Astérisque

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Astérisque, tome 113-114 (1984), p. 206-211

<http://www.numdam.org/item?id=AST_1984__113-114__206_0>

The homology of nilpotent Lie groups made discrete.

by André Haefliger

Let $\mathcal{N}$ be the category of torsion free divisible nilpotent groups and let $\mathcal{L}$ be the category of nilpotent Lie algebras over the rationals $\mathbb{Q}$.

Malcev theory establishes an equivalence between those two categories. To a nilpotent Lie algebra $\underline{n}$ over $\mathbb{Q}$ corresponds on $\underline{n}$ a structure of nilpotent group N using the Campbell-Hausdorff formula. The identity map of $\underline{n}$ gives the exponential map exp : $\underline{n} \to N$.

If the Lie algebra $\underline{n}$ is defined over $\mathbb{R}$ and is finite dimensional, then the corresponding nilpotent group N is the simply connected Lie group whose Lie algebra is $\underline{n}$.

Let $H_*^{\mathbb{Z}}(\underline{n})$ be the homology of $\underline{n}$ considered as a Lie algebra over $\mathbb{Z}$. It is the homology of the exterior algebra $\Lambda^{\mathbb{Z}}(\underline{n})$ of $\underline{n}$ over $\mathbb{Z}$ (in degree > 0, it is the same as the exterior algebra over $\mathbb{Q}$), with the boundary operator defined by

$$\partial(X_1\wedge\ldots\wedge X_k) = \sum_{r<s} (-1)^{r+s+1}[X_r,X_s]\wedge X_1\wedge\ldots\wedge\hat{X}_r\wedge\ldots\wedge\hat{X}_s\wedge\ldots\wedge X_k.$$

If $\underline{n}$ is defined over an extension K of $\mathbb{Q}$, we denote by $H_*^K(\underline{n})$ the homology of $\underline{n}$ considered as a Lie algebra over K (in the above definition, we consider the exterior algebra of $\underline{n}$ over K). We have a natural homomorphism $H_*^{\mathbb{Z}}(\underline{n}) \longrightarrow H_*^K(\underline{n})$ induced by the natural mapping $\Lambda^{\mathbb{Z}}(\underline{n}) \longrightarrow \Lambda^K(\underline{n})$.

If N is a nilpotent Lie group, one can define the continuous cohomology $H_c^*(N;\mathbb{R})$ of N as the cohomology of the subcomplex of continuous Eilenberg-Mac-Lane cochains on N. Obviously this cohomology maps in the real cohomology $H^*(N;\mathbb{R})$ of the discrete group N. When N is 1-connected, Van Est [2] has proved that the continuous cohomology of N is canonically isomorphic to the real cohomology $H^*(\underline{n};\mathbb{R})$ of the Lie algebra $\underline{n}$ of N considered as a Lie algebra over $\mathbb{R}$.

Given an integral homology class $a \in H_k(N;Z)$ of the group N and a continuous cohomology class $\alpha \in H_c^k(N;R)$, the evaluation of α on a gives a map

$$H_k(N;\mathbb{Z}) \times H_c^k(N;\mathbb{R}) \to \mathbb{R}$$

which is $\mathbb{Z}$-linear in the first factor and $\mathbb{R}$-linear in the second one.

One basic problem is to determine what real values one can get by evaluating continuous cohomology classes on integral classes. This can be formulated more precisely as follows.

Problem : For a simply connected nilpotent Lie group N, what is the image of the homomorphism

$$H_k(N;\mathbb{Z}) \to \mathrm{Hom}_{\mathbb{R}}(H_c^k(N;\mathbb{R}),\mathbb{R}) \approx H_k^{\mathbb{R}}(\underline{n})$$

associated to the evaluation map ?

On the right hand side, the isomorphism is given by the Van Est theorem. Note that the image of this homomorphism generates $H_*^{\mathbb{R}}(\underline{n})$ iff the continuous cohomology $H_c^k(\underline{n};\mathbb{R})$ injects in $H^k(N;\mathbb{R})$.

The following theorem translates this problem in a problem on Lie algebras.

Theorem. Let N be a torsion free divisible nilpotent group and let n be its corresponding Lie algebra over $\mathbb{Q}$:

a) There is a functorial isomorphism of coalgebras

$$H_*(N;\mathbb{Z}) \to H_*^{\mathbb{Z}}(\underline{n})$$

b) If N is a nilpotent 1-connected Lie group, then via the above isomorphism, the map given by evaluating continuous cohomology classes of N on integral classes is the natural map

$$H_k(N;\mathbb{Z}) \approx H_k^{\mathbb{Z}}(\underline{n}) \longrightarrow H_k^{\mathbb{R}}(\underline{n}) .$$

Proof. We first observe, following Milnor [4], that the reduced homology $\widetilde{H}_*(N;\mathbb{Z})$ is uniquely divisible. Indeed it is sufficient to check it for those N for which the corresponding Lie algebra $\underline{n}$ over $\mathbb{Q}$ is finite dimensional. We can argue by induction on the dimension n of $\underline{n}$. For n=1, then N = $\mathbb{Q}$ and $H_k(\mathbb{Q};\mathbb{Z})$ is $\mathbb{Q}$ for k=1 and zero otherwise. One can find a surjective homomorphism φ of N on $\mathbb{Q}$; the Lie algebra of its kernel is of dimension n-1. So using the

Hochshild-Serre spectral sequence of φ, we see that the reduced homology of N is uniquely divisible if it is so for Ker_{φ}.

Hence the natural map $H_*(N;\mathbb{Z}) \to H_*(N;\mathbb{Q})$ induced by the inclusion of $\mathbb{Z}$ in $\mathbb{Q}$ is an isomorphism. It is well known to people working in rational homotopy theory that $H_*(N;\mathbb{Q})$ is isomorphic to $H_*^{\mathbb{Q}}(\underline{n})$; this follows easily from Malcev theory [3], [6] and a theorem of Nomizu [5]. We briefly recall the argument (for an alternate treatment, see [1] and for a purely algebraic proof, see [7]).

By Malcev theory, given a finitely generated subgroup Γ of N, then $\Gamma_0 = \exp^{-1}(\Gamma)$ generates a finite dimensional Lie subalgebra $\underline{n}_0^{\Gamma}$ over $\mathbb{Q}$. It will be sufficient to prove there is a functorial isomorphism $H_*(\Gamma;\mathbb{Q}) \to H_*^{\mathbb{Q}}(\underline{n}_0^{\Gamma})$, because $N = \varinjlim \Gamma$ and $\underline{n} = \varinjlim \underline{n}_0^{\Gamma}$.

Let $\underline{n}^{\Gamma} = \underline{n}_0^{\Gamma} \otimes \mathbb{R}$ and let N^{Γ} be the 1-connected Lie group with Lie algebra $\underline{n}^{\Gamma}$. Then $\exp : \underline{n}^{\Gamma} \to N^{\Gamma}$ maps Γ_0 on a discrete cocompact subgroup of N identified with Γ. As N is contractible, the cohomology of $X = {}_{\Gamma}\backslash N^{\Gamma}$ is the cohomology of Γ.

A differential form on N^{Γ} will be called rational if its pull back by the exponential map is a polynomial form on $\underline{n}^{\Gamma}$ with respect to a linear system of coordinates taking rational values on $\underline{n}_0^{\Gamma}$. A differential form on X is rational if its pull back on N^{Γ} is rational.

<u>Lemma</u>. <u>The integral of a rational form α on $X = {}_{\Gamma}\backslash N^{\Gamma}$ over a differentiable integral k-cycle is a rational number</u>.

<u>Proof of the lemma</u>. A polynomial rational map of the k-simplex $\Delta_a^k = \{t = (t_1,\dots,t_{k+1}) \in R^{k+1},\ 0 \le t_i,\ \Sigma t_i = 1\}$ in X is a continuous map $G : \Delta^k \to X$ whose liftings $\tilde{G}$ in N are such that $\exp^{-1}\cdot\tilde{G}$ is a polynomial map with rational coefficients. It is clear that the integral of α on such a simplex is a rational number.

Given a singular integral cycle C on X, we can find a finite complex K carrying an integral simplicial cycle C' and a map $f : K \longrightarrow X$ such that f_*C' is homologous to C. So the lemma follows from the fact that f is homotopic to a map f' such that its restriction to each simplex of K is a polynomial rational map. One can construct f' step by step over the skeletoons of K using the known fact that any rational polynomial map of the boundary of Δ^k in $\underline{n}^\Gamma$ can be extended to a rational polynomial map of Δ^k. □

Let $C^*(\underline{n};\mathbb{R})$ (resp. $C^*(\underline{n}_0,\mathbb{Q})$) be the DG-algebra of multilinear forms on $\underline{n}^\Gamma$ (resp. $\underline{n}_0^\Gamma$). We have a natural inclusion of $C^*(\underline{n}_0^\Gamma:\mathbb{Q})$ in $C^*(\underline{n}^\Gamma;\mathbb{R}) = C^*(\underline{n}_0^\Gamma;\mathbb{Q})\otimes_{\mathbb{Q}}\mathbb{R}$.

The DG-algebra of Γ-invariant forms on N^Γ contains the DG-algebra of N-invariant forms on N^Γ which is naturally isomorphic to $C^*(\underline{n}^\Gamma;\mathbb{R})$.

The theorem of Nomizu [5] asserts that the inclusion $C^*(\underline{n}^\Gamma;\mathbb{R}) \longrightarrow A^*(X)$ induces an isomorphism in cohomology. As the subcomplex $C^*(\underline{n}_0^\Gamma;\mathbb{Q})$ is mapped on rational forms, the lemma implies that by restriction we get a map inducing an isomorphism of $H^*(\underline{n}_0^\Gamma;\mathbb{Q})$ in $H^*(X;\mathbb{Q}) = H^*(\Gamma;\mathbb{Q})$, because we get an isomorphism after tensorization with $\mathbb{R}$.

By duality, we get the functorial isomorphism

$$H_*(\Gamma;\mathbb{Q}) \longrightarrow H_*^{\mathbb{Q}}(\underline{n}_0^\Gamma) \ .$$

This proves part a) of the theorem.

When N is a simply connected nilpotent Lie group, the identity map of Γ extends uniquely as a homomorphism of Lie groups $N^\Gamma \longrightarrow N$ (cf. [6]); the induced morphism $\underline{n}^\Gamma \to \underline{n}$ on Lie algebras over $\mathbb{R}$ is the extension of the inclusion $\underline{n}_0^\Gamma \subset \underline{n}$.

Via the van Est isomorphism, the map

$$H^*(\underline{n};\mathbb{R}) = H_c^*(N;\mathbb{R}) \to H^*(\Gamma;\mathbb{R})$$

is induced by the composition of maps

$$C^*(\underline{n};\mathbb{R}) \longrightarrow C^*(\underline{n}^\Gamma;\mathbb{R}) \longrightarrow A^*(X) \ .$$

This proves part b) of the theorem.

Remarks and examples.

For any Lie algebra $\underline{g}$ over R, the map

$$\rho_k : H_k^{\mathbb{Q}}(\underline{g}) \longrightarrow H_k^{\mathbb{R}}(\underline{g})$$

is an isomorphism for k=1 (both groups are isomorphic to $\underline{g}/_{[\underline{g},\underline{g}]}$), and is surjective for k=2.

If $\underline{g}$ is defined over $\mathbb{Q}$, i.e. if there is a Lie subalgebra $\underline{g}_0$ over $\mathbb{Q}$ such that $\underline{g} = \underline{g}_0 \otimes_{\mathbb{Q}} \mathbb{R}$, then the image of ρ_k generates $H_k^{\mathbb{R}}(\underline{g})$ over $\mathbb{R}$, because it contains $H_k^{\mathbb{Q}}(\underline{g}_0) \otimes 1 \subset H_k^{\mathbb{Q}}(\underline{g}_0) \otimes \mathbb{R} = H_k^{\mathbb{R}}(\underline{g})$.

Via the theorem, this corresponds to the well known fact that, for a nilpotent Lie group N defined over $\mathbb{Q}$, the continuous cohomology injects in the real cohomology of N, because N has a cocompact discrete subgroup.

It is also easy to see that, when $\underline{g}$ is defined over $\mathbb{Q}$, and has a strictly positive graduation, then ρ_k is surjective. This is the case for instance for the group of upper triangular matrices, or for a two step nilpotent Lie group defined over $\mathbb{Q}$.

When $\underline{g}$ is defined over a Galois extension of $\mathbb{Q}$, P. Deligne has shown to me that the image of ρ_k still generates $H_k^{\mathbb{R}}(\underline{g})$.

It is expected that, in general, the image of ρ_k does not generate $H_k^{\mathbb{R}}(\underline{g})$.

To end up, let us consider the nilpotent Lie group N of real upper triangular matrices of order 3. Its Lie algebra is of dimension 3 with center $\underline{c}$ of dimension 1. Then

$$H_2^{\mathbb{Q}}(\underline{n}) = [H_1^{\mathbb{R}}(\underline{n}/\underline{c}) \otimes_{\mathbb{R}} H_1^{\mathbb{R}}(\underline{c})] \oplus \mathrm{Ker}(\underline{n} \otimes_{\mathbb{Q}} \underline{c} \longrightarrow \underline{n} \otimes_{\mathbb{R}} \underline{c}).$$

The first summand is isomorphic to $H_2^{\mathbb{R}}(\underline{n})$. A typical element in the second summand is represented by a cycle of the form $\lambda X \wedge Y - X \wedge \lambda Y$, where $X, Y \in \underline{n}, \lambda \in \mathbb{R}$.

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