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AN EXTENSION OF PIZZETTI'S FORMULA TO RIEMMANIAN MANIFOLDS

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This note summarizes joint work of Alfred Gray and myself. Let $M_m(r,f)$ denote the mean-value of a real-valued integrable function f over a sphere with centre m and radius r in n-dimensional eucldean space R^n. Then the formula of Pizzetti [PI.1], [PI.2], [CH,P.287] states that

$$(1.1.)\qquad M_m(r,f,) = \Gamma(\tfrac{1}{2})n \sum_0^\infty \left[\frac{r}{2}\right]^{2k} \frac{(\Delta^k f)_m}{k!\Gamma(\frac{1}{2})n+k)} .$$

Formula (1.1) can be written more compactly as a Bessel function in $\sqrt{-\Delta}$ [ZA]. We have

$$(1.2)\qquad M_m(r,f) = \left[j_{(n/2)-1}\,(r\sqrt{-\Delta})f\right]\ (m)$$

where

$$j_\ell(z) = 2^\ell\Gamma(\ell + 1)J_\ell(z)/z^\ell$$

and J^1 is the Bessel function of the first kind of order .

In this paper we generalize (1.1) to arbitrary Riemannian manifolds. Our formula also generalizes the mean-value theorem for harmonic spaces [WI]. Complete proofs will appear elsewhere. In the Riemannian case we define $M_m(r,f)$ as the mean-value of f over a geodesic sphere with centre m and radius r in an n-dimensional Riemannian manifold M. More precisely we have

$$(1.3.)\qquad M_m(r,f) = \int_{\exp_m(S^{n-1}(r))} f\omega \;/V(\exp_m(S^{n-1}(r)))$$

where $\exp_m$ is the exponential map of M at m.

The exponential map $\exp_m$ can be used to transfer formulas from M to the tangent space M_m. Let g_{ij} be the components of the metric tensor with respect to a system of normal coordinates, $(x_1,\ldots,x_n)$ and put

$$\theta = \sqrt{\det(g_{ij})}\ , \qquad \tilde{\Delta}_m = \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2} .$$

Both θ and $\tilde{\Delta}_m$ are independent of the choice of normal coordinates at m.

We have

<u>Theorem 1</u>.

$$(1.4)\qquad M_m(r,f) = \frac{\sum_{k=0}^\infty \left(\frac{r}{2}\right)^{2k} \frac{1}{k!\Gamma(\frac{1}{2}n + k)} \tilde{\Delta}_m^{\,k}\,[f\theta]_m}{\sum_{k=0}^\infty \left(\frac{r}{2}\right)^{2k} \frac{1}{k!\Gamma(\frac{1}{2}n + k)} \tilde{\Delta}_m^{\,k}[\theta]_m}$$

$$= \frac{J_{(n/2)-1}(r\sqrt{-\tilde{\Delta}_m})[f\theta]_m}{J_{(n/2)-1}(r\sqrt{-\tilde{\Delta}_m})[\theta]_m}$$

The operator $\tilde{\Delta}_m$ is of some interest in itself. We prove that there is a globally defined differential operator L_{2k} of degree 2k which coincides with the k^{th} power of $\tilde{\Delta}_m$ at m. For example we find that

(1.5) $$(\tilde{\Delta}_m f)_m = (\Delta f)_m$$

(1.6) $$(\tilde{\Delta}_m{}^2 f)_m = (\Delta^2 f + \frac{1}{3} < df, d\tau > + \frac{2}{3} < \nabla^2 f, \rho >)_m$$

where Δ denotes the ordinary Laplacian, τ is the scalar curvature, ρ the Ricci curvature and $\nabla^2 f$ the Hessian of f. Using (1.4), (1.5) and (1.6) we prove

Theorem 2.

$$M_m(r,f) = f(m) + A(m)r^2 + B(m)r^4 + O(r^6)$$

as $r \to 0$, where

$$A = \frac{1}{2n}\Delta f$$

$$B = \frac{1}{24n(n+2)}(3\Delta^2 f - 2 < \nabla^2 f, \rho > - 3 < \nabla f, \nabla\tau > + \frac{4\tau}{n}\Delta f).$$

We have also computed the coefficient of r^6 but it is too complicated to write down here.

2. A CHARACTERIZATION OF EINSTEIN MANIFOLDS. As an immediate corollary to theorem 2 we have

(2.1) $$M_m(r,f) = f(m) + O(r^4) \text{ as } r \to 0$$

if and only if f is harmonic near m. We prove

Theorem 3.

Let M be an Einstein manifold and let $m \in M$. Then for small $r > 0$, every function harmonic near m has the mean-value property

(2.2) $$M_m(r,f) = f(m) + O(r^6) \text{ as } r \to 0$$

Conversely, we have

Theorem 4. Let M be can analytic manifold and let $m \in M$. If for small $r > 0$, every function harmonic near m has the mean-value property

(2.3) $$M_m(r,f) = f(m) + o(r^6) \text{ as } r \to 0.$$

then M is Einstein.

The analyticity is required in theorem 4 because our proof depends upon the Gauchy-Kowalewski existence theorem for elliptic operators. We thus obtain a characterization of Einstein manifolds by the mean-value property (2.2). A similar but more complicated characterization of Einstein manifolds is given in [FR].

3. A CHARACTERIAZATION OF SUPER-EINSTEIN MANIFOLDS. We denote by $\dot{R}$ the symmetric tensor field given by

$$\dot{R}(x,y) = \sum_{i,j,k=1}^{n} R(e_i,e_j,e_k,x)R(e_i,e_j,e_k,y)$$

for $x,y \in M_m$, where $e_1,\ldots,e_n$ is an orthonormal basis of M_m. We define a manifold to be super-Einstein if it is Einstein and in addition satisfies the condidition

(3.1) $$\dot{R}(x,y) = \frac{1}{n}\,||R||^2 < x,y > \qquad \text{for } n > 4,$$

$$||R||^2 = \text{constant for } n = 4.$$

This definition is suggested in [BE, p.165]. It is easy to see that an irreducible symmetric space is super-Einstein . There exist metrics on spheres of dimension $4n + 3$ which are Einstein but not super-Einstein.

We prove

Theorem 5. Let M be a super-Einstein manifold, and let $m \in M$. Then for small $r > 0$, every function harmonic near m has the mean-value property

$$M_m(r,f) = f(m) + O(r^8) \text{ as } r \to 0.$$

The proof depends upon the explicit calculation of the coefficient of r^6 in the expansion of $M_m(r,f)$. Conversely, we have

Theorem 6. Let M be an analytic manifold, and let $m \in M$. If for small $r \to 0$, every function harmonic near m has the mean-value property

$$M_m(r,f) = f(m) + O(r^8) \text{ as } r \to 0, \text{ then M is super-Einstein.}$$

Again we make use of the Gauchy-Kowalewski theorem to prove theorem 6. We thus obtain a characterization of super-Einstein manifolds.

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