2[\frac{1}{p} - 1]$. We regard r as fixed for the rest of this section. Condition (4.9) and the obviously closely related (4.7) and (4.8) are special cases of (2.7) of [3]. An exposition of the properties of such functions in this context, for example the fact that a molecule can be written as a sum of atoms, is in [26].

We emphasize that we are considering the atomic Hardy spaces built from molecules (ar atoms) with vanishing moments. This is slightly different from the point of view in [3].

We will prove

<u>Proposition 4.10.</u> <u>If</u> f <u>is in</u> $A^{p,\alpha}$ <u>then</u> $f = f_0 + \Sigma\lambda_i M_i$ <u>where</u> f_0 <u>is a polynomial of degree</u> r, <u>the</u> M_i <u>are</u> p-<u>molecules, and</u> λ_i <u>scalars, with</u>

$$\|f_0\|^p_{p,\alpha} + \Sigma|\lambda_i|^p \leq c\|f\|^p_{p,\alpha} .$$

<u>Corollary 4.11.</u> $A^{p,\alpha}$ <u>is contained in the atomic</u> H^p <u>space associated with the space of homogeneous type</u> (D, ν, m).

<u>Proof of the proposition.</u> Write $f = f_0 + z^r g$ where f_0 is the sum of the first r terms in the Taylor series of f. f_0 satisfies the required estimate. g is in $A^{p,\alpha}$ and $\|g\| \leq c\|f\|$. We now apply Theorem 2 to g.

$$g = \Sigma \lambda_i \left(\frac{(1-|\zeta_i|^2)^2}{(1-\overline{\zeta}_i z)^4}\right)^{(1+\alpha)/p}$$

It now suffices to show that the functions

$$M_i(z) = \left(\frac{(1-|\zeta_i|^2)^2}{(1-\overline{\zeta}_i z)^4}\right)^{(1+\alpha)/p} = \left(\frac{B(z,\zeta_i)^2}{B(\zeta_i,\zeta_i)}\right)^{(1+\alpha)/p}$$

satisfies (4.9). Before doing that we need the following estimate on $m(z\ ,\ \zeta)$.

$$m(z\ ,\ \zeta) \leq c \max\ (1 - |\zeta|\ ,\ |z - \zeta|)^{2+2\alpha} \tag{4.10}$$

This estimate is obtained making direct estimates for each of the several possible geometric situations. The estimate for the first integral in (4.9), i.e.,

$$I_1 = \int_D |B(z\ ,\ \zeta)|^{4(1+\alpha)/p}\ B(\zeta\ ,\ \zeta)^{-2(1+\alpha)/p}\ B(z\ ,\ z)^{-\alpha}\ dxdy\ .$$

follows from the disk case of Lemma 2.2. This yields

$$|I_1|^p \leq c\ B(\zeta\ ,\ \zeta)^{(1+\alpha)(2-p)}\ .$$

To estimate

$$I_2 = \int_D |M(z)|^2\ m(z\ ,\ \zeta)^{2/p}\ d\nu(z)$$

we write $I_2 = I_2' + I_2''$ where I_2' is an integral over $B = \{|z - \zeta| \leq 1 - |\zeta|\}$ and I_2'' is the remainder. Using (4.10) gives

$$I_2' \leq \int_B |B(z\ ,\ \zeta)|^{4(1+\alpha)/p}\ B(\zeta\ ,\ \zeta)^{-4(1+\alpha)/p}\ B(z\ ,\ z)^{-\alpha}\ dxdy$$

which, by Lemma 2.2, can be estimated by

$$|I_2'|^{2-p} \leq c\ B(\zeta\ ,\ \zeta)^{-(1+\alpha)(2-p)}\ .$$

A similar argument yields a similar estimate for I_2'' . These estimates together show that M satisfies (4.9) and the proof is complete.

The corollary follows directly from the proposition and the results of Taibleson and Weiss in [26]. As a consequence of the corollary, if f is is $A^{p,\alpha}$ then f can be written as a sum of atoms with the atoms <u>supported on D</u> . This is an interesting companion to Theorem 2.28 of [3] which establishes a similar local decomposition for distributions supported on the unit interval.

Many of the general results in [3] can now be used on the $A^{p,\alpha}$ spaces. For example.

Proposition 4.12. Suppose $\alpha > -\frac{1}{2}$, $0 < p_1 < 1 < p_2 < \infty$. Suppose T is a linear map of holomorphic functions, and T is a bounded map of $A^{p,\alpha}$ to itself and $A^{p_2,\alpha}$ to itself. Then T is a bounded map of $A^{p,\alpha}$ to itself for $p_1 < p < p_2$.

Proof outline. By Lemmas 2.1 and 2.8 we may pick β so large that

$$Kh(z) = \int_D B(z\ ,\ \zeta)^{\beta}\, B(\zeta\ ,\ \zeta)^{-\beta+1}\, h(\zeta)d\zeta$$

is a bounded projection of $L^p(B(\zeta\ ,\ \zeta)^{-\alpha}\, d\nu(\zeta))$ onto $A^{p,\alpha}$ for $1 < p \leq p_2$. K will not be a bounded for $p \leq 1$ but is a bounded projection of the Hardy space associated with the space of homogeneous type $(D\ ,\ \nu\ ,\ m)$ onto $A^{p,r}$. We now apply the Marcninkiewicz type interpolation theorem (Theorem D) of [3] to the composite operator TK and conclude that TK is bounded on the atomic H^p space if $p_1 \leq p \leq 1$ and on L^p if $1 < p \leq p_2$. Since $TK = T$ on $A^{p,\alpha}$ and $A^{p,\alpha}$ is contained in the appropriate H^p or L^p, the conclusion follows.

There are two details of this argument which are straightforward but lengthy and have been omitted. First, it must be proved that K is bounded on the atomic Hardy spaces. This follows from arguments similar to those used in [3] on pages 598-600. Second, the interpolation theorem, Theorem D of [3], must be extended to the case of H^p spaces defined in terms of atoms with vanishing moments.

4.5. Automorphic forms.

In this section we again suppose D is the unit disk in the complex plane.

One method of obtaining automorphic forms on D is to form the Poincaré series of functions in appropriate Bergman spaces. In some cases, when this is combined with the representation of the Bergman spaces given in Theorem 2 a particularly simple expression results. One can then show, for example, that certain spaces of automorphic forms are finite dimensional. We now present this in a simple case. The applicability of Theorem 2 to the general theory of automorphic forms has not been investigated. As general references for the theory of automorphic forms we refer to the books of Kra [15] and Baily [1].

Let $\Gamma = \{A_i\}_{i=1}^{\infty}$ be a discrete group of conformal automorphisms of D with A_1 the identity transformation. Let q be an integer, $q \geq 2$. A holomorphic function f defined on D is said to be an automorphic form of weight q (with respect to Γ) if

$$f(A_i(z))(A_i'(z))^q = f(z) \quad i = 1, 2 \dots ; z \in D .$$

We denote the space of all such f for which

$$\|f\|_\Gamma = \iint_R |f(z)| \, B^{1-\frac{q}{2}}(z, z) dV(z)$$

is finite by $A^{1,\frac{q}{2}-1}(D, \Gamma)$. Here R is any fundamental domain for Γ. In particular, if Γ is the trivial group then $A^{1,\frac{q}{2}-1}(D, \Gamma) = A^{1,\frac{q}{2}-1}(D)$.

For f holomorphic on D we define the Poincaré series of f by

$$(\Theta f)(z) = (\Theta_{q,\Gamma} f)(z) = \sum_i f(A_i(z))A_i'(z)^q \tag{4.11}$$

Our starting point is the following result (see, for example, Chapter 3 of [15]).

<u>Theorem.</u> <u>If</u> f <u>is in</u> $A^{1,\frac{q}{2}-1}(D)$ <u>then (4.11) converges uniformly and absolutely on compact subsets.</u> <u>In fact,</u> Θ <u>is a continuous map of</u> $A^{1,\frac{q}{2}-1}(D)$ <u>onto</u> $A^{1,\frac{q}{2}-1}(D, \Gamma)$.

Now suppose that Γ has a compact fundamental domain B. For a given ϵ, choose points $\zeta_1, \dots, \zeta_N$ in B so that each point in B is within invariant distance ϵ of some ζ_i. Let $\zeta_{ij} = A_i(\zeta_j)$. It is straightforward to check that the points ζ_i could be chosen so that the ζ_{ij} form a $c\epsilon$-lattice for some constant c which does not depend on ϵ. We will suppose that the points have been chosen so that that is true.

We now apply Theorem 2 to the space $A^{1,\frac{q}{2}-1}(D)$ using the points ζ_{ij} as the required η-lattice. Let

$$H_{ij}(z) = B(z, \zeta_{ij})^q \, B(\zeta_{ij}, \zeta_{ij})^{-q/2} .$$

By Theorem 2, if f is in $A^{1,\frac{q}{2}-1}(D)$ then

$$f = \Sigma \lambda_{ij} H_{ij}$$

with $\Sigma|\lambda_{ij}| \leq c\|f\|$.

A direct computation using the invariance properties of the Bergman kernel, (2.9), shows

$$\theta H_{ij} = \alpha_{ij} \theta H_{1j}$$

with α_{ij} a constant of modulus one. Thus, setting $\mu_j = \sum_i \alpha_{ij} \lambda_{ij}$, we have

$$\theta f = \sum_{j=1}^{n} \mu_j \theta H_{1j} .$$

Hence,

<u>Proposition 4.13.</u> $A^{1,\frac{q}{2}-1}(D , \Gamma)$ <u>is spanned by the functions</u> $\theta(H_j)$ $j = 1 , \ldots , n$.

Since it is known that $\dim (A^{1,\frac{q}{2}-1}(D , \Gamma))$ grows linearly with q , and since the number n obtained in the previous argument is roughly η^{-2} , the quantity η_0 in Theorem 2 must satisfy $\eta_0 = O(q^{-1/2})$ for large q and $p = 1$.

One can carry the analysis further and also show that the space of automorphic forms is finite dimensional even if it is only assumed that B has finite invariant area. We outline the argument. If f is in $A^{1,\frac{q}{2}-1}(D , \Gamma)$ then

$$f = \theta(P(\chi_B f))$$

where χ_B is the characteristic function of B and P is the projection from $L^{1,\frac{q}{2}-1}(D)$ onto $A^{1,\frac{q}{2}-1}(D)$ used in the proof of Theorem 2. (See Chapter 3 of [15]). Thus, as in the proof of Theorem 2, it suffices to show that $P(\chi_B f)$ can be well approximated by sums of the type $\sum_{j=1}^{n} \lambda_j H_{ij}(z)$ with the ζ_{ij} appropriately chosen points in B . This is done exactly as in the proof of Theorem 2, but there is an error term that cannot be estimated by a "Riemann sum" argument. That term is estimated using the fact that such an f must have the limiting value zero at each cusp in an appropriately chosen fundamental domain.

4.6. Hankel Operators.

Before proving the theorem which was stated in the introduction about trace class Hankel operators, we introduce some notation. For a function g defined on the positive real axis, let $\check{g}$ be the inverse Fourier transform of the function on the real axis which is zero for negative argument and which agrees with g for positive argument. Thus

$$\check{g}(x) = \int_0^\infty g(t)\, e^{2\pi i t x}\, dt \, .$$

Such a function $\check{g}$ has a natural extension to a holomorphic function in the upper half plane. (For discussion of all these issues, see [24].) We shall denote this extension by $\check{g}(z)$. Furthermore,

$$\lim_{y\to 0} \int_{-\infty}^{\infty} |\check{g}(x+iy)|^2\, dx = \int_0^\infty |g(t)|^2\, dt \, ,$$

that is, if g is in L^2 of the half line, then $\check{g}$ is in H^2 of the half space.

Proposition 4.14. There is an η-lattice $\{z_i\}$ in the upper half plane such that the following conditions are equivalent:

(a) The linear map of L^2_+ to itself given by

$$(Hf)(x) = \int_0^\infty k(x+y) f(y)\, dy$$

is of trace class.

(b) There are numbers λ_i with $\Sigma|\lambda_i| < \infty$ such that

$$k(t) = \Sigma\, \lambda_i (\operatorname{Im} z_i) e^{-i\bar{z}_i t}$$

(c) The function $(\check{k}(z))''$ is in the Bergman space A^1 of the upper half plane.

Proof. As was noted in the introduction to this volume, it is elementary that (b) implies (a). The fact that (c) implies (b) is a direct application of Theorem 2 followed by a Fourier transform calculation. We write $\check{k}(z)''$ using formula (2.1) with the indicies $p = 1$, $r = 0$, $\theta = -1/2$. Note that if the domain being considered is the upper half plane then the Bergman kernel is $B(z, w) = C(z - \bar{w})^{-2}$. Let $y_i = \mathrm{Im}(z_i)$. The decomposition we obtain is for the function $\check{k}''$. After integrating twice we obtain

$$\check{k}(z) = \Sigma \lambda_i \frac{y_i}{(z-\bar{z}_i)}$$

for a summable sequence λ_i . Taking Fourier transforms on both sides of this equation yields the required representation.

We now wish to show that (a) implies (c). The theory of trace class operators on L^2 spaces insures that $k(x + y)$ can be written in the following form

$$k(x + y) = \sum_{i=1}^{\infty} u_i(x)v_i(y) \quad x , y \geq 0 \tag{4.12}$$

with u_i and v_i in L^2_+ and $\Sigma\|u_i\|_2 \|v_i\|_2 < \infty$. (Whether this representation is a theorem or a definition depends on the point of view taken.) Multiply both sides of this equality by $x^{1/2} y^{1/2}$ and apply the inverse Fourier transform (i.e., the mapping from g to $\check{g}$) twice, first in the x variable and then in the y variable. We then evaluate the resulting expression on the diagonal $x = y$. We denote the new variable by s . The right hand side of (4.12) becomes $\Sigma(x^{1/2} u)^{\vee}(s)(y^{1/2}v)^{\vee}(s)$. However, up to a constant factor which we ignore $(x^{1/2} u)^{\vee}$ is the inverse Fourier transform of the half-order derivative of the function $u^{\vee}$. Thus the transformed version of the right hand side of (4.12) is $\Sigma(D^{1/2} u_i)(s)(D^{1/2} v_i)(s)$. The half order differentiation operator is a constant multiple of a unitary map from the Hardy space $H^2(\mathbb{R})$ to the Bergman space A^2 of the upper half plane. (This is easy to check using Fourier transforms.) Hence the right hand side of (4.12) is transformed to a sum of the products of elements of A^2 with the sum of the products of the norms finite. By the Cauchy-Schwartz inequality, this transformed function is in A^1 .

To finish the proof we now analyze the behaviour of the left hand side of (4.12) under this transformation. Let $\mathfrak{F}_x$ denote the operation which sends g to

$\check{g}$ with x as the variable in g. $\mathfrak{F}_y$ is defined similarly. The proof will be finished as soon as we prove the following lemma.

Lemma. $\mathfrak{F}_y \mathfrak{F}_x \, x^{1/2} y^{1/2} k(x+y)\big|_{x=y} = c\check{k}''$.

Proof. The left hand side is

$$\int_0^\infty \int_0^\infty r^{1/2} t^{1/2} k(r+t) e^{2\pi i(tx+ry)} \, dtdr\Big|_{x=y}$$

$$= \int_0^\infty \int_0^\infty r^{1/2} t^{1/2} k(r+t) e^{2\pi ix(t+r)} \, dtdr$$

If we let r and $s = r + t$ be new integration variables we obtain

$$\int_0^\infty \int_0^s r^{1/2} (s-r)^{1/2} k(s) \, e^{2\pi ixs} \, drds .$$

Computing the r integral yields

$$c \int_0^\infty s^2 \, k(s) e^{2\pi ixs} \, ds$$

$$= c\Big(\int_0^\infty k(s) e^{2\pi ix} \, dx\Big)''$$

which is the required formula.

It should be emphasized that we obtained this proof only after we learned that V.V. Peller had proved the equivalence of (a) and (c) in the discrete case (i.e., for Hankel matrices acting on $\mathbb{Z}^+$). Peller also announces a characterization of those Hankel matrices which give operators in the Schatten p class, with $p > 1$. His condition is that $\check{k}(z)'$, which is a function on the disk, be in the space $A^{p,(p-1)/2}$. This result is compatible with the speculation that condition (b) of Proposition 4.14 characterizes the Hankel operators in the Schatten p class for $0 < p < \infty$ if the condition $\Sigma|\lambda_i| < \infty$ is changed to $\Sigma|\lambda_i|^p < \infty$.

The choice of the power $1/2$ in the Lemma was not really necessary for the proof. Other positive powers would have led to results which are equivalent (via fractional integration) to the proposition. The choice of the power zero does not give the full result. A direct calculation of the Fourier transform of (4.12) shows that $\check{k}'(z)$ must be in H^1. This result, due to Howland and Rosenblum, is not a sufficient condition.

A similar calculation can be carried out for Hankel type operators on the L^2 space of any homogeneous self-dual cone. A combination of Fourier transform theory for such cones ([11], [24]) and Theorem 2 yields results similar to the previous proposition. If the L^2 space of the cone is replaced by the L^2 space of the cone with an appropriate weight, the result that is obtained is a nuclearity criterion for Hankel type operators defined on the spaces $A^{2,r}$. This extends the results of [5].

Appendix. We now outline the proofs of Lemma 2.1, 2.2 and 2.3. We prove the first two lemmas by carrying further some of the ideas presented by Gindikin in [11]. We will make free use of the results and arguments of [11], especially of Section 5. The proofs are quite computational and use the machinery of special functions on Siegel domains of the second kind. However the basic ideas are quite straightforward. The following three ideas are the basis for these computations. First, the theory of the Bergman kernel can be developed using Fourier transform techniques. Second, the domains being considered have a large group of automorphisms. Using this group, many computations can be replaced by homogeneity considerations. Finally the Riemann-Liouville fractional integrals are unitary maps between the various Hilbert spaces $A^{2,r}$. This allows the results for $A^{2,0}$ to be used as a basis for other results.

We restrict our presentation to the case when the base cone V is self-dual (i.e., D is symmetric). We start by recalling that

$$B(z\ ,\ \zeta) = c\left(\frac{z_1-\overline{\zeta}_i}{2} - F(z_2\ ,\ \zeta_2)\right)^{2d-q} \tag{A.1}$$

where $z = (z_1\ ,\ z_2)$, $z_1 \in \mathbb{C}^n$, $z_2 \in \mathbb{C}^m$ and $(\ \ )^p$ is the "compound power function" with multi-index p. $d = (d_1\ ,\ \ldots\ ,\ d_\ell)$ is a vector of non-negative integers $q = (q_1\ ,\ \ldots\ ,\ q_\ell)$ is a vector of positive integers and ℓ is the rank of V. We also need the result that if $\widetilde{iy} = (iy\ ,\ 0)$ for y in V then

$B(iy , iy) = cy^{2d-q}$ is the density of the measure on D which is invariant under the automorphism group.

Gindikin presents the basic Fourier transform description of the space $A^{2,0}$ and the Plancherel-Parseval formula (Proposition 5.5). His proof extends directly to the weighted spaces $A^{2,r}$ and gives the following

Proposition A.1. There is a positive constant ϵ_D which depends only on D so that if $r > -\epsilon_D$ then $G(z)$ is in $A^{2,r}(D)$ if and only if G has a representation

$$G(z) = G(z_1 , z_2) = \int_V \tilde{g}(\lambda , z_2)e^{i(\lambda, z\ )} d\lambda$$

with $\tilde{g}(\lambda , z_2)$ defined on $V \times \mathbb{C}^m$, entire in z_2 and satisfying

$$\int_V \int_{\mathbb{C}^m} |\tilde{g}(\lambda , z_2)|^2 e^{-2(\lambda, F(z_2, z_1))} \lambda^{d-(2d-q)r} dz_2 d\bar{z}_2\, d\lambda < \infty$$

In this case the previous integral equals $c_r\|G\|^2_{2,r}$.

Proof. The basic idea is to reduce to the classical case. The details are exactly those of the proofs of Propositions 5.4 and 5.5 of [11].

Once the proposition is verified, Lemma 2.1 follows in the same way in which Gindikin's Theorem 5.1 follows from his proposition 5.5.

Another consequence of Proposition A.1 should be mentioned. The expression for the norm of G involves a factor of $\lambda^{-(2d-q)r}$. Hence, the operation of multiplication of $\tilde{g}$ by various compound powers of λ will produce isometries between the various spaces $A^{2,r}$. More precisely, the Riemann-Liouville fractional integral operator $\mathcal{R}^p_V$ defined in Section 5 of [11] ($\mathcal{R}^p_V$ involves multiplication of $\tilde{g}$ by $\lambda^{-\rho}$) is an isometry from $A^{2,r}$ to $A^{2,r'}$ with $r' = r + \alpha$, $\rho = -\frac{1}{2}\alpha(2d - q)$. From this point of view, our Lemma 2.1 is very close to Theorem 5.5 of [11].

We now prove Lemma 2.2. We must evaluate

$$\int_D |B(z , \zeta)|^{1+\alpha} B(\zeta , \zeta)^{-\beta} dV(\zeta) \tag{A.2}$$

Let $z = (x + iy , z_2)$ and $\zeta = (\xi + i\eta , \zeta_2)$. So, by (A.1),

$$B(z , \zeta) = c\left(\frac{x-\xi}{2i} + \frac{y+\eta}{2} - \mathrm{Re}F(z_2 , \zeta_2) - i\,\mathrm{Im}F(z_2 , \zeta_2)\right)^{2d-q} .$$

Note that the factor $B(\zeta , \zeta)^{-\beta}$ doesn't depend on the variable ξ . Hence, when we evaluate (A.2) by doing the integration in ξ first, we are led to evaluate integrals of the form

$$\text{(A.3)} \qquad \int_{\mathbb{R}^n} \left|(i\xi + s)^{2d-q}\right|^{1+\alpha} d\xi .$$

Here we have made a linear change of variable and have set $s = y + \eta - 2\mathrm{Re}F(z , \xi)$. (Notice that $y - F(z_2 , z_2)$ and $\eta - F(\zeta_2 , \zeta_2)$ are both in the cone V and hence so is s .) To evaluate this integral, we use the integral representation for the generalized power function. By (2.6) and (2.29) of [11]

$$\text{(A.4)} \qquad (i\xi + s)^{(2d-q)\frac{1+\alpha}{2}} = c\int_V e^{-i\xi\cdot\lambda} e^{-s\lambda} \lambda^{-(2d-q)(\frac{1+\alpha}{2})} \lambda^d d\lambda .$$

Hence we may evaluate (A.3) using Plancherel's theorem for the cone V . This gives

$$c\, s^{(2d-q)(1+\alpha)-d} .$$

Let $p_1 = (2d - q)(1 + \alpha) - d$ and $p_2 = -\beta(2d - q)$. With this notation (A.2) equals

$$c\int_{\mathbb{C}^m} \int_{\eta-F(\zeta_2,\zeta_2)\in V} \left(\frac{y+\eta}{2} - \mathrm{Re}F(z_2 , \zeta_2)\right)^{p_1} (\eta - F(\zeta_2 , \zeta_2))^{p_2} d\eta d\zeta_2 d\bar{\zeta}_2$$

Set $\eta' = \eta - F(\zeta_2 , \zeta_2)$ and use the polarization identity for $\mathrm{Re}F(z_2 , \zeta_2)$. The integral becomes

$$c\int_{\mathbb{C}^m} \int_V (y - F(z_2 , z_2) + F(\zeta , \zeta) + \eta')^{p_1} (\eta')^{p_2} d\eta' d\zeta_2 d\bar{\zeta}_2 .$$

We now perform the η' integration using the homogeniety of the cone V . This produces a new constant factor (involving the cone beta function) and gives

$$c\int_{\mathbb{C}^m} (y - F(z_2 , z_2) + F(\zeta , \zeta))^{p_1+p_2-d} d\zeta d\bar{\zeta} .$$

By an argument similar to that which proves Proposition 2.8 in [11], this integral equals

$$c(y - F(z_2 , z_2))^{p_1+p_2-d-q} .$$

Since $p_1 + p_2 - d - q = (2d - q)(\alpha - \beta)$, this is the required result. The restriction $\alpha > \beta > -\epsilon_D$ comes from the requirement that the various factors absorbed into the constant c be absolutely convergent integrals.

It follows from (A.1) and the Fourier transform representation of the generalized power function (A.4), that the operator $\mathcal{D} = \Re_V^{2d-q}$ maps the various powers of the Bergman kernel to each other. That is

$$\mathcal{D}^a B(z , \zeta)^b = c_{ab} B(z , \zeta)^{a+b}$$

($\mathcal{D}$ is being regarded as an operator in the variable z.) We use this fact to produce a more general form of Lemma 2.2. By Lemma 2.1 applied to the function $F(\cdot) = B(\cdot , z_0)^m$ we obtain

$$B(z , z_0)^m = c\int B(z , \zeta)^{1+r} B(\zeta , z_0)^m B(\zeta , \zeta)^{-r} dV(\zeta) .$$

If we apply the operator $\mathcal{D}^a$ in the variable z this produces the equation

$$B(z , z_0)^{m+a} = c\int B(z , \zeta)^{1+r+a} B(\zeta , \zeta_0)^m B(\zeta , \zeta)^{-r} dV(\zeta) .$$

when $z = z_0$ and $m = 1 + r + a = (1 + \alpha)/2$ this becomes Lemma 2.2. However, in order to justify this formal computation, we need Lemma 2.2 to insure that the right hand side of the equation is an absolutely convergent interval. Using Lemma 2.2 and Holder's inequality with the measure $B(\zeta , \zeta)^{-r} dV(\zeta)$ we obtain

<u>Lemma 2.2'.</u> <u>Suppose</u> α_1 , α_2 <u>are positive and</u> $\alpha_1 + \alpha_2 > r + 1 > -\epsilon_D + 1$ <u>then</u>

$$B(z , z_0)^{\alpha_1+\alpha_2-r-1} = c_{\alpha_1,\alpha_2,r} \int B(z , \zeta)^{\alpha_1} B(\zeta , z_0)^{\alpha_2} B(\zeta , \zeta)^{-r} dV(\zeta) .$$

It should be noted that this formula is not conformally invariant and is false in the bounded realization of a domain. Also, the derivation of this formula suggests that it is the derivatives of the kernel functions which should play a role in the generalizations of this result, not the powers. This is also suggested by the results in Section 3.

We now prove Lemma 2.3. We thank A. Korányi for showing us this proof.

First note that D has a transitive group of linear automorphisms. If g is a linear automorphism of D , then (2.9), the invariance property of the Bergman kernel becomes

$$B_D(g(z)\ ,\ g(\zeta))\ |\det g'|^2 = B_D(z\ ,\ \zeta)\ ,$$

and the factor $|\det g'|^2$ does not depend on z or ζ. Hence, it suffices to prove Lemma 2.3 for a single fixed ζ_0 with a constant C_D which does not depend on z or ζ. Pick and fix ζ_0.

We now reduce to the case of a bounded domain. Let T be a biholomorphic map of D to a bounded domain R which has the property that if $(z_1, \ldots, z_n)$ is in R and α_i are numbers with $|\alpha_i| \leq 1$, $i = 1, \ldots, n$ then $(\alpha_1 z_1, \alpha_2 z_2, \ldots, \alpha_n z_n)$ is also in R. We further suppose that $T(\zeta_0) = \zeta$. (The existence of such a T and R is insured by results in [30].) We denote by B_R the Bergman kernel function for R. We wish to show

$$\left|\frac{B(z,\zeta_0)}{B(z,\zeta)} - 1\right| \leq C_{\mathbf{D}} d(\zeta\ ,\ \zeta_0)$$

if $d(z\ ,\ \zeta_0)$ is small. By (2.9) this is equivalent to showing that

$$\left|\frac{B_R(Tz,0)}{B_R(Tz\ ,T\zeta)}\left(\overline{\frac{\det T'(\zeta_0)}{\det T'(\zeta)}}\right) - 1\right| \leq C_D d(\zeta\ ,\ \zeta_0)\ .$$

Now note that d is the invariant distance on D. Hence, denoting by d_R the invariant distance on R, $d_R(Tz\ ,\ 0) = d(z\ ,\ \zeta_0)$. However we are only interested in z with $d(\zeta\ ,\ \zeta_0) \leq 10$. Hence Tz will be in a compact subset of B. On such a subset the invariant distance is comparable to the ordinary Euclidean distance. Also, the ratio $\det (T'(\zeta_0))/\det (T'(\zeta)$ is of the form $1 + O(|T\zeta|)$ for ζ near ζ_0. Hence we must show that

$$\left|\frac{B_R(w,0)}{B_R(w,v)} - 1\right| \leq C|v|$$

for all w in R and all v in a fixed compact subset K contained in R.

The simple shape of R makes it possible to write down the form of the Bergman kernel. Let z^α be the homogeneous monomial with exponent given by the multi-index α. We claim that there are positive constants c_α so that

$$B_R(z\ ,\ \zeta) = \sum_i c_\alpha z^\alpha \overline{\zeta}^\alpha\ .$$

To show this, it suffices to show that the z^α are pairwise orghogonal and that their span is dense in $A^2(R)$. The orthogonality follows from the rotational symmetry of R. The density of the polynomials is a consequence of the fact that R is starshaped. More specifically, if $f(z)$ is in $A^2(R)$ and $0 < r < 1$ then $fr(z) = f(ra)$ is in A^2 and f is the norm limit of the f_r. However f_r has a Taylor series representation using the z^α and the coefficients of the series tend to zero at least as fast as $r^{-|\alpha|}$. Hence the f_r and thus f are in the norm closure of the polynomials.

One consequence of this is that $B(0, \zeta) = C_0$ is not zero. Hence, since R has a transitive group of automorphisms, by (2.9) $B(z, \zeta)$ doesn't vanish. Thus it suffices to show

$$|B_R(w, 0) - B_R(w, v)| \leq c|v|B_R(w, v)$$

for all w in R and all v in K. However this is immediate as soon as we note that $B_R(w, v)$ is actually real analytic in an open neighborhood of $\bar{R} \times K$. The reason that $B_R(\cdot, v)$ extends analytically past $\bar{R}$ (when v is in K) is the identity

$$B_R(\tfrac{1}{r}w, rv) = B_R(w, v)$$

which is a direct consequence of the formula for B_R.

It would be interesting to know if Lemma 2.3 remains valid for other classes of domains - for instance, homogeneous domains.

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