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SOME ELEMENTARY RESULTS ON THE COHOMOLOGY OF GRADED LIE ALGEBRAS

by

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Introduction.

The results in this paper were suggested by problems in the study of the homology of local rings (and by similar problems in rational homotopy theory) and are formal analogues of well known theorems in group theory.Thus I characterize solvable graded Lie algebras of finite global dimension,and prove an analogue of Serre´s theorem on the equality of virtual global dimension and the ordinary global dimension of torsion free groups.Finally with these results it is easy to prove the more technical result that the center of a raded Lie algebra of global dimension two has to lie nicely (see below). This completes a recent result of Calle Jacobsson and Clas Löfwall and gives that for a flat and local ring homomorphism between commutative rings, with the fiber artinian of exponent 3 (i e $m^3=0$,m the maximal ideal of the fiber),the corresponding long exact sequence of the cotangent spaces splits into short exact sequences.This has also been obtained independently by Y.Felix and J.-C.Thomas (5),with methods from rational homotopy theory.

The technique used is rather different from the group case; the methods successful there do not see to work here.However the grading allows reduction to the nilpotent case and a natural use of induction.

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1. The definition of torsion free Lie algebras.

In the sequel all Lie algebras are assumed to be strictly positively graded, $g = \sum_{n>0} g_n$, with the Lie bracket satisfying $[x,y] = (-1)^{\deg x \deg y}[y,x]$. (See e g Roos (10)). In particular if x is a homogeneous element of odd degree, $[x,x]$ is not of necessity zero ; in the enveloping algebra it will equal $2x^2$. Thus $x^2 \varepsilon\, g$ if $x \varepsilon\, g_{odd}$, and char $k \neq 2$. (k is always supposed to denote the groundfield. In characteristic 2, one demands the existence of a square operation satisfying certain rules, which are trivially satisfied in char $k \neq 2$ for the usual square. Fix the notation $g_{>n} = \sum_{s>n} g_s$, etc. If $\{ x_a \; a \; \varepsilon \; I\}$ is a set of elements in a Lie algebra h, let $\langle x_a \rangle$ denote the Lie subalgebra generated by this set. An abelian Lie algebra, finitely generated by elements of odd degree, will be called a finite Lie algebra - such Lie algebras are characterized by having a finite dimensional (over k) enveloping algebra. They also exhibit a marked similarity to finite groups ; in addition to the finite dimensionality of the enveloping algebra, they are exactly the self-injective Lie algebras and moreover they are the only ones who have finitistic global dimension zero (but infinite global dimension). (Proofs of these well - known facts may be pieced together from Bass (1) and Farkas (4)).
Note that a Lie algebra g such that gl.dim g is finite, cannot contain a finite Lie subalgebra, and that, since the global dimension is independent of the groundfield, this must also be true for $\bar{g} := \bar{k} \otimes_k g$, where $\bar{k}$ is the algebraic closure of k.

Definition. A Lie algebra is called torsion free if it does not contain any non-zero finite Lie subalgebra. (This is equivalent to : $x^2=0$ implies $x=0$.) It is called absolutely torsion free if $\bar{g}$ is torsion free.

Absolute torsion-freeness is of course not implied by torsion-freeness.
An example is given by $U(g) = Q\langle x,y\rangle \;/\; (x^2-y^2, [x,y])$, which is torsion free over Q , but not when extended to an algebra over the complex numbers.
As noted above, homological finite-dimensionality implies absolute torsion-

freeness;conversely the next proposition shows that torsion-freeness has homological effects.

Proposition 1.Suppose that g is an absolutely torsion free,finitely generated Lie algebra,and that U(g) is 2-homogeneous (see below).Then the subalgebra $\langle Ext^1_{U(g)}(k,k)\rangle$ of $\langle Ext^*_{U(g)}(k,k)\rangle$,generated by the one-dimensional elements,is local and artinian.

Proof:The property that U(g) is 2-homogeneous (Löfwall(7)) means that U(g)= $k \oplus V \oplus V^2 \oplus I^3$,where I is the augmentation ideal of U(g) ,$V=I/I^2$,$V^2=I^2/I^3$. If one writes U(g) as the quotient of a free tensor algebra $U(g) = T(V)/\mathfrak{a}$, this means that all relations in $\mathfrak{a} \cap I^2(T(V))$ which do not lie in $I^3(T(V))$ are homgeneous (with respect to the tensor-degree).
Then absolute torsion-freeness implies that $q^{-1}(0) =\{0\}$, q the square-map in the commutative diagram below :

$$(D_1) \qquad \begin{array}{ccc} V \otimes V & \xrightarrow{\phi} & V^2 \\ \delta \updownarrow & \nearrow_{q} & \\ V & & \end{array} \qquad \begin{array}{l} \phi(x \otimes y) = xy \\ \delta(x) = x \otimes x \end{array}$$

Corresponding to the polynomial map q,there is a ring map A_q of the affine rings(i e the symmetric algebras of respective vector space) associated to V and V^2 : $\quad A_q : A_{V^2} \longrightarrow A_V$.
I claim that,in general - ignoring the torsion free condition - the fiber ring of $A_q = k \otimes_{A_{V^2}} A_V$ is isomorphic to $\langle Ext^1_{U(g)}(k,k)\rangle$.
Suppose that this is granted.Then the proposition follows quickly,since the hypothesis of an algebraically closed groundfield and elementary algebraic geometry imply that Krull-dimension of the fiberring = dim $q^{-1}(0)$,where the last dimension is the dimension of $q^{-1}(0)$ as an algebraic variety.In particular if $q^{-1}(0) = \{0\}$,the fiberring is artinian and local.
Proof of the claim: A description of $\langle Ext^1_{U(g)}(k,k)\rangle$ is given in Löfwall (7); slightly differently formulated it reads as follows.Denote by k<W> the free

non-commutative algebra on the vector space W. Consider the diagram :

$$(D_2)\qquad \begin{array}{ccc} k\langle V^* \otimes V^*\rangle & \xleftarrow{k\langle \phi^*\rangle} & k\langle (V^2)^*\rangle \\ \eta \downarrow & \swarrow \tau & \\ k\langle V^*\rangle & & \end{array},$$

where $(-)^*$ is the functor $\mathrm{Hom}_k(-,k)$, and η is defined on a basis $T_i \otimes T_j$ of $V^* \otimes V^*$ as $\eta(T_i \otimes T_j) = T_iT_j$ and ϕ is the multiplication map defined in (D_1). Finally τ is defined by commutativity.

According to (7) the fiber ring of $\tau = k\langle V^*\rangle/(\overline{\mathrm{im}}\tau) = k\langle V^*\rangle/\ (\eta(\overline{\mathrm{im}}\,\phi^*)) = \langle \mathrm{Ext}^1_{U(g)}(k,k)\rangle$. (Here $\overline{\mathrm{im}}$ indicates the image of the augmentation ideal.)

On the other hand, (D_1) induces a commutative diagram

$$(D_3)\qquad \begin{array}{ccc} A_{V\otimes V} & \xleftarrow{A_\phi} & A_{V^2} \\ A_\delta \downarrow & \swarrow A_q & \\ A_V & & \end{array}$$

of affine rings.

The natural maps $k\langle V^*\rangle \longrightarrow A_V$,etc are easily checked to induce a diagram map : $D_2 \longrightarrow D_3$. Thus D_3 is simply the commutativization of D_2. But since $U(g)$ is cocommutative it is well known that $\langle \mathrm{Ext}^1_{U(g)}(k,k)\rangle$ is commutative in the usual sense (see(7)),and so an easy diagram chase gives that this algebra = fiber ring of τ = fiber ring of A_q. QED.

2. The main results.

It is well known that a nilpotent finitely generated group has finite global dimension iff it is torsion free. Betting on the inertia of truth,one might inquire whether this remains true for graded Lie algebras. The group-theoretic proof is easy,working with induction on the upper central series, and using that quotients of the groups in this series are torsion free if the whole group is;it is not however ,translatable. Instead one has to use more homological trickery,in the form of Proposition 1,to establish that it is in fact true.

Denote by $|V|$ the k-dimension of the vector space V.

The following lemma is a result of Felix-Halperin-Thomas (12,thm1.2):

<u>Lemma 1.</u> A solvable Lie algebra g of finite global dimension is finitely generated,and moreover $|g|$ is finite.

We have thus one direction of the equivalence in the following theorem.

<u>Theorem 1.A solvable Lie algebra g has finite global dimension iff it is finitely generated,nilpotent and absolutely torsion free.</u>

<u>The proof</u> will take the rest of this section.The difficult part arises when the nilpotence index i(g) (i e the length of the upper central series) is two.So we will prove the theorem in this case first and then see that the other cases follow easily.Suppose then that g is finitely generated, absolutely torsion free and i(g) = 2.We can obviously also assume that k is algebraically closed.

We will use induction on $|g|$.

If $|g| = 2$, then g has to be free on one odd generator,or abelian on two even generators,and in either case it has finite global dimension.

So suppose that $|g| > 2$ and that the theorem is true for all Lie algebras, satisfying the hypothesis of the theorem and with i(g) = 2,of smaller k-dimension.Let s be a lifting of the surjection $g \to g/[g,g]$ and take a one-dimensional subspace V' of $V := g/[g,g]$ and let V" be a vector space complement of V' in V.Define h" as the normal Lie subalgebra of g generated by s(V") and h' as the Lie subalgebra generated by s(V') ,and note that since g is graded both are properly contained in g.By the induction hypothesis they both then have finite global dimension.Since h" is an h'-module, the semi-direct sum $h'' \propto h'$ is defined and has finite global dimension.Now note that there is a map $t: h''\propto h' \twoheadrightarrow g$ defined by $t(x'',x') = t''(x'') + t'(x')$ if $x'' \in h''$,$x' \in h'$, and t',t" are the inclusions of the respective algebra into g.

The kernel $\{(x,-x) : x \in h' \cap h''\}$ has k-dimension one or zero ($|h'| = 1$ or 2), and lies,since i(g)=2 ,in the center of $h'' \propto h'$.

The point of this roundabout way to use the induction hypothesis is that it now suffices to study a situation with a manageable Hochschild-Serre spectral sequence:

Lemma 2. If h is nilpotent of index 2,and gl.dim.h$<\infty$, $x \in Z(h)$ and $g:=h/\langle x\rangle$ is absolutely torsion free,then also gl.dim.g$<\infty$.

Proof.If $\bar{x} \neq 0$ in $h/[h,h]$ then, since $x \in Z(h)$, we have $h = kx \times g$ and so g must have finite global dimension.So it can be assumed that $x \in [h,h]$.

In the Hochschild-Serre spectral sequence for the extension

$$kx \hookrightarrow h \twoheadrightarrow g$$

$$E_2^{pq} = \mathrm{Ext}^p_{U(g)}(k, \mathrm{Ext}^q_{U(kx)}(k,k)) \quad \Longrightarrow \quad \mathrm{Ext}^*_{U(h)}(k,k)$$

we have $E_2^{pq}=0$ if $q \neq 0$ or 1

$$E_2^{p0}=E_2^{p1} = \mathrm{Ext}^p_{U(g)}(k,k) \text{ and}$$

$$d_2 : \mathrm{Ext}^p_{U(g)}(k,k) \quad \mathrm{Ext}^{p+2}_{U(g)}(k,k) \quad .$$

The extension is given by an element ξ in $\mathrm{Ext}^2_{U(g)}(k,kx) \cong \mathrm{Ext}^2_{U(g)}(k,k)$, and a theorem of Hochschild-Serre says that the differential d_2 is given by the Yoneda product in $\mathrm{Ext}^*_{U(g)}(k,k)$ with this element.At least the theorem says so for ungraded Lie algebras,and it goes through in the graded case (see (9)).It is also possible to give a direct argument.

If we now knew that $\xi \in \langle \mathrm{Ext}^1_{U(g)}(k,k)\rangle$ then it would follow from the absolute torsion-freeness of g and Proposition 1 that $\xi^{n_0}=0$ for some n_0.Since gl.dim.h $<\infty$,the differential d_2 must be an isomorphism for large values of the p-coordinate,i e $\mathrm{Ext}^n_{U(g)}(k,k) \xrightarrow{\cdot\xi} \mathrm{Ext}^{n+2}_{U(g)}(k,k)$ and so $\mathrm{Ext}^n_{U(g)}(k,k) = 0$, for large values of n.Thus we would have proved the lemma.(Note that we use the fact that a Lie algebra g with i(g)=2 is trivially 2-homogeneous,in order to be able to apply Prop.2.)

So it remains to prove that $\xi \in \langle \mathrm{Ext}^1_{U(g)}(k,k)\rangle$. There is in addition to the usual grading,another grading on U(g) ; write U(g) as aquotient $T(V)/\mathfrak{a}$ of a free tensor algebra T(V) ,and consider the grading induced on U(g) by the tensor degree of the tensor algebra.This grading induces a grading on $\mathrm{Ext}^*_{U(g)}(k,k)$ and one has $\langle \mathrm{Ext}^1_{U(g)}(k,k)\rangle = \sum_i \mathrm{Ext}^i_{U(g)}(k,k)_i$,where the lower index denotes the tensor-grading.(See (7) cor.1.1.).Thus it suffices to show that ξ has the correct degree 2.But how does one construct ξ? If s is a

lifting of the surjection $h \twoheadrightarrow g$,then the extension induces a factor set ξ' : $g \times g \to kx$,defined by $\xi'(y,z) = [s(y),s(z)] - s([x,y])$ .Now ,since $x \in [h,h]$, ξ' is in fact a map from $V \times V$ to kx ,where $V:=g/[g,g]$,and this is precisely what is meant by the assertion that ξ has tensor -degree 2.
Thus the theorem is proved in the case where $i(g) = 2$.

In order to conclude the proof of Theorem 1,we first prove the analogue to the theorem of Serre mentioned in the introduction.(See Bieri (2).)

<u>Theorem 2.If h is a Lie subalgebra of the Lie algebra g,such that gl.dim h< ∞, and such that h has finite index in g (i e U(g) is a finitely generated U(h)-module),then,if g is absolutely torsion free,it has finite global dimension.</u>

<u>Remark.</u>In fact,the global dimensions of g and h,in the theorem,will coincide. This is easily seen from the Hochschild-Serre spectral sequence.

<u>Proof.</u>We will use the already proven part of Thm 1.
If $x \in g \setminus h$ is a homogeneous element,it has to be of odd degree,since otherwise,by Poincaré-Birkhoff-Witt,the index of h in g would be infinite.From the grading it follows that h is normal in $\langle h,x \rangle$, and by induction it then suffices to study extensions

$$h \hookrightarrow g \twoheadrightarrow kx \qquad \text{where degx is odd.}$$

Note that if $h \hookrightarrow l \twoheadrightarrow m$ is an arbitrary extension,gl.dim.l ≤gl.dim h + + gl.dim. m .
Let deg x = b and $\min\{ \deg t \; ; \; t \in g/[g,g]\} =: c$ and use induction on b-c.If b=c ,let $l := \langle g_n \; ; \; n \neq b, 2b \rangle$.Then l is an ideal of g and g/l is absolutely torsion free and nilpotent of index 2.So,by the proven part of Thm 1, g/l has finite global dimension,as has $l \subseteq h$ and thus gl.dim. g is finite.Suppose now that b=c+1.Then c is even,and by the preceding case gl.dim $g_{>c}$ is finite. But $g/g_{>c}$ is an abelian Lie algebra on even generators,and so has finite global dimension ,and so gl.dim.g < ∞.

If b-c > 1,then by induction gl.dim $g_{>c} < \infty$,and we can substitute $g_{>c}$ for h.Since g can be obtained by a finite series of extensions,each time with one element ,from $g_{>c}$, and in each extension the induction parameter is 1,it follows that g has finite global dimension.

The end of the proof of Thm 1.Use induction on i(g).Suppose that i(g) > 2, then i([g,g]) < i(g) and so gl.dim.[g,g]<∞.Thus gl.dim.g is finite since, as in the last part of the proof of Thm.2,g is the end result of a finite series of extensions by elements of odd degree (use Thm 2) or even degree (trivial).QED.

For the last section I need

Corollary.If gl.dim.g < ∞ and x ε Z(g),then if g/<x> is absolutely torsion free ,it also has finite global dimension.

Proof.Induction on min{degt ; t ε g/[g,g]} - deg x.If this difference is zero then g = kx X g/<x> and the result is clear.Otherwise,consider $g_{>c}$, where c is min{degt ; t ε g/[g,g] } and proceed exactly as in the proof of Thm 2.

3.The center of a Lie algebra of global dimension 2.

Theorem 3. Suppose that k,the ground field,is algebraically closed,and that g is a finitely generated Lie algebra of global dimension 2,and that z ε Z(g) is a non-zero homogeneous element of the center of g.Then there exists a presentation of g such that z is either a generator or the square of a generator.

For the proof I need a result,which is true in a more general context than the present one.

Proposition 2. Suppose that g is an arbitrary Lie algebra (graded) such that Z(g)≠0 and gl.dim.g is finite.Then if $|\mathrm{Tor}_i^{U(g)}(k,k)|$ is finite for

$0 \leq i <$ gl.dim.g ,then also $|\mathrm{Tor}_i^{U(g)}(k,k)|$ is finite for $i =$ gl.dim.g .Moreover $X(g) := \sum_i (-1)^i |\mathrm{Tor}_i^{U(g)}(k,k)| = 0$.

Compare this proposition with Stallings (11).

Proof.Note that it follows from the assumptions on g that it is finitely generated; if gl.dim.g =0,it cannot have a non-trivial center.
The localization $U(g)_z$ exists if z is a central element,and is flat over U(g), and then $U(g)_z \otimes k = 0$.If $F^\bullet \xrightarrow{\sim} k$ is a free resolution of k over U(g) it follows that $F^\bullet_z := F^\bullet \otimes_{U(g)} U(g)_z \xrightarrow{\sim} 0$, and so $F^\bullet_z$ splits.The proposition would now follow,if it were possible to count the ranks of the free $U(g)_z$-modules in $F^\bullet_z$.Such counting is possible for all rings that have a noetherian quotient ring (Cohn (3)).If degz =n then $U(g)_z \twoheadrightarrow U(g)_z/\langle g_{>n}\rangle = U(g/g_{>n})_z$ which is a noetherian ring,since $g/g_{>n}$ is finite dimensional as a vector space.

In particular it follows that an algebra g as in the hypothesis of theorem 3 is finitely generated,and so the Hilbert series $H_{U(g)}(t)=1/P(t)$,where P(t) is apolynomial to be described later.

Proof of Thm 3.Suppose that $z \in Z(g)$ and that g/<z> is torsion free.Then, according to the corollary of section 2,it has finite global dimension.From the Hochschild-Serre spectral sequence of the extension

$$kz \hookrightarrow g \twoheadrightarrow g/\langle z\rangle$$

one draws the conclusion that $\mathrm{Ext}^2_{U(g)}(k,k)=\mathrm{Ext}^4_{U(g)}(k,k) = \ldots = 0$.
So g/<z> is free and the extension splits and the theorem is true in this case.
Now suppose that g/<z> has torsion.Then z is a square and degz=2n ,n odd.
Consider $h:=\langle g_{>n}, g_{even}\rangle$;this is a subalgebra of finite index in g.Its Hilbert series is $H_{U(h)}(t)= (1+t)^{-e_1} \cdot \ldots \cdot (1+t^n)^{-e_n} \cdot H_{U(g)}(t)$.
Since $z \in h$ and h/<z> is torsion free the previous case gives that

$H_{U(h)}(t) = (1-t^{2n})^{-1} H_{U(f)}(t)$,where f:= h/<z>.Moreover,since h is finitely generated ,$H_{U(f)}(t) = 1/Q(t)$;Q a polynomial with all coefficients of strictly positive degree negative.

Altogether this gives that $(1-t^{2n})$ divides $(1+t)^{e_1} \cdot \ldots \cdot (1+t^n)^{e_n} \cdot P(t)$.

($P(t) = 1/H_{U(g)}(t)$).Since n is odd ,$(1-t^n)$ is relatively prime to $(1+t^m)$ for all integers m and so $(1-t^n)$ divides P(t).

If then g is acounter example to the theorem,also the subalgebra $\langle g_{<n} \rangle$ is a counter example,since it contains z. (Remember that z was a square of degree 2n).So we can suppose that g is generated by elements of degree less than n.This means that

$$P(t) = 1-f_1 t - f_2 t^2 - \ldots - f_{n-1} t^{n-1} + d_2 t^2 + \ldots + d_b t^b \quad , \qquad (E_1)$$

where f_i and d_i denotes the number of generators and relations of degree i,in a minimal presentation of g.

We just showed that $P(t) = (1-t^n)(1+\sum_i a_i t^i) = 1+\ldots+(a_n-1)t^n+\ldots+(a_{2n}-a_n)t^{2n}+\ldots +a_{mn}t^{mn}+\ldots$ where $m := \lceil b/n \rceil$.

Comparison with (E_1) gives that $a_n - 1 \geq 0, a_{2n} - a_n \geq 0, a_{mn} \geq 0$, which implies that -1 is greater than zero.QED.

<u>Corollary</u>.(Of proof).Suppose that g is a Lie algebra of finite global dimension, such that $|\mathrm{Tor}_i^{U(g)}(k,k)|$ is finite for all i.Then if $z \in Z(g)$ and degz=b, $(1-t^b) \mid 1/H_{U(g)}(t)$.

<u>Remark.</u> The following ,easily proved,fact is needed for the proof:that the condition on the $\mathrm{Tor}^i_{U(g)}$ is inherited for U(g/<z>).I suppose the corollary is wellknown ;it is the Felix-Halperin conjecture on Gottlieb-elements in the coformal case,for a finite CW-complex.

<u>Corollary.</u>If g is a Lie algebra of global dimension 2,with non-trivial center,and generated by elements of degree 1,then the center has to lie in degree 2.

The proof of this theorem given by Y.Felix-J.-C.Thomas is less ad hoc.The analogue in group theory is stated in Bieri (2).For the application of the corollary, mentioned in the introduction,to the cotangent spaces of local rings,see Jacobsson (6) and Löfwall(8).

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