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Astérisque, tome 113-114 (1984), p. 278-287
<http://www.numdam.org/item?id=AST_1984__113-114__278_0>

P. A. SMITH THEORY VIA DEFORMATIONS

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It has been shown in [10] that the theory of deformations of algebras can be a useful tool in studying cohomology of transformation groups. The essential observation is a corollary of the following proposition (s.[9]) which is in turn a consequence of Borel's approach to study transformation groups using the localization theorem in equivariant cohomology.

Let X be a finite, connected G-CW-complex, $G = S^1$, F it's fixed point set and $X \to X_G \to B_G$ the fibration associated with the universal principal G-bundle $G \to E_G \to B_G$; $H^*(-)$ denotes cohomology with coefficients in the rationals $\mathbb{Q}$, $R := H^*(B_G) \cong \mathbb{Q}[t]$, degree $t = 2$ and $\mathbb{Q}^\varepsilon$ denotes $\mathbb{Q}$ together with the R-module structure given by $R \cong \mathbb{Q}[t] \to \mathbb{Q}$, $t \mapsto \varepsilon \in \mathbb{Q}$.

PROPOSITION 1: *If* $\varepsilon \neq 0$, *then* $\mathbb{Q}^\varepsilon \otimes_R H^*(X_G)$ *is naturally isomorphic to* $H^{ev}(F) \oplus H^{odd}(F)$ *as* $\mathbb{Z}/2\mathbb{Z}$-*graded* $\mathbb{Q}$-*algebra*.

COROLLARY (1.1) (s.[7]): *If* X *is* TNHZ (*i.e. totally non-homologous to zero*) *in* X_G, *which is equivalent to* $\sum_{i\geq 0} \dim H^i(X) = \sum_{i\geq 0} \dim H^i(F)$), *then* $H^*(F)$ *as a* $\mathbb{Z}/2\mathbb{Z}$-*graded* $\mathbb{Q}$-*algebra is a deformation of* $H^*(X)$.

This corollary allows to interpret results on deforma-

tions of algebras in terms of cohomology of transformation groups, e.g. the well known fact that the embedding dimension of a local, noetherian algebra cannot increase under deformation gives a proof of Bredon's conjecture that the minimal number of algebra-generators of $H^*(F_\nu)$, F_ν being a component of the fixed point set F, is not larger than the minimal number of algebra generators of $H^*(X)$ (under the assumption that X is TNHZ). This and other results of this type are discussed in [10]. We add one further result in this direction which follows from the work of Avramov (s.[3]) in local algebra. We give a direct proof here which is adapted to the given graded situation.
Let cid(A) denote the complete intersection defect of the local algebra A (s.[3]) which in case of the cohomology algebra of a connected, finite CW-complex Y is just the difference between the minimal number of relations and the minimal number of algebra generators of $H^*(Y)$ (since the Krull dimension of $H^*(Y)$ is zero), then one has:

<u>COROLLARY (1.2)</u>: <u>If</u> X <u>is</u> TNHZ <u>in</u> X_G <u>then</u>
$\mathrm{cid}(H^*(X)) \geq \mathrm{cid}(H^*(F_\nu))$ <u>for each component</u> $F_\nu \subset F$.

If $H^{odd}(X) = 0$ then $\mathrm{cid}(H^*(X))$ and $\mathrm{cid}(H^*(F_\nu))$ are always non negative. Hence in this case one gets in particular that $H^*(F_\nu)$ is a complete intersection ($\mathrm{cid}(H^*(F_\nu)) = 0$) if $H^*(X)$ is; a result which in the context of deformation theory has been known for some time.

<u>Proof of (1.2)</u>. We use the following notation: If A is an augmented commutative graded algebra over a ring k, then $\rho_A\colon A \to k$ denotes the augmentation and $Q(A) := \bar{A}/\bar{A}^2$ the k-module of indecomposables where $\bar{A} := \ker \rho_A$ is the augmentation ideal. Let $\Lambda(V)$ be the free commutative graded $\mathbb{Q}$-algebra over the graded $\mathbb{Q}$-vector space V (with the canonical augmentation). Choose a minimal (grading preserving) presentation $\Lambda(W) \xrightarrow{\beta} \Lambda(V) \xrightarrow{\alpha} H^*(X) \to 0$ of the connected commutative graded algebra $H^*(X)$ (i.e. α is surjective and induces an isomorphism of $\mathbb{Q}$-vector spaces
$Q(\alpha)\colon Q(\Lambda(V)) \cong V \to Q(H^*(X))$ and β induces an isomorphism

$Q(\beta): Q(\Lambda(W)) \cong W \to \ker \alpha/_{\ker \alpha \cdot \overline{\Lambda(V)}}$). It is well known (compare e.g. [7], chap. 1 for the non commutative case and the Lie algebra case)

that $\dim_{\mathbb{Q}} V$ = minimal number of generators of $H^*(X)$

$\dim_{\mathbb{Q}} W$ = minimal number of relations of $H^*(X)$

$\dim_{\mathbb{Q}} W - \dim_{\mathbb{Q}} V = \mathrm{cid}(H^*(X))$.

The algebra $H^*(X_G)$ is a graded algebra over the graded ring $R \cong \mathbb{Q}[t]$ which as a graded R-module is isomorphic to $R \otimes H^*(X)$ ("$\otimes$" means "$\otimes_{\mathbb{Q}}$") but carries a "twisted" multiplicative structure (s.[10]) such that Proposition 1 holds and for $\varepsilon = 0$ one has $\mathbb{Q}^{\varepsilon} \otimes_R H^*(X_G) = H^*(X_G)/_{tH^*(X_G)} \cong H^*(X)$.

By choosing a base point for a component F_ν of F one gets an augmentation $\rho_{H^*(X_G)}: H^*(X_G) \to R$ such that $\mathrm{id}_{\mathbb{Q}^\varepsilon} \otimes_R \rho_{H^*(X_G)}$ corresponds to $\rho_\nu: H^*(F) \to H^*(F_\nu) \to \mathbb{Q}$ for $\varepsilon \neq 0$ and to $\rho_{H^*(X)}: H^*(X) \to \mathbb{Q}$ for $\varepsilon = 0$.

One can "lift" α and β to morphisms of augmented graded R-algebras

$\tilde{\alpha}: R\otimes\Lambda(V) \to H^*(X_G)$ and $\tilde{\beta}: R\otimes\Lambda(W) \to R\otimes\Lambda(V)$

(the algebra structure of $R\otimes\Lambda(V)$ resp. $R\otimes\Lambda(W)$ is given by the usual algebra structure of the tensor product) such that

(i) $\mathrm{id}_{\mathbb{Q}^0} \otimes_R \tilde{\alpha} = \alpha$, $\mathrm{id}_{\mathbb{Q}^0} \otimes_R \tilde{\beta} = \beta$

(ii) $\tilde{\alpha} \circ \tilde{\beta}(R\otimes\overline{\Lambda(W)}) = 0$

(iii) $\tilde{\alpha}: R\otimes\Lambda(V) \to H^*(X_G)$ (hence also $Q(\tilde{\alpha}): Q(R\otimes\Lambda(V)) \cong R\otimes V \to Q(H^*(X_G))$) and $Q(\tilde{\beta}): Q(R\otimes\Lambda(W)) \cong R\otimes W \to \ker \tilde{\alpha}/_{\ker \tilde{\alpha} \cdot R\otimes\overline{\Lambda(V)}}$ are surjective.

(In fact, $R\otimes\Lambda(W) \xrightarrow{\tilde{\beta}} R\otimes\Lambda(V) \xrightarrow{\tilde{\alpha}} H^*(X_G) \longrightarrow 0$ can be viewed as a presentation of the R-algebra $H^*(X_G)$ (compare [4], but we don't need this explicitly).

To construct $\tilde{\alpha}$ one chooses a $\mathbb{Q}$-linear map $V \to \overline{H^*(X_G)}$ such that the composition with the projection $H^*(X_G) \to \mathbb{Q}^0 \otimes_R H^*(X_G) = H^*(X)$ coincides with $\alpha|_V$. Then there is a unique extension to an R-algebra morphism $\tilde{\alpha}: R\otimes\Lambda(V) \to H^*(X_G)$.

By construction $\mathrm{id}_{\mathbb{Q}^0} \underset{R}{\otimes} \tilde{\alpha} = \alpha$ and

$R\otimes\Lambda(V) \xrightarrow{\tilde{\alpha}} H^*(X_G) \to H^*(X_G)/_{t\cdot H^*(X_G)} = Q^0 \underset{R}{\otimes} H^*(X_G)$ is surjective.

It follows by induction that

$R\otimes\Lambda(V) \xrightarrow{\tilde{\alpha}} H^*(X_G) \to H^*(X_G)/_{t^q H^*(X_G)}$ is surjective for each $q \in \mathbb{N}$.

Hence $\tilde{\alpha}$ (and $Q(\tilde{\alpha})$) is surjective since $\tilde{\alpha}$ preserves degree and $(t^q\circ H^*(X_G))^n = 0$ for $q > \frac{n}{2}$.

The construction of $\tilde{\beta}$ is similar. One chooses a $\mathbb{Q}$-linear map $W \to \ker\tilde{\alpha} \subset R\otimes\Lambda(V)$ such that the composition with the projection $R\otimes\Lambda(V) \to \mathbb{Q}^0 \underset{R}{\otimes} (R\otimes\Lambda(V)) = \Lambda(V)$ equals $\beta|_W$. This is possible since $\tilde{\alpha}$ has an R-linear splitting ($H^*(X_G)$ is a free R-module). To show that $Q(\tilde{\beta})\colon R\otimes W \to \tilde{K} := \ker\tilde{\alpha}/_{\ker\tilde{\alpha}\cdot R\otimes\overline{\Lambda(V)}}$ is surjective it suffices (by an argument similar to that above) to observe that the composition $R\otimes W \xrightarrow{Q(\tilde{\beta})} \tilde{K} \to \tilde{K}/_{t\tilde{K}}$ is surjective.

Tensoring the sequence $R\otimes\Lambda(W) \xrightarrow{\tilde{\beta}} R\otimes\Lambda(V) \xrightarrow{\tilde{\alpha}} H^*(X_G) \to 0$ with $\mathbb{Q}^\varepsilon$ over R ($\varepsilon \neq 0$) one gets a sequence of augmented $\mathbb{Z}/2\mathbb{Z}$-graded (!) $\mathbb{Q}$-algebras

$$\mathbb{Q}^\varepsilon \underset{R}{\otimes} (R\otimes\Lambda(W)) \cong \Lambda(W) \xrightarrow{\tilde{\beta}^\varepsilon} \mathbb{Q}^\varepsilon \underset{R}{\otimes} (R\otimes\Lambda(V)) \cong \Lambda(V) \xrightarrow{\tilde{\alpha}^\varepsilon} \mathbb{Q}^\varepsilon \underset{R}{\otimes} H^*(X_G) \cong H^*(F) \to 0$$

such that $\tilde{\alpha}^\varepsilon := \mathrm{id}_{\mathbb{Q}^\varepsilon} \underset{R}{\otimes} \tilde{\alpha}$ (and hence $Q(\tilde{\alpha}^\varepsilon)$) and

$Q(\tilde{\beta}^\varepsilon) = Q(\tilde{\beta})^\varepsilon\colon W \to \ker\tilde{\alpha}^\varepsilon/_{\ker\tilde{\alpha}^\varepsilon\,\overline{\Lambda(V)}}$ are surjective.

Let $\alpha_\nu\colon \Lambda(V) \to H^*(F_\nu)$ denote the following composition

$$\Lambda(V) \cong \mathbb{Q}^\varepsilon \underset{R}{\otimes} (R\otimes\Lambda(V)) \xrightarrow{\alpha^\varepsilon} \mathbb{Q}^\varepsilon \underset{R}{\otimes} H^*(X_G) \cong H^*(F) \to H^*(F_\nu)\ (\varepsilon \neq 0)$$

which is a surjective morphism of augmented $\mathbb{Z}/2\mathbb{Z}$-graded algebras. By standard arguments (s.e.g. [7], Theorem (1.2.1)) one gets (even though α_ν does not neccessarily preserve the $\mathbb{Z}$-grading)

$\dim_{\mathbb{Q}} V \geq \dim Q(H^*(F_\nu))$ = minimal number of generators of $H^*(F_\nu)$

$\dim_{\mathbb{Q}}(\ker\alpha_\nu/_{\ker\alpha_\nu\cdot\overline{\Lambda(V)}}) - \dim_{\mathbb{Q}} V \geq \operatorname{cid} H^*(F_\nu)$.

To finish the proof it suffices to show that $\dim_{\mathbb{Q}} W \geq \dim_{\mathbb{Q}}(\ker\alpha_\nu/_{\ker\alpha_\nu\cdot\overline{\Lambda(V)}})$. Since $Q(\tilde{\beta}^\varepsilon)$ is surjective one has $\dim_{\mathbb{Q}} W \geq \dim_{\mathbb{Q}}(\ker \tilde{\alpha}^{\varepsilon}/_{\ker \tilde{\alpha}^{\varepsilon}\cdot\overline{\Lambda(V)}})$. We claim that $\ker \alpha^{\varepsilon} \subset \ker \alpha_{\nu}$ induces a surjection $\ker \tilde{\alpha}^{\varepsilon}/_{\ker \tilde{\alpha}^{\varepsilon}\cdot\overline{\Lambda(V)}} \longrightarrow \ker \alpha_{\nu}/_{\ker \alpha_{\nu}\cdot\overline{\Lambda(V)}}$. The algebra $H^*(F)$ can be written as a direct product $H^*(F) = H^*(F_{\nu}) \times H^*(F')$ where $F' = F \setminus F_{\nu}$. Then $\ker \alpha_{\nu} = (\tilde{\alpha}^{\varepsilon})^{-1}(0,H^*(F'))$. Choose $e \in \Lambda(V)$ such that $\tilde{\alpha}^{\varepsilon}(e) = (0,1)$. If $x \in \ker \alpha_{\nu}$ then $x = (x-ex)+ex$ shows that $x \in \ker \tilde{\alpha}^{\varepsilon} + \ker \alpha_{\nu}\cdot\overline{\Lambda(V)}$. This gives the desired result.

It might be worthwhile to remark that the above proof also reproves known results on the minimal number of generators and relations for $H^*(F_{\nu})$ in comparison with those for $H^*(X)$ (s.[4],[6],[9],[10]). In fact the proof given here is in a sense an extension of the proof for the number of generators given in [9] and simplifies the arguments given in [4] for the number of relations.

These results generalize to torus actions (by induction on the dimension of the torus) and analogous results can be obtained for $(\mathbb{Z}/p\mathbb{Z})^d$-actions (p prime) and cohomology with $\mathbb{Z}/p\mathbb{Z}$ coefficients by the same method.

The theory of minimal models (s.[5],[11]) allows to use deformation theory of algebraic structures (different from algebra multiplication) to obtain results on rational homotopy (and cohomology) of transformation groups.

We assume X to be a finite, 1-connected (to avoid complications with the fundamental group) G-CW-complex ($G=S^1$) with $\sum_{i>0} \dim_{\mathbb{Q}}(\pi_i(X)\otimes\mathbb{Q}) < \infty$. If $M(X)$ is a minimal model of X, a model for the fibration $X \to X_G \to B_G$ is given by

$$R=\mathbb{Q}[t]=M(B_G) \to M(X)[t]=\mathbb{Q}[t] \otimes M(X)=M(X_G) \to M(X) \quad \text{(s. e.g.[5])}$$

where $M(X_G)$ is equipped with a differential which can be considered a one-parameter family of deformations of the differential of $M(X)$ $(d(m) = d_0(m) + t\, d_1(m) + \ldots + t^j\, d_j(m) + \ldots,$ where $m \in M(X) \subset M(X)[t]$, $d_j\colon M^q(X) \to M^{q+1-2j}(X)$ is a derivation, $d_0 = d_{M(X)}$, $d^2 = 0)$.

The following proposition follows from Allday's results on rational homotopy of torus actions (s.[1],[2]). It relates to Allday's rational homotopy analogue of the localization theorem in cohomology in the same way as Proposition 1 relates to the later.

Let $\eta_\nu : M(F) \to \mathbb{Q}$ $(M(F) := \underset{\nu}{\times} M(F_\nu))$ denote the augmentation corresponding to the inclusion of a point $*$ into a non-empty component F_ν of F, and

$$\bar{\eta}_\nu^\varepsilon : \mathbb{Q}^\varepsilon \underset{R}{\otimes} M(X_G) \longrightarrow \mathbb{Q}^\varepsilon \underset{R}{\otimes} M((F)_G) \xrightarrow{\mathrm{id}\otimes(\eta_\nu)_G} \mathbb{Q}^\varepsilon \underset{R}{\otimes} R \cong \mathbb{Q}$$ the induced augmentation on $\mathbb{Q}^\varepsilon \underset{R}{\otimes} M(X_G)$.

<u>PROPOSITION 2</u>: <u>If</u> $\varepsilon \neq 0$, <u>then</u> $(\mathbb{Q}^\varepsilon \underset{R}{\otimes} M(X_G), \bar{\eta}_\nu^\varepsilon)$ <u>and</u> $(M(F), \eta_\nu)$ <u>are weakly equivalent as augmented</u> $\mathbb{Z}/2\mathbb{Z}$-<u>graded, differential</u> $\mathbb{Q}$-<u>algebras</u>.

<u>In particular one has</u>:

(i) $H^*(\mathbb{Q}^\varepsilon \underset{R}{\otimes} M(X_G)) \cong H^*(F)$ <u>as</u> $\mathbb{Z}/2\mathbb{Z}$-<u>graded algebras</u>.

(ii) $\Pi^*(\mathbb{Q}^\varepsilon \underset{R}{\otimes} M(X_G), \bar{\eta}_\nu^\varepsilon) \cong \Pi^*(F, \eta_\nu) = \Pi^*(F_\nu, \eta_\nu)$.

($\Pi^*(A,\eta)$ of an augmented, differential algebra is defined by $\Pi^*(A,\eta) := H(\ker \eta / (\ker \eta)^2)$ where the homology is taken with respect to the differential $\bar{d}$ on $Q(A,\eta) := \ker \eta / (\ker \eta)^2$ induced by the differential d of A.)

$\Pi^*(F_\nu, \eta_\nu)$ is the pseudo-dual rational homotopy of F_ν, in particular if F_ν is 1-connected $\Pi^*(F_\nu, \eta_\nu)$ is the dual (over $\mathbb{Q}$) of $\pi_*(F_\nu) \otimes \mathbb{Q}$.

<u>Proof of Proposition 2</u>: The isomorphism between $\mathbb{Q}^\varepsilon \underset{R}{\otimes} H^*(X_G)$ and $\mathbb{Q}^\varepsilon \underset{R}{\otimes} H^*(F_G) \cong H^*(F)$ (as $\mathbb{Z}/2\mathbb{Z}$-graded algebras) is induced by the inclusion $F \hookrightarrow X$ (s. [9], compare Prop. 1). Since the tensor product with $\mathbb{Q}^\varepsilon$ over R ($\varepsilon \neq 0$!) commutes with homology $F \hookrightarrow X$ (resp. $\mathbb{Q}^\varepsilon \underset{R}{\otimes} M(X_G) \to \mathbb{Q}^\varepsilon \underset{R}{\otimes} M(F_G) \cong M(F)$) induces an isomorphism of $\mathbb{Z}/2\mathbb{Z}$-graded augmented algebra

$$H^*(\mathbb{Q}^\varepsilon \underset{R}{\otimes} M(X_G)) \cong \mathbb{Q}^\varepsilon \underset{R}{\otimes} H^*(X_G) \to \mathbb{Q}^\varepsilon \underset{R}{\otimes} H^*(F_G) \cong H^*(\mathbb{Q}^\varepsilon \underset{R}{\otimes} M(F_G)) \cong H^*(M(F)).$$

Applying (2.3) and (2.4) of [1] then gives the desired result.

Since $\mathbb{Q}^{\varepsilon} \underset{R}{\otimes} M(X_G) \cong M(X)$ for $\varepsilon=0$ one has immediately

<u>COROLLARY (2.1)</u>: <u>Up to weak equivalence</u> $(M(F),\eta_\nu)$ <u>is a deformation of</u> $(M(X),\eta)$ $(\eta\colon M(X) \to M(F) \xrightarrow{\eta_\nu} \mathbb{Q})$.

Of course, here the differential of $M(X)$ and not the algebra structure is deformed.

The cohomology of F and the pseudo-dual rational homotopy of F_ν can be calculated from the deformation of the model $(M(X),\eta)$. Therefore one gets immediately:
$\chi_\pi(X) := \chi(\Pi^*(X)) = \chi(\Pi^*(F_\nu)) = \chi_\pi(F_\nu)$ (s.[1]).

From semi-continuity of the dimension of homology groups of a chain complex under deformation of the boundary one has:

<u>COROLLARY (2.2)</u>: $\sum_{i>o} \dim \Pi^i(X) \geq \sum_{i>o} \dim \Pi^i(F_\nu)$ <u>for all</u> $F_\nu \subset F$ (<u>s</u>.[1]).

Using the fact that $(M(X),\eta)$ can be considered a filtered $\mathbb{Z}/2\mathbb{Z}$-graded algebra $(\mathcal{F}^k(M(X),\eta) := \bigoplus_{j=o}^{k} M^j(X))$ in such a way that the deformation preserves this filtration one easily gets the following refinement of Corollary (2.2):

<u>Corollary (2.2)'</u>: $\sum_{i>o} \dim \Pi^{k+2i}(X) \geq \sum_{i>o} \dim \Pi^{k+2i}(F_\nu)$ <u>for all</u> $F_\nu \subset F$ <u>and all</u> $k \in \mathbb{N}$ (compare [1] and s.[12] for a different proof).

<u>Proof of (2.2)'</u>. The morphism (of R-algebras) $M(X_G) \cong R\otimes M(X) \to M(F_G) \cong R\otimes M(F)$ induced by $F \subset X$ preserves degrees and hence also the filtration given by $\mathcal{F}^k(M(X_G)) \cong R \otimes \bigoplus_{j=o}^{k} M^j(X)$ resp. $\mathcal{F}^k(M(F_G)) \cong R \otimes \bigoplus_{j=o}^{k} M^j(F)$.

Therefore $\mathbb{Q}^{\varepsilon} \underset{R}{\otimes} M(X_G) \to \mathbb{Q}^{\varepsilon} \underset{R}{\otimes} M(F_G) \cong M(F)$ $(\varepsilon \neq 0)$ preserves the induced filtrations. For the indecomposables (with respect to the augmentations coming from η_ν) one has the corresponding situation, i.e. $\mathbb{Q}^{\varepsilon} \otimes_R Q(M(X_G),\bar{\eta}_\nu) \to Q(M(F),\eta_\nu) = Q(M(F_\nu),\eta_\nu) = \Pi^*(F_\nu,\eta_\nu)$ preserves the induced filtrations.

This morphism is surjective and hence induces a surjective morphism

$$\mathbb{Q}^{\varepsilon} \otimes_R Q(M(X_G),\bar{\eta}_\nu)/_{F^k(Q^{\varepsilon} \otimes_R Q(M(X_G)),\bar{\eta}_\nu)} \longrightarrow Q(M(F_\nu),\eta_\nu)/_{F^k(Q(M(F_\nu),\eta_\nu)}.$$

As $\mathbb{Q}$-vector spaces the source of this morphism is isomorphic to $\Pi^*(X)/_{F^k\Pi^*(X)} \cong \bigoplus_{j>k} \Pi^j(X)$ and the target is isomorphic to $\Pi^*(F)/_{F^k\Pi^*(F)} \cong \bigoplus_{j>k} \Pi^j(F)$.

Since "everything in sight" is (at least) $\mathbb{Z}/2\mathbb{Z}$-graded the assertion (2.2)' follows.

The well known analogous result for cohomology can be obtained by similar arguments.

As above these results generalize to torus actions.

The analogous assumption to "X being TNHZ in X_G" is the assumption " $\sum_{i>o} \dim \Pi^i(X) = \sum_{i>o} \dim \Pi^i(F_\nu)$" and in fact under this hypothesis one gets a result which is analogous to Corollary (1.1) (compare [2]) namely:

<u>COROLLARY (2.3)</u>: <u>If</u> $\sum_{i\geq o} \dim \Pi^i X = \sum_{i\geq o} \dim \Pi^i(F_\nu)$ <u>then the co-Lie algebra</u> $\Pi^*(F_\nu)$ <u>is a deformation of the co-Lie algebra</u> $\Pi^*(X)$.

(The co-Lie algebra structure corresponds to the dual of the Whitehead product. To get graded co-Lie algebras $L^*(X)$, resp. $L^*(F_\nu)$ one has to shift the degree by one, i.e. $L^i(X) := \Pi^{i+1}(X)$, $L^*(X) = \Sigma^{-1}\Pi^*(X)$).

<u>Proof of (2.3)</u>. It is shown in [2] that under the above hypothesis $(M(X_G),\bar{\eta}_\nu)$ gives rise to a co-Lie algebra $L(M(X_G),\bar{\eta}_\nu)$ over $\mathbb{Q}[t] = R$. The quadratic part of the differential of $M(X_G) = R \otimes M(X)$ (obtained by deforming the differential of $M(X)$ in-

duces the co-Lie algebra structure on $L(M(X_G),\bar{\eta}_\nu) \cong R \otimes \Sigma^{-1} \Pi^*(X)$ (as R-modules but with twisted co-Lie algebra structure). Since this construction is natural with respect to the inclusion $F \to X$ and since the isomorphism given in Prop. 2 is induced by this inclusion, one has $\mathbb{Q}^\varepsilon \otimes_R L(M(X_G),\bar{\eta}_\nu) \cong L(F_\nu)$ for $\varepsilon \neq 0$. On the other hand it is immediate from the construction of $L(M(X_G),\bar{\eta}_\nu)$ that $\mathbb{Q}^0 \otimes_R L(M(X_G)\bar{\eta}_\nu) \cong L(X)$.

If F_ν is 1-connected one has the corresponding results for rational homotopy Lie algebras. Therefore the already developed theory of deformation of (graded) Lie algebras (s. e.g. [8]) is at hand to give further results on the rational homotopy of torus actions.

I have learned at the conference that C. Allday has drawn similar conclusions. We plan to pursue some further questions in this direction in a joint paper.

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