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by Morihiko Saito

This note is a supplement to "Gauss-Manin system and mixed Hodge structure"(cited as [Sa]), which is submitted for publication in Proceedings of the Japan Academy. In this supplement, we discuss the following questions, which we could not discuss in full detail in the paper:

1) the necessity of a unipotent base change in the formulation of the result of Scherk and Steenbrink (e.g., counter-examples to the first formulation of Scherk, Steenbrink and Pham, cf.[Ph]),

2) the diffference between the limit Hodge filtration of Schmid (which is obtained using a unipotent base change) and the limit of Hodge filtration which is obtained without a base change.

§1. The main point of the paper [Sa] is the following: in the formulation of the result of Scherk and Steenbrink, it is necessary to take a unipotent base change. We give two examples in which the first formulation of Scherk, Steenbrink and Pham as stated in the first version of [Ph] does not apply.

(1.1) First we review the notations in [Sa],[SS] and [Ph].

Let $f:\mathbb{C}^{n+1},o \to \mathbb{C},o$ be a holomorphic function with an isolated singularity, and let $f:X \to S$ be a Milnor fibration so that $H_X := R^n f_* \mathbb{C}_X|_{S^*}$ is a local system on $S^* = S-\{0\}$. There is a natural extension $\mathcal{A}$ of H_X to the origin as a locally free $\mathcal{O}_S$-Module with a regular singular connection ∇, such that the eigenvalues of $\operatorname{res}(t\nabla_{d/dt})$ are in $(-1,0]$. ($\mathcal{A}$ is denoted by $\mathcal{L}_X$ in [Sa,(1.3)].) There is another extention $\mathcal{H}_X^{(o)}$, which we call the Brieskorn lattice. $\mathcal{H}_X^{(o)}$ is a locally free $\mathcal{O}_S$-Module with a a regular singular connection such that $\mathcal{H}_{X,o}^{(o)} \cong \Omega_{X,o}^{n+1} / df \wedge d\Omega_{X,o}^{n-1}$. It is known that there is a natural inclusion $\mathcal{H}_{X,o}^{(o)} \hookrightarrow \mathcal{A}$ (by Malgrange), which is $\mathcal{O}_S$-linear, preserves the connection and induces an isomorphism on S^*. $\mathcal{H}_{X,o}^{(o)}$ is also a free $\mathbb{C}\{\{\partial_t^{-1}\}\}$-module of rank μ, where $\mathbb{C}\{\{\partial_t^{-1}\}\} = \{ \sum_{i\geq 0} a_i \partial_t^{-1} : \sum a_i r^i / i! < \infty \ \exists r>0 \}$ and $\partial_t = \nabla_{d/dt}$ (Malgrange, Pham).

The Gauss-Manin system $\int_f^{\bullet} \mathcal{O}_X$ is defined as an integration of system (cf. [Ph],[Sa]). $\int_f^{\bullet} \mathcal{O}_X$ contains $\mathcal{A}$ and $\mathcal{H}_{X,o}^{(o)}$ naturally, and it is a holonomic system on S such that $DR(\int_f^{\bullet} \mathcal{O}_X) = R^n f_* \mathbb{C}_X$. ($\int_f^{\bullet} \mathcal{O}_X$ is denoted by $\mathcal{H}_X$ in [Sa].)

Let $X_\infty := X^* \times_{S^*} U$ be a base change of X^* by the universal covering $p: U \to S^*$. We set $H_\infty := H^n(X_\infty, C) (\simeq \Gamma(U, \rho^* H_X))$, i.e., H_∞ is the set of multivalued horizontal sections of H_X.

We have an isomorphism $H_\infty \xrightarrow{\sim} \mathcal{A}_o / t\mathcal{A}_o$, by $u \to \exp(-\log t \log M / 2\pi\sqrt{-1})\, u$, where M is the monodromy of H_X and the eigenvalues of $\log M$ are in $[o,1)$. Here we regard $\mathcal{A}$ as a subsheaf of $j_*(\mathcal{O}_{S^*} \otimes H_X)$, where $j: S^* \to S$ is an inclusion.

(1.2) The first formulation of Scherk, Steenbrink and Pham (cf.[Ph]) asserts the following.

Let $\{F_{St}^{\cdot}\}$ be the Hodge filtration of Steenbrink on H_∞, then we have

(1.2.1) $F_{St}^p = \partial_t^{n-p} \mathcal{H}^{(o)} \cap \mathcal{A}_o / \partial_t^{n-p} \mathcal{H}^{(o)} \cap t\mathcal{A}_o \ (\subset \mathcal{A}_o / t\mathcal{A}_o \simeq H_\infty)$

for any p, where we set $\mathcal{H}^{(o)} := \mathcal{H}_{X,o}^{(o)}$ and take intersections in $\int_f^{\bullet} \mathcal{O}_X$.

By a result of Steenbrink, $\{F^{\cdot}_{St}\}$ is compatible with the monodromy decomposition $H_\infty = \oplus_\lambda H_{\infty,\lambda}$, where $H_{\infty\lambda} = \{u \in H_\infty : (M-\lambda)^{n+1} u = 0\}$. First we give an example for $n = p = 1$ that $\mathcal{H}^{(o)} / \mathcal{H}^{(o)} \wedge t\mathcal{S}_o$ is not compatible with the decomposition (hence (1.2.1) does not hold.)

(1.3) Example 1. $f = x^5/5 + y^5/5 + a x^3y^3/3$.

This is the first example in which b-function changes under a μ-constant deformation (i.e., $b(s) = (s+1)\,\Pi_{i=2}^{8}(s + i/5)$ for $a=0$, and $b(s) = (s+1)\,\Pi_{i=2}^{7}(s + i/5)$ for $a\neq 0$ (by T. Miwa).)

We assume now $a\neq 0$.

We have a $\mathbb{C}\{\{\partial_t^{-1}\}\}$-basis $\{w_{ij} = x^{i-1}y^{j-1}dx\wedge dy\}_{i,j=1,\cdots,4}$ of $\mathcal{H}^{(o)}$. Let $\tilde{\mathcal{H}}^{(o)} = \sum_{i=0}^{1}(\partial_t t)^i\mathcal{H}^{(o)}$ be the saturation of $\mathcal{H}^{(o)}$. Then we have

$$\tilde{\mathcal{H}}^{(o)} = \sum_{(i,j)\neq(4,4)}\mathbb{C}\{\{\partial_t^{-1}\}\}w_{ij} + \mathbb{C}\{\{\partial_t^{-1}\}\}\partial_t w_{44}$$

Set

$$V^o := \sum_{j=1}^{3}\mathbb{C}\{\{\partial_t^{-1}\}\}w_{jj} + \mathbb{C}\{\{\partial_t^{-1}\}\}\partial_t w_{44},$$

$$V^k := \sum_{i-j\equiv k(\mathrm{mod}5)} \mathbb{C}\{\{\partial_t^{-1}\}\}w_{ij} \quad \text{for } k=1,\cdots,4.$$

We can verify that for $k=o,\cdots,4$, V^k is an $\mathcal{E}(o)$-submodule of $\tilde{\mathcal{H}}^{(o)}$. ($\mathcal{E}(0) \doteqdot \mathbb{C}\{t\}\{\{\partial_t^{-1}\}\}$) For there is a decomposition

$$\mathbb{C}\{x,y\}dx\wedge dy = \oplus_{k=0}^{4}\{\sum_{i-j\equiv k(\mathrm{mod}5)} a_{ij}x^{i-1}y^{j-1}dx\wedge dy\}$$

of $\Omega^2_{X,o}$ which induces the one on $\mathcal{H}^{(o)}$ such that the action of t and ∂_t^{-1} are compatible with it.

Hence there is a decomposition $\mathcal{S} = \oplus_{i=0}^{4}\mathcal{S}^i$ (resp. $H_X = \oplus H_X^i$, resp. $H_\infty = \oplus H_\infty^i$) as locally free $\mathcal{O}_S$-Modules with connection (resp. as local systems, resp. as vector spaces with monodromy action) such that $V^i = \tilde{\mathcal{H}}^{(o)} \wedge \mathcal{S}^i$ (resp. $\mathcal{S}^i$ is an extension of H_X^i, resp. $H_\infty^i = \Gamma(U,\rho^*H_X^i)$).

The action of $\partial_t t$ on V^o / tV^o is given by the following matrix,

	w_{11}	$\partial_t w_{44}$	w_{22}	w_{33}
w_{11}	2/5	0	0	0
$\partial_t w_{44}$	-a/15	3/5	0	0
w_{22}	*	*	4/5	0
w_{33}	*	*	*	6/5

This implies

$$w_{33} \equiv 0 \qquad (\mathrm{mod}\ t\mathscr{V})$$
$$w_{22} \equiv t^{-1/5} \otimes u_4 \qquad (\mathrm{mod}\ t\mathscr{V})$$
$$\partial_t w_{44} \equiv t^{-2/5} \otimes u_3 \qquad (\mathrm{mod}\ t\mathscr{V} + \mathbb{C}w_{22})$$
$$w_{11} \equiv t^{-3/5} \otimes u_2 - (a/3)\, t^{-2/5} \otimes u_3 \qquad (\mathrm{mod}\ t\mathscr{V} + \mathbb{C}w_{22})$$

where $\{u_i\}_{i=1,\cdots,4}$ is a basis of H^o_∞ such that $M u_i = \exp(-2\pi\sqrt{-1}\ i/5)\ u_i$.

Thus we have $(\mathcal{H}^{(o)}/\ \mathcal{H}^{(o)} \cap t\mathscr{S}_o) \cap H^o_\infty = \mathbb{C}u_4 + \mathbb{C}(u_2-(a/3)u_3)$, hence $\mathcal{H}^{(o)}/\ \mathcal{H}^{(o)} \cap t\mathscr{S}_o$ is not compatible with the monodromy decomposition, because we have $\mathcal{H}^{(o)} = \oplus_i\ \mathcal{H}^{(o)} \cap \mathscr{S}^i$.

<u>Remark</u>. We have $F^1_{St} \cap H^o_\infty = \mathbb{C}u_2 + \mathbb{C}u_4$, because we have

$$\tilde{t}^3 \pi^* w_{11} \equiv 1 \otimes u_2\ (\mathrm{mod}\ \tilde{t}\,\tilde{\mathscr{S}}),$$

where $\pi:\tilde{S} \to S$ is a 5-fold covering such that $\pi^* t = \tilde{t}^5$ and $\tilde{\mathscr{S}}$ (= $\tilde{\mathcal{L}}_X$ in [Sa]) is an extention of $\pi^* H_X$ as in (1.1) (cf.[Sa(3.2)]).

(1.4) <u>Example</u> 2.

Let $f:\mathbb{C}^2,o \to \mathbb{C},o$ be a holomorphic function such that $\{f=0\}$ is an irreducible and reduced curve. We show that $F^1_{St} \neq \mathcal{H}^{(o)}/\ \mathcal{H}^{(o)} \cap t\mathscr{S}_o$ if f is not quasi-homogeneous.

Proof) By a result of Lê and A'Campo, the local monodromy is semi-simple and $H_{\infty,1} = \{0\}$. Suppose $F^1_{St} = \mathcal{H}^{(o)}/\mathcal{H}^{(o)} \cap t\mathscr{S}_o$ holds.

There is a basis $\{u_j\}_{j=1,\cdots,\mu}$ of H_∞ such that $F^1_{St} = \sum_{j=1}^{\mu/2} \mathbb{C}u_j$ and $M u_j = \exp(-2\pi\sqrt{-1}\ \alpha_j)\ u_j$, for F^1_{St} is compatible with the monodromy decomposition. We may assume that $-1 < \alpha_j < 0$ $(1\leqq j\leqq \mu/2)$, $0 < \alpha_j < 1$ $(\mu/2<j\leqq\mu)$ and $\alpha_j + \alpha_{\mu+1-j} = 0$ by the duality of exponents.

We set $v:= t^{\alpha_j} \otimes u_j \in \mathcal{S}$ for $j=1,\cdots,\mu$ and $V:= \sum_{j=1}^{\mu} \mathbb{C}\{t\}v_j \subset \mathcal{S}$. V is a free $\mathcal{O}_S$-Module containing $\mathcal{H}_X^{(o)}$, because of $F^1_{St} = \mathcal{H}^{(o)}/\ \mathcal{H}^{(o)} \cap t\mathcal{S}_o$.

Let $\{\gamma_i(t)\}_{i=1,\cdots,\mu}$ be a multivalued horizontal basis of $\bigsqcup_{t\in S^*} H_1(X_t,\mathbb{C})$ and $\{w_j\}_{j=1,\cdots,\mu}$ be a $\mathcal{O}_S$-basis of $\mathcal{H}_X^{(o)}$. Then $(\det(\int_{\gamma_i(t)} v_j))^2$ and $(\det(\int_{\gamma_i(t)} w_j))^2$ are both nowhere vanishing holomorphic functions on S, due to the duality of exponents and a lemma of Kyoji Saito.

Then we have $V = \mathcal{H}^{(o)}$, for there is a basis $\{e_i\}$ of V such that $\{t^{m_i}e_i\}$ is a basis of $\mathcal{H}^{(o)}$ $(m_i\geqq 0)$.

It is clear that $\mathcal{H}^{(o)} = V$ is saturated (i.e., $t\partial_t V \subset V$). Hence f is quasihomogeneous by a result of Kyoji Saito. Q.E.D.

Remark. In general, we can show the following.
Let $f:\mathbb{C}^2,o \to \mathbb{C},o$ be a holomorphic function with an isolated singularity. We assume that the local monodromy of f is semi-simple. Then $\mathcal{H}^{(o)}/\ \mathcal{H}^{(o)} \cap t\mathcal{S}_o$ is compatible with the monodromy decomposition, if and only if f is quasihomogeneous.

Problem. For n=1, does the subspace $\mathcal{H}^{(o)}/\ \mathcal{H}^{(o)} \cap t\mathcal{S}_o$ of H_∞ determine the local moduli of f in the family of μ-constant deformation? In general, does $\mathcal{H}^{(o)} \subset \mathcal{S}$ determine the local moduli of f in the μ-constant family?

§2. The examples in §1 mean that the proof of the formulation of Scherk, Steenbrink and Pham such as stated in the first version of [Ph] is not complete. This contradiction comes from the following.

(2.1) Let $(H_{\mathbb{Z}}, \mathcal{F}^{\cdot})$ be a polarizable variation of Hodge structure of weight n on S^*: i.e., $H_{\mathbb{Z}}$ is a local system on S^*, $\mathcal{F}^{\cdot}$ are holomorphic subbundles of $\mathcal{O}_{S^*} \otimes H_{\mathbb{Z}}$ such that $\partial_t \mathcal{F}^p \subset \mathcal{F}^{p-1}$, and there is a bilinear form $H_{\mathbb{Z}} \otimes H_{\mathbb{Z}} \to \mathbb{Z}$ such that they induce a polarized Hodge structure on $H_{\mathbb{C},t}$ for $\forall t \in S^*$. Here $H_{\mathbb{C}} = R^n\bar{f}_*\mathbb{C}_Y|_{S^*}$ and $\bar{f} : Y \to S$ is a compactification of a Milnor fibration $f : X \to S$, cf. [Sa,(1.1)].

Then $\mathcal{F}^{\cdot}$ can be extended to the origin as subbundles $\hat{\mathcal{F}}^{\cdot}$ of $\mathcal{J}$, where $\mathcal{J}$ is an extension of $H_{\mathbb{C}} = \mathbb{C} \otimes H_{\mathbb{Z}}$ as in (1.1). But the limit filtration $\hat{\mathcal{F}}^{\cdot}|_{t=0}$ of $H_{\mathbb{C},\infty} \simeq \mathcal{J}/t\mathcal{J}$ is different from the filtration $F^{\cdot}_{\infty}$ of Schmid, which is obtained using a unipotent base change by Steenbrink. ($H_{\mathbb{C},\infty} := \Gamma(U, \rho^* H_{\mathbb{C}})$, cf.(1.1))

(2.2) First we show the existence of the extension $\hat{\mathcal{F}}^{\cdot}$.

We fix the coordinates t and z of S and U such that $S = \{|t|<1\}$, $U = \{\mathrm{Im}\, z>0\}$ and $\rho^* t = \exp(2\pi\sqrt{-1}\, z)$.

A natural isomorphism $H_{\mathbb{C},\infty} = \Gamma(U,\rho^* H_{\mathbb{C}}) \xrightarrow{\sim} (\rho^* H_{\mathbb{C}})_z$ induces a Hodge filtration $F^{\cdot}_z$ on $H_{\mathbb{C},\infty}$, which depends holomorphically on z. As we have $F^{\cdot}_{z+1} = M^{-1} F^{\cdot}_z$ for $\forall z \in U$, $\exp(z \log M)\, F^{\cdot}_z$ are filtrations on $H_{\mathbb{C},\infty}$ which depend only on $t = \exp(2\pi\sqrt{-1}\, z)$.

Let $M = M_s M_u$ be the Jordan decomposition of M and set $N := \log M_u$ (N is nilpotent). As M_s has a finite order e (cf. [Sc,(6.1)]), $\exp(z N)\, F^{\cdot}_z$ depends on $\tilde{t} := \exp(2\pi\sqrt{-1}\, z/e)$.

The Theorem of Schmid [Sc, 6.16] assures that there exists a limit $F_\infty^\cdot = \lim_{\mathrm{Im}\ z \to \infty} \exp(zN)\, F_z^\cdot$ in the flag manifold of $H_{\mathbb{C},\infty}$, such that the Hodge filtration $\{F_\infty^\cdot\}$ and the monodromy weight filtration $\{W.\}$ determine a mixed Hodge structure on $H_{\mathbb{Z},\infty}$.

Using this theorem we show the existence of $\hat{\mathcal{F}}^\cdot$.

If we choose an $\mathcal{O}_S$-basis of $\mathcal{J}$, the subbundles $\mathcal{F}^\cdot$ determine a holomorphic map $\Phi: S^* \to \mathrm{Flag}(\mathbb{C}^m)$, and the existence of $\hat{\mathcal{F}}^\cdot$ is equivalent to the extension of Φ on S.

Let $\{u_{ij}\}_{i=1,\cdots,\ell,\ j=0,\cdots,r_i-1}$ be a basis of $H_{\mathbb{C},\infty}$ such that $-N/(2\pi\sqrt{-1})\, u_{ij} = u_{i,j-1}$ $(u_{i,-1} := 0)$, $M_s u_{ij} = \exp(2\pi\sqrt{-1}\, a_i/e)\, u_{ij}$ $(a_i \in [0,e-1])$.

Then $\{v_{ij} = \exp(-\log t \log M/2\pi\sqrt{-1})\, u_{ij}\}_{ij}$ (resp. $\{\tilde{v}_{ij} = \exp(-\log \tilde{t}\ eN/2\pi\sqrt{-1})\, u_{ij}\}_{ij}$) is a $\mathcal{O}_S$-(resp. $\mathcal{O}_{\tilde{S}}$-) basis of $\mathcal{J}$ (resp. $\tilde{\mathcal{J}}$), where the eigenvalues of $\log M$ are in $[0,1)$. We remark that in general we have $\tilde{\mathcal{J}} \neq \pi^*\mathcal{J}$, i.e., there is a natural inclusion $\tilde{\mathcal{J}} \subset \pi^*\mathcal{J}$ such that $\pi^* v_{ij} = \tilde{t}^{-a_i}\, \tilde{v}_{ij}$.

Using these basis, $\mathcal{F}^\cdot$ (resp. $\tilde{\mathcal{F}}^\cdot := \pi^*\mathcal{F}^\cdot$) can be identified with a holomorphic map $\Phi: S^* \to \mathrm{Flag}(H_{\mathbb{C},\infty})$ (resp. $\tilde{\Phi}: \tilde{S}^* \to \mathrm{Flag}(H_{\mathbb{C},\infty})$) such that $\Phi(t) = \exp(z \log M)\, F_z^\cdot$ (resp. $\tilde{\Phi}(\tilde{t}) = \exp(zN)\, F_z^\cdot$), for $t = \exp(2\pi\sqrt{-1}\, z)$ (resp. $\tilde{t} = \exp(2\pi\sqrt{-1}\, z/e)$).

Using Plücker coordinates, we can regard Φ (resp. $\tilde{\Phi}$) as $\Phi = (\phi_0(t):\cdots:\phi_k(t)): S^* \to \mathbb{P}^k$ (resp. $\tilde{\Phi} = (\tilde{\phi}_0(\tilde{t}):\cdots:\tilde{\phi}_k(\tilde{t})): \tilde{S}^* \to \mathbb{P}^k$), where ϕ_i (resp. $\tilde{\phi}_i$) are holomorphic functions on S^* (resp. $\tilde{S}^*$). Moreover, there are holomorphic functions g_i on $\tilde{S}$ such that $\phi_i(\pi(\tilde{t})) = g_i(\tilde{t})\, \tilde{\phi}_i(\tilde{t})$, because we have $\pi^* v_{ij} = \tilde{t}^{-a_i}\, \tilde{v}_{ij}$ and vector bundles on S^* are trivial.

By the result of Schmid, $\tilde{\Phi}$ can be extended to the origin holomorphically. Hence there is a nowhere vanishing holomorphic function h on $\tilde{S}^*$ such that $h \cdot \tilde{\phi}_i$ and $h \cdot \pi^* \phi_i$ are holomorphic at the origin. Let $h(\tilde{t}) = \sum_{j=0}^{e-1} h_j(\pi(\tilde{t}))\cdot\tilde{t}^j$ be a decomposition of h such that h_j are holomorphic functions on S^*. Then $h_j\cdot\phi_i$ are extended to the origin, and also is Φ. Q.E.D.

(2.3) The reason why $\hat{\mathcal{F}}^{\cdot}|_{t=0} \neq F_{\infty}^{\cdot}$ is obvious from the proof. If $\tilde{\Phi} = \Phi\circ\pi$, they coincide, but this does not hold in general.

Example 3. Let $H_{\mathbb{Z}}$ be a local system on S^*, having a multi-valued basis $\{e_1, e_2\}$ such that $M\,e_1 = e_2$, $M\,e_2 = -e_1 - e_2$, where M is the monodromy of $H_{\mathbb{Z}}$.($M^3=1$) We define a skew symmetric bilinear form $\langle,\rangle$ on $H_{\mathbb{Z}}$ by $\langle e_1, e_2\rangle = 1$, and a Hodge subbundle $\mathcal{F}^1 := \mathcal{O}_{S^*}\,v \subset \mathcal{O}\otimes H_{\mathbb{Z}}$ by $v := g(t)\otimes e_1 + h(t)\otimes e_2$, where $g(t) := -a\,t^{-1/3} + \zeta\,t^{-2/3}$, $h(t) := a\,\zeta\,t^{-1/3} - t^{-2/3}$, $\zeta^3 = 1$, $\operatorname{Im}\zeta > 0$, $a \in \mathbb{C}$, $a \neq 0$ and $|a| \ll 1$.

It is easy to see that they form a polarized variation of Hodge structure of weight 1. (We set $\mathcal{F}^0 := \mathcal{O}\otimes H_{\mathbb{Z}}$, $\mathcal{F}^2 := \{0\}$.) For example, $\sqrt{-1}\,\langle v, \bar{v}\rangle = -2\operatorname{Im} g\,\bar{h} > 0$ comes from $\operatorname{Im}\zeta > 0$ and $|a| \ll 1$.

We define another basis $\{u_1, u_2\}$ of $H_{\mathbb{C},\infty} = \Gamma(U, \rho^* H_{\mathbb{C}})$ by $u_1 := -e_1 + \zeta\,e_2$, $u_2 := \zeta\,e_1 - e_2$ such that $M\,u_1 = \zeta\,u_1$, $M\,u_2 = \zeta^{-1}\,u_2$ and $v = a\,t^{-1/3}\otimes u_1 + t^{-2/3}\otimes u_2$.

Then we have

$$\Phi(t) = \mathbb{C}\,(a\,u_1 + u_2) \quad (\subset H_{\mathbb{C},\infty}) \quad \text{for } {}^{\forall}t \in S^*,$$

$$\tilde{\Phi}(\tilde{t}) = \mathbb{C}\,(a\,\tilde{t}\,u_1 + u_2) \quad (\subset H_{\mathbb{C},\infty}) \quad \text{for } {}^{\forall}\tilde{t} \in \tilde{S}^*.$$

Hence $\Phi(0) = \mathbb{C}\,(a\,u_1 + u_2) \neq \tilde{\Phi}(0) = \mathbb{C}\,u_2$ ($\because a \neq 0$).

§3. Some remarks.

(3.1) The use of the Gauss-Manin system $\int^{\bullet}\mathcal{O}_X$ in the formulation of the result of Scherk-Steenbrink was first claimed by F. Pham (cf. [Ph]). One might think that $\int^{\bullet}\mathcal{O}_X$ and $\int^{0}\mathcal{O}_X \otimes_{\mathcal{O}_S} \mathcal{O}_S[t^{-1}] = \mathcal{A} \otimes_{\mathcal{O}_S} \mathcal{O}_S[t^{-1}]$ would produce the same filtration, because we are considering the limit of the filtration on $S^* = S-\{0\}$. But this is not true, because the fundamental short exact sequence

$$0 \longrightarrow \oplus\mathcal{O}_S \longrightarrow \int^{0}\mathcal{O}_Y \longrightarrow \int^{0}\mathcal{O}_X \longrightarrow 0$$

does not split as $\mathcal{D}_S$-Modules in general, and we have an inclusion $\int^{0}\mathcal{O}_Y \hookrightarrow \int^{0}\mathcal{O}_Y \otimes_{\mathcal{O}_S} \mathcal{O}_S[t^{-1}]$ (cf. [Sa (2.5),(3.5)]). (The above exact sequence was found independetly by F. Pham (cf.[Ph 4.1]).)

(3.2) The rest of the proof of Theorem (3.2) in [Sa] is almost the same as Lemma 2 in [Va]. It is possible to prove the theorem without using it. For we can show the following. Let $\overline{Y} \to Y$ be a modification which is isomorphic on S^*. ($\overline{Y}$ is smooth) Then $\int\mathcal{O}_Y$ is a direct factor of $\int\mathcal{O}_{\overline{Y}}$ as a filtered complex (cf.[Sa]').

(3.3) Let R be the residue of $t\partial_t: \tilde{\mathcal{H}}^{(o)} \to \tilde{\mathcal{H}}^{(o)}$. Then $\exp(-2\pi\sqrt{-1}\,R)$ and the monodromy M are conjugate to each other as matrices for $n = 1$ (i.e., $\{f=0\}$ is a plane curve). Combined with the result of Malgrange (Springer Lect. Note, 459, p. 115, Theorem (5.4)), we have the following. Let $b(s) = (s+1)\,\Pi_i(s+\alpha_i)^{m_i}$ be the b-function of f, and let $a(s) = \Pi_j(s-\lambda_j)^{r_j}$ be the minimal polynomial of the monodromy. Then we have $r_j = \max\{\, m_i : \exp(-2\pi\sqrt{-1}\,\alpha_i) = \lambda_j\}$ for $n = 1$.

In fact, let $\{u_{ij}\}_{ij}$ be a basis of H_∞ such that $\{u_{ij}\}_{j=1,\dots,\ell_i}$ is a basis of $Gr_i^W H_\infty$ for $i = 0,1,2$, where W is the weight filtration [St]. Since F and W are compatible with the monodromy decomposition, we may assume that $u_{ij} \in F^1 H_\infty$ for $i=1$, $j>\ell_1/2$ or $i=2$, and $M_s u_{ij} = \exp(-2\pi\sqrt{-1}\,\alpha_{ij})\, u_{ij}$ with $\alpha_{ij} \in (-1,0]$, where $M = M_s M_u$ is the Jordan decomposition. Since $N = \log M_u$ acts on H_∞ as the morphism of type $(-1,-1)$, we have $N u_{ij} = 0$ for $i \le 1$, and we may assume that $-N/(2\pi\sqrt{-1})\, u_{2j} = u_{0j}$ for $j \le \ell_0$ and $N u_{2j} = 0$ (hence $\alpha_{2j} = 0$) for $j > \ell_0$.

We set
$$v_{ij} = \exp(-\log t \log M/2\pi\sqrt{-1})\, u_{ij} = \begin{cases} t^{\alpha_{ij}}\, u_{ij} & \text{for } i\le 1 \text{ or } i=2,\ j>\ell_0 \\ t^{\alpha_{2j}}\, u_{2j} + t^{\alpha_{2j}}(\log t)u_{0j} & \text{for } i=2,\ j\le\ell_0, \end{cases}$$

so that $\{v_{ij}\}$ is a $\mathbb{C}\{\{\partial_t^{-1}\}\}$-basis of $\mathcal{L}$.

By [Sa,(3.2)], there is an element $w_{ij} \in \mathcal{H}^{(o)}$, such that
$$\tilde{t}^{-e\alpha_{ij}}\, \pi^*(v_{ij} - w_{ij}) \in \tilde{t}\,\tilde{\mathcal{L}} \qquad \text{for } u_{ij} \in F^1 H_\infty$$
and
$$\tilde{t}^{-e\alpha_{ij}}\, \pi^*(v_{ij} - \partial_t w_{ij}) \in \tilde{t}\,\tilde{\mathcal{L}} \qquad \text{for } u_{ij} \notin F^1 H_\infty,$$
where $\pi: \tilde{S} \ni \tilde{t} \mapsto t = \tilde{t}^e \in S$ is a unipotent base change and $\tilde{\mathcal{L}}$ is the canonical extension for $\pi^* H_X$. Hence

$\{\partial_t w_{ij}\}_{i=0 \text{ or } i=1, j\le\ell_1/2} \cup \{w_{ij}\}_{i=1,j>\ell_1/2 \text{ or } i=2}$ is a $\mathbb{C}\{\{\partial_t^{-1}\}\}$-basis of $\mathcal{L}$, and we have $\partial_t^{-1}\mathcal{L} \subset \mathcal{H}^{(o)}$. Then by the induction on the eigenvalue α_{ij}, we can show that v_{0j}, v_{1j} $(j>\ell_1/2)$ and v_{2j} are contained in $\tilde{\mathcal{H}}^{(o)}$.

For example, let $v = \sum v_i$ be an element of $\tilde{\mathcal{H}}^{(o)}$, such that $(t\partial_t - \alpha_i)^2 v_i = 0$ and $\alpha_i \neq \alpha_j$ for $i \neq j$. We may assume that $(t\partial_t - \alpha_i) v_i = 0$ for $i \geq 2$ by the induction hypothesis, where $v = \sum v_i$ is the expansion of w_{2j} $(j \leq \ell_o)$ modulo $\partial_t^{-1}\mathcal{A} + \sum_{\alpha_{2j'} > \alpha_{2j}} \mathbb{C}\, v_{2j'}$. Then v_i and $(t\partial_t - \alpha_i) v_i$ are contained in $\tilde{\mathcal{H}}^{(o)}$, because we have the following identity:

$$\det \begin{pmatrix} 1, & 0 & , 1, & \cdots & , 1 \\ \alpha_1, & 1 & , \alpha_2, & \cdots & , \alpha_k \\ \alpha_1^2, & 2\alpha_1 & , \alpha_2^2, & \cdots & , \alpha_k^2 \\ \vdots & \vdots & \vdots & & \vdots \\ \alpha_1^k, & k\alpha_1^{k-1} & , \alpha_2^k, & \cdots & , \alpha_k^k \end{pmatrix}$$

$$= \pm \Pi_{i<j} (\alpha_i - \alpha_j)\, \Pi_{i>1} (\alpha_i - \alpha_1).$$

Thus $\tilde{\mathcal{H}}^{(o)}$ has a basis $\{v_{0j}\} \cup \{v_{1j}\}_{j>m} \cup \{v_{2j}\} \cup \{\partial_t^{-1} v_{1j}\}_{j \leq m}$ with $m \leq \ell_1/2$ (by changing $\{u_{1j}\}$ if necessary), which gives the desired result.

In general, we have that $\partial_t^{-n}\mathcal{A} \subset \mathcal{H}^{(o)}$, which implies that $f^{n+1} \in \sum \mathcal{O}\, \partial f/\partial x_i$. But I do not know whether $\exp(-2\pi\sqrt{-1}\, R)$ is conjugate to M for $n \geq 2$.

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