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FREE RESOLUTIONS OF TENSOR FORMS

H. Andreas Nielsen

Introduction

Let $\{T_{ij}\}$ be a $m \times n$ matrix of indeterminates. For all t we construct bounded free resolutions of the ideals generated by t-minors of $\{T_{ij}\}$ in the polynomial ring $\mathbb{Z}[\{T_{ij}\}]$. If $m = n$ and $T_{ij} = T_{ji}$ (resp. $T_{ij} = -T_{ji}$), we construct, by the same procedure, bounded free resolutions of the ideals generated by t-minors (reps. 2t-pfaffians). Also Plücker and Veronese embeddings are treated.

The constructions are global in the sense that they are carried out for a graded symmetric algebra of a locally free module over a noetherian scheme. A base change along a given cosection provide us with locally free resolutions in the perfect, depth generic cases. If the base scheme is defined over the field of rational numbers, a homotopy construction gives rise to minimal resolutions similar to those previously obtained by Józefiak and Pragacz [6], Lascoux [10,11] and Nielsen [12].

In section 1 we give the general graded construction which is of interest in itself. Sections 2 and 3 contain the applications and examples mentioned above.

1. Graded complexes

Objects M, N are graded by the integers $\mathbb{Z}$ with $[M]^n$ the n'th homogeneous component. Graded maps $f: M \to N$ have degree 0 and n'th homogeneous component $[f]^n: [M]^n \to [N]^n$. $M(m)$ and $f(m)$ denote the same object and map with shifted grading $[M(m)]^n = [M]^{m+n}$, $f(m): M(m) \to N(m)$ with $[f(m)]^n = [f]^{m+n}$. Graded modules M over a graded ring A satisfy $[A]^m[M]^n \subseteq [M]^{m+n}$.

For a module V, SV (ΛV) denote the symmetric (exterior) algebra with $[SV]^n = S^nV$ ($[\Lambda V]^n = \Lambda^n V$) being n'th symmetric (exterior) product of V for $n \geq 0$ and the zero module for $n < 0$.

Definition 1.1. Let V be a module on a scheme (X, O_X) and let M be a graded SV-module. For all $p, q \in \mathbb{Z}$, we define graded SV-modules

$$E_0^{pq} = [M]^q \otimes_{O_X} \Lambda^{-p-q} V \otimes_{O_X} SV(p) \tag{1.2}$$

and graded SV-linear maps

$$d_0^{pq}: E_0^{pq} \to E_0^{pq+1}; \qquad d_1^{pq}: E_0^{pq} \to E_0^{p+1q} \tag{1.3}$$

satisfying: $d_0^{pq+1} d_0^{pq} = 0$, $d_1^{p1+q} d_1^{pq} = 0$, $d_1^{pq+1} d_0^{pq} + d_0^{p+1q} d_1^{pq} = 0$;

in non-trivial cases given by

$$[d_0^{pq}]^n (m^q \otimes v_1 \wedge \ldots \wedge v_{-p-q} \otimes w_1 \otimes \ldots \otimes w_{p+n})$$

$$= \Sigma_{i \in [1,-p-q]} (-1)^{i-1} v_i m^q \otimes v_1 \wedge \ldots \hat{v}_i \ldots \wedge v_{-p-q} \otimes w_1 \otimes \ldots \otimes w_{p+n}$$

$$[d_1^{pq}]^n (m^q \otimes v_1 \wedge \ldots \wedge v_{-p-q} \otimes w_1 \otimes \ldots \otimes w_{p+n})$$

$$= \Sigma_{i \in [1,-p-q]} (-1)^{i-1} m^q \otimes v_1 \wedge \ldots \hat{v}_i \ldots \wedge v_{-p-q} \otimes v_i \otimes w_1 \otimes \ldots \otimes w_{p+n}.$$

Altogether we have a double complex

(1.4) $$(E_0^{pq}, d_0^{pq}, d_1^{pq})_{p,q\in\mathbb{Z}}$$

with total complex

(1.5) $$(E_0^r, d_0^r)_{r\in\mathbb{Z}}; \qquad E_0^r = \coprod_{p+q=r} E_0^{pq}, \qquad d_0^r = \coprod_{p+q=r} d_0^{pq} + d_1^{pq}.$$

Definition 1.6. A (double) complex is called bounded if only finitely many chain objects are $\neq 0$. A complex $(K^r, d^r)_{r\in\mathbb{Z}}$ is called a resolution (of M) if, $K^r = 0$ for $r > 0$, $H^r(K^\cdot) = 0$ for $r \neq 0$ (and $H^o(K^\cdot) \cong M$).

Theorem 1.7. Let V be a coherent locally free module on a noetherian scheme (X, O_X) and assume M to be a graded coherent SV-module with all $[M]^q$ locally free O_X-modules.

The complex (1.5) is a bounded resolution of M with locally free graded SV-modules.

Put $d = \sup\{p \in \mathbb{Z} \mid [\mathrm{Tor}.^{SV}(M, SV/VSV]^p \neq 0\}$ and denote by

(1.8) $$(E^{pq}, d_0^{pq}, d_1^{pq})_{p\geq -d}$$

the bounded double complex having the same chains and differentials as (1.4) for $p \geq -d$ and 0 else.

The total complex of (1.8)

(1.9) $$(E^r, d^r); \qquad E^r = \coprod_{p\geq -d} [M]^{r-p} \otimes_{O_X} \Lambda^{-r} V \otimes_{O_X} SV(p)$$

is a bounded resolution of M with E^r coherent locally free graded SV-modules.

Proof. For fixed $n \in \mathbb{Z}$ the graded component $([E_0^{pq}]^n, [d_0^{pq}]^n, [d_1^{pq}]^n)_{p,q\in\mathbb{Z}}$ of (1.4) is a bounded double complex with column cohomology $H^q([E_0^{p\cdot}]^n,[d_0^{p\cdot}]^n) \cong [\mathrm{Tor}^{SV}_{-p-q}(M,SV/VSV)]^{-p}\otimes_{O_X} S^{p+n}V$ and rows $([E_0^{pq}]^n,[d_1^{pq}]^n)_{p\in\mathbb{Z}}$ exact for $q \neq n$ with cohomology $H^{-n}([E_0^{\cdot n}]^n,[d_1^{\cdot n}]^n) \cong [M]^n$ and zero else. The conclusions follow easily now.

Proposition 1.10. Given homotopy equivalences $(E^{pq},d_0^{pq})_{q\in\mathbb{Z}} \cong (H^q(E^{p\cdot},d_0^{p\cdot}))_{q\in\mathbb{Z}}$ of each column in the double complex (1.8) to its own cohomology, we may construct a bounded complex with coherent locally free graded chains

$$E_1^r = \coprod_{q\in\mathbb{Z}}[\mathrm{Tor}^{SV}_{-r}(M,SV/VSV)]^q \otimes_{O_X} SV(r-q) \tag{1.11}$$

and differentials $d_1^r\colon E_1^r \to E_1^{r+1}$ satisfying $d_1^r \otimes_{SV} \mathrm{id}_{SV/VSV} = 0$ together with a homotopy equivalence between the complex $(E_1^r,d_1^r)_{r\in\mathbb{Z}}$ and the resolution (1.9).

Proof. Since (1.11) is column cohomology, we only need the induction step after column index p. Given a diagram of maps of complexes

$$\begin{array}{ccc} D_1 & \xrightarrow{\ f\ } & D_0 \\ {\scriptstyle p_1}\downarrow\uparrow{\scriptstyle i_1} & & {\scriptstyle p_0}\downarrow\uparrow{\scriptstyle i_0} \\ C_1 & & C_0 \end{array}$$

and homotopies $d_k s_k + s_k d_k = p_k i_k - \mathrm{id}$; $d_k t_k + t_k d_k = i_k p_k - \mathrm{id}$, $k = 0,1$. Then (mapping cone $C^n(f) = D_1^{n+1} \oplus D_0^n$, $d^n = -d_{D_1}^{n+1} + d_{D_0}^n + f^{n+1}$) there are maps $C^\cdot(f) \underset{i}{\overset{p}{\rightleftarrows}} C^\cdot(p_0 f i_1)$ and homotopies $ds + sd = pi - \mathrm{id}$; $dt + td = ip - \mathrm{id}$, given by $p = p_1 + p_0 + p_0 f t_1$ and i, s and t

are determined by chase in the following commutative diagram coming from the long exact homotopy sequences ([,] denotes homotopy classes of maps of complexes, [ ] shift, cf.[14])

$$\begin{array}{ccccccccc} [C_0[1],C^\cdot(f)] & \to & [C_1[1],C^\cdot(f)] & \to & [C^\cdot(p_0fi_1),C^\cdot(f)] & \to & [C_0,C^\cdot(f)] & \to & [C_1,C^\cdot(f)] \\ \downarrow -\circ p_0[1] & & \downarrow -\circ p_1[1] & & \downarrow -\circ p & & \downarrow -\circ p_0 & & \downarrow -\circ p_1 \\ [D_0[1],C^\cdot(f)] & \to & [D_1[1],C^\cdot(f)] & \to & [C^\cdot(f),C^\cdot(f)] & \to & [D_0,C^\cdot(f)] & \to & [D_1,C^\cdot(f)] \end{array}$$

Unfortunately, this provides us with complicated looking formulas, e.g. one gets an i with 11 terms.

Corollary 1.12. Let $s: V \to O_X$ be a cosection of V and denote O_X regarded as (nongraded) SV-module through s by ${}_sO_X$.

If $\mathrm{Tor}_i^{SV}(M,{}_sO_X) = 0$ for $i \neq 0$, then the complex

$$(1.13) \qquad ({}_sE^r, {}_sd^r) = (E^r \otimes_{SV} {}_sO_X, d^r \otimes_{SV} \mathrm{id}_{{}_sO_X})$$

$$ {}_sE^r = \coprod_{p+q=r} [M]^q \otimes_{O_X} \Lambda^{-p-q}V$$

is a bounded resolution of $M \otimes_{SV} {}_sO_X$ with coherent locally free O_X-modules.

If moreover the assumptions of Proposition 1.10 are satisfied, then the complex

$$(1.14) \qquad ({}_sE_1^r, {}_sd_1^r) = (E_1^r \otimes_{SV} {}_sO_X, d_1^r \otimes_{SV} \mathrm{id}_{{}_sO_X})$$

$$ {}_sE_1^r = \mathrm{Tor}_{-r}^{SV}(M, SV/VSV)$$

is a bounded coherent O_X-locally free resolution homotopy equivalent to (1.13).

Lemma 1.15. If for each maximal $x \in \operatorname{Supp} M \otimes_{SV} {}_sO_X$ depth $O_{X,x} \geqq \operatorname{pd}_{SV \otimes_{O_X} O_{X,x}} M \otimes_{O_X} O_{X,x}$, then $\operatorname{Tor}_i^{SV}(M, {}_sO_X) = 0$ for $i \neq 0$.

Proof. Lemme d'acyclicité.

2. Minors of a general matrix

Partitions $\chi \vdash n$ of a natural number $n \in \mathbb{N}$ are functions $\chi\colon \mathbb{N} \to \mathbb{N}_0$ such that $\chi(i) \geqq \chi(i+1)$ and $\Sigma_{i \in \mathbb{N}} \chi(i) = n$. We introduce $\omega_n \vdash n$, $\omega_n(i) = 1$ for $i \leqq n$, the Young dual of $\chi \vdash$, $\chi^{\sim} = \Sigma_{i \in \mathbb{N}} \omega_{\chi(i)}$, the rank $\delta(\chi) = \sup\{i \in \mathbb{N} \mid \chi(i) \geqq i\}$, and the length $l(\chi) = \sup\{i \in \mathbb{N} \mid \chi \geqq \omega_i\}$, where $\chi \geqq \chi' \Leftrightarrow \chi(i) \geqq \chi(i')$ for all $i \in \mathbb{N}$. The Ferrers-Sylvester graph of χ is $\Gamma_\chi = \{(i,j) \in \mathbb{N} \times \mathbb{N} \mid j \leqq \chi(i)\}$.

Let E, F denote coherent locally free modules on a noetherian scheme (X, O_X). Set $V = E \otimes_{O_X} F$ and for $\chi \vdash n$, let $I_\chi \subseteqq SV$ be the SV-ideal generated locally by elements $\Pi_{j \in \mathbb{N}} \det\{e_{ij} \otimes f_{kj}\}_{i,k}$ for all indexing $\Gamma_\chi \to E_{|U}, (i,j) \to e_{ij}$ and $\Gamma_\chi \to F_{|U}, (k,j) \mapsto f_{kj}$, ($I_{\omega_t}$ ideal generated locally by t-minors).

Theorem 2.1. Suppose E, F have constant $\operatorname{rk} E = m$, $\operatorname{rk} F = n$ on X. Set $V_{\mathbb{Z}} = \mathbb{Z}^m \otimes_{\mathbb{Z}} \mathbb{Z}^n$ and $d = \sup\{p \in \mathbb{Z} \mid [\operatorname{Tor}_{\cdot}^{SV_{\mathbb{Z}}}(SV_{\mathbb{Z}}/I_{\omega_t}, \mathbb{Z})]^p \neq 0\}$.

In case $V = E \otimes_{O_X} F$, $M = SV/I_{\omega_t}$ defined above, the bounded double complex (1.8), $(E^{pq}, d_0^{pq}, d_1^{pq})_{p \geqq -d}$, has total complex with chains the coherent locally free graded SV-modules

$$(2.2) \qquad E^r = \coprod_{p \geq -d} [S\,E \otimes F/I_{\omega_t}]^{r-p} \otimes_{O_X} \Lambda^{-r} E \otimes F \otimes_{O_X} S\,E \otimes F(p)$$

and (E^r, d^r) is a bounded resolution of $S\,E \otimes F/I_{\omega_t}$

Proof. Follows from Theorem 1.7 once we note that $[M]^q$ are locally free and defined functorial with respect to base change, [1].

Remark 2.3. Unfortunately no calculation of the d in Theorem 2.1 is available. From Proposition 2.8 below, treating the case $\mathbb{Q} \subseteq O_X$, we get the lower bound $d \geq mn - \sup\{m,n\}(t-1)$.

It would be of great interest to know if $\mathrm{Tor}^{SV_{\mathbb{Z}}}(SV_{\mathbb{Z}}/I_{\omega_t}, \mathbb{Z})$ are free $\mathbb{Z}$-modules, in which case we would have equality above, or not.

Corollary 2.4. The complex (E^r, d^r) of Theorem 2.1 is a complex of functors on the category of pairs of quasi-coherent O_X-modules, giving a resolution of $S\,E \otimes F/I_{\omega_t}$ in case E, F are coherent locally free of constant $\mathrm{rk}\,E = m$, $\mathrm{rk}\,F = n$.

If we delete the summation restriction $p \geq -d$, then we get a functorial complex giving a resolution for all coherent locally free E, F.

Proposition 2.5. Let $E \to F^{\vee}$ be an O_X-linear map and let $s\colon E \otimes F \to O_X$ denote the induced cosection. Under the assumptions of theorem 2.1 we set $O_X/I_t = SV/I_{\omega_t} \otimes_{SV\,s} O_X$ and suppose that for all maximal $x \in \operatorname{Supp} O_X/I_t$, depth $O_{X,x} \geqq (m-t+1)\ (n-t+1)$, then = holds and the complex $({}_sE^r,\ {}_sd^r)$

$$(2.6) \qquad {}_sE^r = \coprod_{p \geqq -d} [SE \otimes F/I_{\omega_t}]^{r-p} \otimes_{O_X} \Lambda^{-r} E \otimes F$$

is a bounded resolution of O_X/I_t with coherent locally free O_X-modules.

Proof. Follows directly from corollary 1.12 and lemma 1.15 using strongly generically perfectness of $SV_{\mathbb{Z}}/I_{\omega_t}$, [4].

Remark 2.7. $l({}_sE^r,\ {}_sd^r) = \sup\{-r \in \mathbb{Z} \mid {}_sE^r \neq 0\} \leqq \inf\ \{d, mn\}$ as one easily sees.

Let us also remark that the graded components $[SE \otimes F/I_{\omega_t}]^q$ have canonical bases. If E, F are free with bases $\{e_1, \ldots e_m\}$, $\{f_1, \ldots, f_n\}$ then the elements $\Pi_{j \in \mathbb{N}} \det\{e_{i|j} \otimes f_{k|j}\}_{i,k}$ for $\chi \vdash q$ $l(\chi) \leqq t$ and all $\Gamma_\chi \to \{e_1, \ldots, e_m\}$, $(i,j) \mapsto e_{i|j}$ and all $\Gamma_\chi \to \{f_1, \ldots, f_n\}$, $(k,j) \mapsto f_{k|j}$ both satisfying $i-1|j < i|j \leqq i|j+1$, constitutes a basis for $[SE \otimes F/I_{\omega_t}]^q$. Cf. [1] for details.

In case $\mathbb{Q} \subseteqq O_X$ a complete description of the chains in the minimal resolution has been given by A. Lascoux. Cf. [10] and [12]. We restate the results here for completness and to give an impression of what "is needed" in the characteristic free cases.

<u>Tensor</u> (or Schur) <u>functors</u> ([10], [12],[13]) are defined for each partition $\chi \vdash n$, T_χ endofunctor on a module category, by $T_\chi E = \otimes_{i \in \mathbb{N}} S^{\chi(i)}E$ modulo submodule generated by elements $\otimes_i(e_{i1} \otimes \ldots \otimes e_{i\chi(i)})$ for all $\Gamma_\chi \to E$, $(i,j) \to e_{ij}$ such that $e_{i'j} = e_{ij}$ for some $(i',j) \neq (i,j)$.

<u>Proposition 2.8</u>. Let E, F be coherent locally free modules on a noetherian scheme (X, O_X) defined over Spec $\mathbb{Q}$. Set $V = E \otimes_{O_X} F$ and $M = SV/I_{\omega_t}$ then each column in the double complex (1,4) $(E_0^{pq}, d_0^{pq}, d_1^{pq})_{p,q \in \mathbb{Z}}$,

$$(E_0^{pq}, d_0^{pq})_{q \in \mathbb{Z}} = ([SE \otimes F/I_{\omega_t}]^q \otimes \Lambda^{-p-q} E \otimes F) \otimes SV(p)$$

is SV-linear homotopy equivalent to its own cohomology.

The resolution of M (1.11) (E_1^r, d_1^r) have chains given functorial

$$(2.9) \quad E_1^r = \coprod_{\chi \vdash -r} (E_1^r)_\chi \otimes SE \otimes F(r-(t-1)\delta(\chi))$$

where $(E_1^r)_\chi = T_{(\chi+(t-1)\omega_{\delta(\chi)})^\sim} E \otimes T_{(\chi^\sim+(t-1)\omega_{\delta(\chi)})^\sim} F$.

For $\mathrm{rk}E = m$, $\mathrm{rk}F = n$ constant on X, $l(E_1^r, d_1^r) = (m-t+1)(n-t+1)$.

<u>Proof</u>. From the description in [2] we see that in the category of endofunctors of $\mathbb{Q}$-modules it is effectively possible (may give an algorithm) to split mono- and and epimorhisms, so the first part follows from Proposition 1.10 using base change. For (2.9) there are calculations in [10], [12] or one may use Bott's theorem on cohomology of line bundles together with the Weyl character formula.

Corollary 2.10. The functorial complex (2.9) is unique up to unique natural isomorphism.

Proof. From the proof of (1.7) follows that the chains in (2.9) are unique up to isomorphism. Since everything is SV-linear it suffices to see for each $\chi \vdash - r$ that the multiplicity of $(E_1^r)_\chi$ in E_1^{r+1} regarded as functors is 1. This is obvious from the Littlewood-Richardson formula, [12].

Corollary 2.11. If under the assumptions of Proposition 2.5 (X, O_X) is defined over Spec $\mathbb{Q}$ then $({}_sE_1^r, {}_sd_1^r)$ with chains

(2.12) $${}_sE_1^r = T_{(\chi+(t-1)\omega_{\delta(\chi)})}{}^{\sim}E \otimes T_{(\chi^{\sim}+(t-1)\omega_{\delta(\chi)})}{}^{\sim}F.$$

is a bounded locally free resolution of O_X/I_t of $l({}_sE_1^r, {}_sd_1^r) = (m-t+1)(n-t+1)$ being minimal in the fibre at $x \in \operatorname{Supp} O_X/I_1$.

3. A list of other cases

The constructions follow the general approach of section 1 using metholds similar to those of section 2.

Throughout this section E will denote a locally free module of constant $\operatorname{rk} E = m$ on a noetherian scheme (X, O_X).

Partitions and tensorfunctors introduced in section 2 will be used.

(3.1) Symmetric matrix ([1], [8] [11])

a) Set $V = S^2E$ and let $I_{2\omega_t}$ be the ideal in SV generated locally by elements $\det\{e_{i1} \otimes e_{i'2}\}_{i,i'}$, for all indexing $\Gamma_{2\omega_t} \to E_{|U}$, $(i,j) \mapsto e_{ij}$, (t-minors).

b) For $M = SV/I_{2\omega_t}$ the complex (1.5) is a bounded resolution of M with locally free graded SV-modules. M has canonical local bases, [1].

c) Put $d = \sup\{p \in \mathbb{Z} \mid [\mathrm{Tor}.^{SS^2\mathbb{Z}^m}(SS^2\mathbb{Z}^m/I_{2\omega_t}, \mathbb{Z})]^p \neq 0\}$ then the complex (1.9) is a bounded resolution of M with coherent locally free graded SV-modules.

d) Given $E \to E^\vee$ locally symmetric, i.e. inducing a cosection $s: V = S^2E \to O_X$, then if $\operatorname{depth} O_{X,x} \geq \frac{1}{2}(m-t+2)(m-t+1)$ for all maximal $x \in \operatorname{Supp} M \otimes_{SV} {}_sO_X$, we get locally free resolutions of $M \otimes_{SV} {}_sO_X = O_X/$ (ideal locally generated by t-minors of $E \to E^\vee$), as in 1.12 - 1.15.

e) Suppose moreover $\mathbb{Q} \subseteq O_X$, then we get locally free (graded) resolutions (1.11), (1.14) of $l(E_1^r) = \frac{1}{2}(m-t+2)(m-t+1)$, and the functorial chains are computed by Lascoux, [11].

(3.2) Alternating matrix ([1], [6], [9])

a) Set $V = \Lambda^2E$ and let $I_{\omega_{2t}}$ be the ideal in SV generated locally by the Pfaffians (of diagonal 2t-submatrices) in $S^t\Lambda^2E$.

b) For $M = SV/I_{\omega_{2t}}$ the complex (1.5) gives a bounded resolution of M with locally free graded SV-modules. M has canonical local bases, [1].

c) If we put $d = \sup\{p \in \mathbb{Z} \mid [\mathrm{Tor}.^{S\Lambda^2\mathbb{Z}^m}(S\Lambda^2\mathbb{Z}^m/I_{\omega_{2t}}, \mathbb{Z})]^p \neq 0\}$, the complex (1.9) gives a bounded resolution with coherent locally free graded SV-modules.

d) Given $E \to E^\vee$ locally alternating, i.e. induces a cosection $s: \Lambda^2E \to O_X$, then if $\operatorname{depth} O_{X,x} \geq \frac{1}{2}(m-2t+2)(m-2t+1)$ for all maximal $x \in \operatorname{Supp} M \otimes_{SV} {}_sO_X$, we get locally free resolutions of $M \otimes_{SV} {}_sO_X = O_X/$(ideal generated locally by Pfaffians of diagonal 2t-submatrices), as in 1.12 - 1.15.

e) Suppose moreover $\mathbb{Q} \subseteq O_X$ then we get resolutions (1.11), (1.14), of $l(E_1^r, d_1^r) = \frac{1}{2}(m-2t+2)(m-2t+1)$ and the functorial chains are computed by Józefiak and Pragacz, [6], in terms of tensor functors.

(3.3) Plücker embedding

a) Set $V = \Lambda^t E$ and let I_t be the ideal in SV generated locally by the Plücker relations in S^2V, [5].

b) For $M = SV/I_t$ the associated complex of (1.5) on $\mathbb{P}(\Lambda^t E)$ gives a locally free resolution of the Plücker embedding $\mathrm{Grass}_t E \rightarrow \mathbb{P}(\Lambda^t E)$, [4].

c) In case $\mathbb{Q} \subseteq O_X$ no calculation of the functorial chains for general t are known to me.

(3.4) Veronese embedding

a) Set $V = S^t E$ and let I_t be the ideal in SV generated by $I_t^2 = \mathrm{Ker}\,(S^2V \rightarrow S^{2t}E)$.

b) For $M = SV/I_t$ the associated complex of (1.5) on $\mathbb{P}(S^t E)$ gives a locally free resolution of the t-uple Veronese embedding $\mathbb{P}E \rightarrow \mathbb{P}\,S^t E$.

c) In case $\mathbb{Q} \subseteq O_X$ no calculation of the functorial chains are known to me, but the computations do not look very complicated. Indeed a calculation of the last non-vanishing chain module shows that the embedding is Gorenstein if and only if t divides $\mathrm{rk}\,E$.

4. Completing remarks

Ad. 1. Using a "reduction to diagonal" argument, e.g. as in the proof of "Tor rigidity" M. Auslander & D. Buchsbaum, Codimension and multiplicity, Ann. of Math. 2nd. ser. 68 (1958), p. 632, the double complex (1.4) could have been defined

$$(E_0^{pq}, d_0^{pq}, d_1^{pq}) = M \otimes_{SV} \Lambda^{\cdot} (V \otimes_{O_X} SV \otimes_{O_X} SV(-1))$$

the latter being the bigraded Koszul complex on the augmentation $V \otimes SV \otimes SV(-1) \to SV \otimes SV$, $v \to v \otimes 1 - 1 \otimes v$.

Ad. 2. During this conference L. L. Avramov pointed out that in case of a perfect module the highest grading of Tor appears at the last Tor. Since in case of determinantal ideals the type is independent of characteristic we have in (2.3) $d = mn - \sup\{m,n\}(t-1)$.

Ad. 3. In (3.2) c) $d = \frac{1}{2}m(m-2t+1)$ by the same reasoning as above. Moreover in characteristic 0 T. Józefiak has proved uniqueness of the minimal functorial complex (3.2)c).

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