Astérisque

Marek Golasinski

Some remarks on the rational homotopy type of diagrams and reduced K_O

Astérisque, tome 113-114 (1984), p. 187-191

<http://www.numdam.org/item?id=AST_1984__113-114__187_0>

SOME REMARKS ON THE RATIONAL HOMOTOPY

TYPE OF DIAGRAMS AND REDUCED K_0.

by

Marek GOLASINSKI

1 - THE RATIONAL HOMOTOPY TYPE OF DIAGRAMS

Let $\mathbb{C}$ be a closed model category in the Quillen's sense (see 5). If $\mathbb{I}$ is a small category $\mathbb{C}^{\mathbb{I}}$ denotes the functors category. A map in $\mathbb{C}^{\mathbb{I}}$ $f : C \to C'$ is a <u>fibration</u>, respectively a <u>weak equivalence</u> if $f(i)$ is a fibration, respectively a weak equivalence for every $i \in \text{ob}\, \mathbb{I}$. A <u>cofibration</u> is a map that has the left lifting property with respect to all trivial fibrations. We have the following result of Quillen-Bousfield-Kan (see 1, p. 313).

<u>THEOREM</u> 1.1.- <u>$\mathbb{C}^{\mathbb{I}}$ equipped as above is a closed model category.</u>

Let $\mathbb{G}$ be a discrete group. $\mathbb{G}$-Set is the category of left $\mathbb{G}$-sets and $\mathbb{I}$ is the full subcategory of $\mathbb{G}$-Set determined by $\mathbb{G}/\mathbb{H}$ as $\mathbb{H}$ varies over all subgroups of $\mathbb{G}$. Denote by $\mathbb{G}$-SS the category of left $\mathbb{G}$-simplicial sets and $\mathbb{G}$-Top the category of left $\mathbb{G}$-topological spaces. Define functors $J : \mathbb{G}\text{-SS} \to SS^{\mathbb{I}}$ by $J(X)(G/\mathbb{H}) = X^{\mathbb{H}}$, where $X^{\mathbb{H}} = \{x \in X ; h x = x , \text{ for all } h \in \mathbb{H}\}$ and $T : SS^{\mathbb{I}} \to \mathbb{G}\text{-SS}$ by $T(F) = F(\mathbb{G})$ provided with its natural $\mathbb{G}$-action acquired from $\mathbb{G}\text{-Set}(\mathbb{G},\mathbb{G}) = \mathbb{G}$.

Let $f : T(F) \to X$ be a map in $\mathbb{G}$-SS. Define $f' : F \to J(X)$ by $f'(\sigma) = f\, F(q)(\sigma)$ for $\sigma \in F(G/\mathbb{H})$ and $q : \mathbb{G} \to \mathbb{G}/\mathbb{H}$ the natural quotient map. It is routine to check that f' is natural. Furthemore if $h : F \to J(X)$ then $h(\sigma) = h^{\vee}\, F(q)(\sigma)$ where $h^{\vee} : F(\mathbb{G}) \to X$ is the $\mathbb{G}$-component of h, i.e. h is determined by $h^{\vee}$. We have thus established :

PROPOSITION 1.2.- *J is full and faithul and right adjoint to T. Furthemore T preserves limits and both T and J preserve tensor products over SS.*

Using J we view G-SS as a subcategory of $SS^{\mathbb{I}}$. A map $f : X \to X'$ of $\mathbb{G}$-SS is said to be a *fibration*, respectively a *weak equivalence* if $J(f)$ is a fibration, respectively a weak equivalence of $SS^{\mathbb{I}}$. A *cofibration* in $\mathbb{G}$-SS is a map of $\mathbb{G}$-SS that has the left lifting property with respect to all trivial fibrations in $\mathbb{G}$-SS. We have

PROPOSITION 1.3.- *$\mathbb{G}$-SS equipped as above is a closed model category. Furthemore each monomorphism of $\mathbb{G}$-SS is a cofibration and thus any object of $\mathbb{G}$-SS is cofibrant.*

Consider the adjoint pair $S : Top \rightleftarrows SS : |\ |$, where S is the singular functor and $|\ |$ is the geometric realization. The functors yield by naturality an adjoint pair $S_{\mathbb{G}} : \mathbb{G}\text{-}Top \rightleftarrows \mathbb{G}\text{-}SS : |\ |_{\mathbb{G}}$ with natural isomorphism $\mathbb{G}\text{-}Top(|F|_{\mathbb{G}}, X) \simeq \mathbb{G}\text{-}SS(F, S_{\mathbb{G}}(X))$.

Let Q-DGA be the category of differential graded Q-algebras and $A^* : SS \rightleftarrows Q\text{-}DGA : F^*$ the pair of de Rham adjoint functors (see 6). These functors determine an adjoint pair $A^{*\mathbb{I}} : SS^{\mathbb{I}} \rightleftarrows Q\text{-}DGA^{\mathbb{I}} : F^{*\mathbb{I}}$. If $fQ\text{-}SS^{\mathbb{I}}_N \subset SS^{\mathbb{I}}$ is the full subcategory given by functors $X \in SS^{\mathbb{I}}$ such that $X(G/\mathbb{H})$ is nilpotent, rational and of finite Q-type for every subgroup $\mathbb{H} \subset \mathbb{G}$ and $fQ\text{-}DGA^{\mathbb{I}} \subset Q\text{-}DGA^{\mathbb{I}}$ is the full subcategory given by those functors $A \in Q\text{-}DGA^{\mathbb{I}}$ that $A(G/\mathbb{H})$ is equivalent to a minimal algebra with finitely many multiplicative generators in each dimension for every subgroup $\mathbb{H} \subset \mathbb{G}$. Then we obtain a generalization of the Sullivan-de Rham result (cf. 6) :

THEOREM 1.4.- *Let $\mathbb{G}$ be a finite group. The adjoint pair $A^{*\mathbb{I}} : SS^{\mathbb{I}} \rightleftarrows Q\text{-}DGA^{\mathbb{I}} : F^{*\mathbb{I}}$ induces an equivalence of homotopy categories*

$$Ho(fQ\text{-}SS^{\mathbb{I}}_N) \underset{\approx}{\rightleftarrows} H_o(fQ\text{-}DGA^{\mathbb{I}}).$$

Let $fQ\ \mathbb{G}\text{-}SS_N$ be the full subcategory of $\mathbb{G}$-SS given by nilpotent, rational and of finite Q-type $\mathbb{G}$-simplicial sets. The functor $J : \mathbb{G}\text{-}SS \longrightarrow SS^{\mathbb{I}}$ is full and faithful, then we have.

COROLLARY 1.5.- *The above equivalence induces a bijection between equivariant rational homotopy types of $fQ\ \mathbb{G}\text{-}SS_N$ on the one hand and isomorphism classes of minimal systems of DGA's in the Triantafillou sense (see 7) on the other.*

2 - REDUCED K_0 OF 0-FORMS ON A FINITE SIMPLICIAL COMPLEX

Sullivan has proved (see 6, cf. also 4) that for a finite simplicial complex X with vertices $v_1,\dots,v_n$ and corresponding barycentric coordinates $b_1,\dots,b_n$ the algebra of rational forms on X

$$A_Q^0X = {Q[b_1,\dots,b_n]\otimes\Lambda(db_1,\dots,db_n)}/I ,$$

where $Q[b_1,\dots,b_n]$ is the ring of rational polynomials in $b_1,\dots,b_n$, $\Lambda(db_1,\dots,db_n)$ is the exterior algebra on $db_1,\dots,db_n$ and I is the ideal generated by $b_1+\dots+b_n-1$, $db_1+\dots+db_n$, $b_{i_1}\dots b_{i_p}\ db_{j_1}\dots db_{j_q}$ if there is no $p+q$ - simplex of X with vertices $v_{i_1},\dots,v_{i_p}, v_{j_1},\dots,v_{j_q}$.

Kan and Miller have shown (see 3) that the weak homotopy type of a finite simplicial set X can be reconstructed from R-algebra A_R^0X of 0-forms on X, when R is a unique factorization domain.

If pro R-$\mathbb{A}$ denote the pro-category of R-algebras then Jardine has proved (see 2) that there are functors $\hat{A} : SS \rightleftarrows \text{pro } R\text{-}\mathbb{A} : \hat{F}$ inducing an equivalence of suitable homotopy categories

$$Ho(SS) \underset{\approx}{\rightleftarrows} Ho(\text{pro } R\text{-}\mathbb{A}) .$$

Our purpose is to show that there exists a simplicial set $G_\infty(\infty)$ (the simplicial Grassman variety) such that for a finite simplicial complex X, $\tilde{K}_0(A_k^0X) = [X, G_\infty(\infty)]$ where $\tilde{K}_0$ is the reduced Grothendieck's group of A_k^0X and k is a field.

The Grassman variety $G_m(n)$ is defined as a functor from the K-algebras category k-$\mathbb{A}$ to the category of sets, for $1 \leq n < m$ and R in k-$\mathbb{A}$ by

$$G_m(n)(R) = \{Q \subset R^m ;\ Q \text{ is } R\text{-split projective of rank } n\}.$$

The assignment $Q \mapsto Q \underset{R}{\otimes} S$ associated to the k-algebra homomorphism $\theta : R \to S$ defines the function $\theta_* : G_m(n)(R) \longrightarrow G_m(n)(S)$.

Let $P(R)$ ($P_n(R)$) be the set of isomorphism classes of R-modules finitely generated and projective over R (of rank n), $\tilde{K}_0(R)$ the Grothendieck's group of $P(R)$ and $K_0(R)$ reduced K_0.
The natural embeding $R^m \longrightarrow R^{m+1}$ induces a map $G_m(n) \longrightarrow G_{m+1}(n)$.
Put $$G_\infty(n) := \underset{n}{\text{colim}}\ G_m(n) .$$
Then there is a natural surjective function $\tau_R : G_\infty(n)(R) \longrightarrow P_n(R)$ which is induced by the assignment $(P \longrightarrow R^m) \longmapsto P$.

Let X be a finite simplicial complex, thought of a member of as a member of the category SS of simplicial sets, and let k be an arbitrary field. Recall that there is a natural simplicial set map

$$\eta_X : X \longrightarrow \mathrm{Spec}(A_k^0 X)(A_k^0\ \Delta_*) = k\text{-}A(A_k^0 X,\ A_k^0\ \Delta_*)$$

where Δ_n is the standard n-simplex.

Let Sch_k denotes the category of schemes over k, thought of as a full subcategory of the functors category from $k\text{-}A$ to Set.
η_X may be used to define a function

$$\vartheta : \mathrm{Sch}_k(\mathrm{Spec}\ A_k^0 X, Y) \longrightarrow SS(X,\ Y(A_k^0\ \Delta_*))$$

for arbitrary k-schemes Y in such a way that ϑ associates to a k-scheme map $f : \mathrm{Spec}\ A_k^0 X \longrightarrow Y$ the composition

$$X \xrightarrow{\eta_X} \mathrm{Spec}(A_k^0 X)\ (A_k^0\ \Delta_*) \xrightarrow{f_*} Y(A_k^0\ \Delta_*)\ .$$

<u>PROPOSITION</u> 2.1.- ϑ <u>induces a bijection</u>

$$\vartheta_* : \mathrm{Sch}_k(\mathrm{Spec}\ A_k^0 X, Y) \xrightarrow{\approx} SS(X,\ Y(A^0\ \Delta_*))$$

<u>for all finite simplicial complexes</u> X <u>and all schemes</u> Y.

Then the above map τ gives rise to a natural surjective function

$$\tau_X : SS(X,\ G_\infty(n)\ (A_k^0\ \Delta_*)) \longrightarrow P_n(A_k^0 X)$$

in view of above theorem and the Yoneda lemma.

<u>THEOREM</u> 2.2.- <u>The map</u> $\tau_X : SS(X,\ G_\infty(n)\ (A_k^0\ \Delta_*)) \longrightarrow P_n(A_k^0 X)$ <u>factors through a bijection</u>

$$(\tau_X)_* : [X,\ G_\infty(n)\ (A_k^0\ \Delta_*)] \xrightarrow{\approx} P_n(A_k^0 X)\quad .$$

The map $P_n(A_k^0 X) \longrightarrow P_{n+1}(A_k^0 X)$ which is defined by $P \longmapsto A_k^0 X \oplus P$ clearly fits into a commutative diagram

$$\begin{array}{ccc} [X,\ G_\infty(n)\ A_k^0\ \Delta_*] & \longrightarrow & [X,\ G_\infty(n+1)\ A_k^0\ \Delta_*] \\ \downarrow{\scriptstyle (\tau_X)_*} & & \downarrow{\scriptstyle (\tau_X)_*} \\ P_n(A_k^0 X) & \longrightarrow & P_{n+1}(A_k^0 X) \end{array}$$

Formal nonsense now shows that

$$[X, G_\infty(\infty)\ (A_k^0\ \Delta_*)] = \operatorname{colim}\ (P_n(A_k^0X) \longrightarrow P_{n+1}(A_k^0X))$$

may be identified with $\tilde{K}_0(A_k^0X)$ via a map which is induced by the assignments $P \longmapsto P - (A_k^0X)^{rkP}$, where $G_\infty(\infty) = \operatorname{colim}_n G_\infty(n)$.

There is also a similar result for finite G-simplicial complexes.

Let G be a finite group such that $\chi(k)\ \chi|G|$, $\chi(k)$ is the characteristic of k, $V_1,\dots,V_\ell$ all irreducible G-modules over k and $V := \bigoplus_{i=1}^{\ell} V_i$. The $\mathbb{G}$-Grassman variety $\mathbb{G}_m^{\mathbb{G}}(n)$ is defined as a functor from the k-algebras category k-$\mathbb{A}$ to the category of sets, for $1 \leq n < m$ and R in k-$\mathbb{A}$ by

$$G_m^{\mathbb{G}}(n)(R) := \{Q \subset R^m \otimes V\ ;\ Q \text{ is R-split projective of rank } n\}.$$

Remark that for a k-$\mathbb{G}$-algebra R the category of R-$\mathbb{G}$-modules is equivalent to the category of $R * \mathbb{G}$-modules, where $R * G$ is the twisted product of R and $\mathbb{G}$. Let $P^{\mathbb{G}}(R)$ denotes the set of isomorphism classes of $R * \mathbb{G}$-modules finitely generated and projective over R, $K_0^{\mathbb{G}}R$ the Grothendieck's group of $P^{\mathbb{G}}(R)$ and $\tilde{K}_0^G(R)$ reduced K_0. Then we have.

<u>THEOREM</u> 2.3.- <u>For a finite group</u> $\mathbb{G}$ <u>such that</u> $\chi(k)\ \chi|\mathbb{G}|$ <u>and a finite</u> G-<u>simplicial complex</u> X

$$K_0^{\mathbb{G}}(A_k^0X) = [X, G_\infty^{\mathbb{G}}(\infty)\ (A_k^0\ \Delta_*)]_{\mathbb{G}}\ .$$

R E F E R E N C E S.

[1] A.K. BOUSFIELD, D.M. KAN - Lecture Notes in Math. 304 1972.

[2] J.F. JARDINE - Algebraic homotopy theory, Can. J. Math. 33 1981, pp. 302-319.

[3] D.M. KAN, E.Y. MILLER - Homotopy types and Sullivan s algebras of 0-forms, Topology vol. 16 1977, pp. 193-197.

[4] M.A. PENNA - On a theorem of Sullivan, Can. Math. Bull. Vol. 21 1978 pp. 201-206.

[5] D.G. QUILLEN - Lecture Notes in Math. 43 1967.

[6] D. SULLIVAN - Differential forms and the topology of manifolds, Proc. of Congress on Manifolds, Tokyo, Japan 1973 pp. 37-49.

[7] G. TRIANTAFILLOU - Equivariant minimal models, Trans. of A.M.S. Vol.274 2 1982, pp. 509-532.