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SULLIVAN-QUILLEN MIXED TYPE MODEL FOR FIBRATIONS AND THE HAEFLIGER MODEL FOR THE GELFAND-FUKS COHOMOLOGY

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1. Introduction (The Bott "Conjecture").

Let M be a paracompact Hausdorff C^{∞}-manifold of dimension $n \geqslant 1$ and L_M be the topological Lie algebra of C^{∞}-vector fields on M. Gelfand-Fuks [1] considered the differential graded algebra (DGA for brevity) $C_c^*(L_M)$ of continuous cochains of L_M, and its cohomolgy $H^*(C_c^*(L_M))$ is called the Gelfand-Fuks cohomology of M.

On the other hand, let $EU_n^{(2n)} \to BU_n^{(2n)}$ be the universal U_n-bundle restricted over the (homotopical) 2n-skelton of the base and

$$(1.1.)\qquad \hat{\gamma}_n : EU_n^{(2n)} \to EU_n^{(2n)} \underset{U_n}{\times} EU_n \to BU_n$$

be the associated fiber bundle over BU_n with fiber $EU_n^{(2n)}$. And let τ_M^C be the complexification of the tangent bundle of M classified by a map $f_M^C : M \to BU_n$. Consider the cross-section space $\Gamma((f_M^C) * (\hat{\gamma}_n))$ of the induced bundle $(f_M^C) * (\hat{\gamma}_n)$ equipped with the compact open topology. Then the Bott "Conjecture" asserts ;

$$(1.2.)\qquad H^*(C_c^*(L_M)) \cong H^*(\Gamma((f_M^C) * (\hat{\gamma}_n));R).$$

A. Haefliger [3], [4] affirmatively solved this conjecture by constructing a Sullivan-Quillen mixed type model for the fibration $(f_M^C) * (\hat{\gamma}_n)$. Here, by a Sullivan-Quillen mixed type model for a fibration, we mean a DG Lie algebra

$L = A^* \otimes \bar{L}$ over a DGA A^* with a differential d, whose restriction $(A^*, d\,|\,A^* = d_A)$ is a model for the base space in the sense of Sullivan and whose quotient $(\bar{L} = R \otimes_{A^*} L,\ 1 \otimes_{A^*} d)$ is a model for the fiber in the sense of Quillen.

The superiority of the mixed type model lies in the following fact. The cochain complex $C^*_A(L)$ over A^* of L is a Sullivan model for the total space of the fibration while the cochain complex $C^*_R(L)$ over R of L is a Sullivan model for the cross-section space of the fibration.

But if we want to construct a mixed type model on the universal level, i.e. a model for $\hat{\gamma}_n$ itself instead of the induced one $(f^C_M)^* * (\hat{\gamma}_n)$, we have no longer a differential on L but a pair (D,χ) of a derivation D on L and the <u>Euler element</u> χ in L_{-2} , χ being the obstruction for D to be a differential and, at the same time, being a representative for the obstruction class to the existence of a cross-section of the fibration.

In this note we give a sketch of the following two subjects, the details of which will appear elsewhere. First we present a general view of the Sullivan-Quillen mixed type model in section 2, generalizing the Haefliger-Silveira theory of mixed type model for fibrations with a given cross-section [7]. In section 3, we exhibit a very explicit description of the mixed type model for the fibration (1.1.), and thus give a complete answer to the algebraic computational problem posed by Haefliger [3]. We remark that partial results to this problem permitted us to deduce the following result.

THEOREM (1.3)([6]) : <u>A closed connected orientable manifold</u> M <u>of dimension</u> ≥ 1 <u>has finitely generated Gelfand-Fuks cohomology</u> (<u>as an</u> R-<u>algebra</u>) <u>if and only if</u> $M = S^1$.

I am greatly indebted to S. Hurder's suggestion for accomplishing my computations of the differential in Haefliger's model. I also owe a great deal to A. Haefliger for suggesting me to generalize the mixed type model theory to fibrations without cross-section. Finally the discussions with H. Sliga clarified me the role of the Euler element in the mixed type model.

2. Sullivan-Quillen mixed type model for fibrations.

Let $A^*(= A_{-*})$ be a positively graded DGA with a differential d_A and L_* be a graded Lie algebra over A^* with the grading deg $(a.y) = \deg(y) - \deg(a)$ for $a \in A^*$ and $y \in L$.

DEFINITION 2.1. : A graded Lie algebra L_* over A^* is <u>an algebraic fibration of mixed type over</u> A^* if $(R \otimes_A L_*)_p = 0$ for $p \leq 0$ and if it is equipped with a pair (χ, D), where χ is an element of L_{-2} called <u>an Euler element</u> and $D : L_* \to L_*$ is an A^*-Lie derivation of degree -1, i.e.

(2.2) $$D(a \cdot [y_1, y_2]) = d_A(a) \cdot [y_1, y_2] + (-1)^{\deg(a)} a . \{ [D(y_1), y_2] + (-1)^{\deg(y_1)} [y_1, D(y_2)] \},$$

satisfying the following <u>trace formulas</u> ;

(2.3) $D(\chi) = 0$, and

(2.4) $(D)^2(y) = [\chi, Y]$ for every $y \in L_*$.

The quotient DG Lie algebra $(R \otimes_A L_*, 1 \otimes_A D)$ is called <u>the fiber</u> of this algebraic fibration.

DEFINITION 2.5. : Let (L_*, χ, D) be an algebraic fibration of mixed type over A^*. Its <u>chain complex</u> $C_*^{A,\chi}(L_*)$ <u>over</u> A^* is the DG coalgebra $(S_*^A(\sigma L_*), d = d_L + D + d_\chi)$, where σL_* is the suspension of L_* (the shift of degree by $+1$), $S_*^A(\sigma L_*)$ denotes the symmetric coalgebra of σL_* taken over A^*, d_L is the usual differential on $s_*^A(\sigma L_*)$ arising from the Lie bracket of L_*, D is the coderivation on $S_*^A(\sigma L_*)$ induced by the derivation on L_* denoted by the same symbol, and d_χ is the differential which is nothing but the multiplication by the suspension $\sigma\chi$ of χ, i.e. $d_\chi(x) = \sigma\chi . x$. We call d_χ <u>the Euler differential in</u> $C_*^{A,\chi}(L_*)$. The trace formulas (2.3) and (2.4) are equivalent to ; $d^2 = 0$ in $S_*^A(\sigma L_*)$. <u>The cochain complex</u> $C^*_{A,\chi}(L_*)$ <u>over</u> A^* <u>of</u> L_* is the A^*-dual of $C_*^{A,\chi}(L_*)$, namely

(2.6) $$\mathrm{Hom}_{A^*}(S_*^A(\sigma L_*), A^*) \cong A^* \otimes S_R^*(R \otimes_A \sigma L_*) \quad ; \quad \mathrm{Hom}_{A^*}(d, 1).$$

This is an algebraic fibration over A^* in the sense of Sullivan.

Conversely, starting from an algebraic fibration $A^* \to E^*$ in the sense of Sullivan, we can construct a mixed type fibration (L_*, D, χ) over A^* with χ being a representative of the obstruction class to the existence for a cross-section in the minimal model for the fibration above.

3. <u>The Haefliger model</u>.

Now we return to the fibration $\hat{\gamma}_n$ of (1.1). The minimal model for the base space BU_n is given by

$$I^n = R\left[\bar{c}_1,\bar{c}_2,\ldots,\bar{c}_n\right] \;;\; \deg \bar{c}_i = 2i, \quad d(\bar{c}_i) = 0. \tag{3.1}$$

A model for the fiber $EU_n^{(2n)}$ is given by

$$\hat{W}_n = E(h_1,h_2,\ldots,h_n) \otimes (R[c_1,c_2,\ldots,c_n]/(\deg > 2n)) \tag{3.2}$$

with $\deg h_i = 2i-1$, $\deg c_i = 2i$, $d(h_i) = c_i$, $d(c_i) = 0$.
A model (in the sense of Sullivan) for the total space is given by

$$I^n \otimes \hat{W}_n \;;\; d(h_i) = c_i - \bar{c}_i \,,\; d(c_i) = d(\bar{c}_i) = 0. \tag{3.3}$$

The fiber $EU_n^{(2n)}$ has the rational homotopy type of a bouquet of spheres and its minimal model (in the sense of Quillen) is

$$L(\sigma^{-1}\tilde{H}^*(\hat{W}_n)') \;;\; d \equiv 0. \tag{3.4}$$

A convenient basis $\{[h_I c_J]$; partitions I and J satisfy certain inequalities$\}$ for $\tilde{H}^*(\hat{W}_n)$ was found by J. Vey [2].

Now $I^n \otimes L(\sigma^{-1}\tilde{H}^*(\hat{W}_n)')$ has the natural graded Lie algebra structure over I^n. We define the Euler element χ in it by

$$\chi = \sum_\omega \bar{c}_\omega \otimes \sigma^{-1}\left[h_{\omega_1} c_{\omega_2} c_{\omega_3} \cdots\right]', \tag{3.6}$$

where the summation runs over all the partitions $\omega = (\omega_1,\omega_2,\omega_3,\ldots)$ such that $1 \leqslant \omega_1 \leqslant \omega_2 \leqslant \ldots$, $\omega_2 + \omega_3 + \ldots \leqslant n$, and that $\omega_1 + \omega_2 + \omega_3 + \ldots > n$. And we define I^n-Lie derivation D as a sum of two differentials d_1 and d_2 ; (c.f. [6], p. 398 for the notations)

$$D = d_1 + d_2 : I^n \otimes L(\sigma^{-1}\tilde{H}^*(\hat{W}_n)') \to I^n \otimes L(\sigma^{-1}\tilde{H}^*(\hat{W}_n)'), \tag{3.7}$$

$$\begin{aligned} d_1(1 \otimes y(I,J)) = &- \sum_{(1)} \operatorname{sign} \prod_{1\leqslant\nu} (\omega_1 - i_\nu)\bar{c}_\omega \otimes y(\omega_1+I;\omega-\omega_1+J) \\ &+ \sum_{(2)} \operatorname{sign} \prod_{1\leqslant\nu} (j_t - i\ )\bar{c}_\omega \otimes y(\omega_1+I-i_1+j_t;\omega-\omega_1+i_1+J-j_t) \,, \end{aligned} \tag{3.8}$$

where $y(I;J) = \sigma^{-1}[h_I c_J]'$, and

(3.9) $d_2(1 \otimes y(I;J))$

$$= \sum_{(1)} (-1)^{|I(1)|} \mathrm{sign} \sum_{1 \leq \mu,\nu} (i_\mu^{(1)} - i_\nu^{(2)}) \bar{c}_\omega \otimes [y(I(1);J(1)), y(\omega_1 + I(2) - i_1^{(2)}; \omega - \omega_1 + i_1^{(2)} + J(2))]$$

$$+ \sum_{(2)} (-1)^{|I(1)|} \mathrm{sign} \prod_{1 \leq \mu,\nu} (i_\mu^{(1)} - i_\nu^{(2)}) \bar{c}_{\omega(1)} \bar{c}_{\omega(2)} \otimes \Big[y(\omega_1^{(1)} + I(1) - i_1^{(1)}; \omega(1) - \omega_1^{(1)} + i_1^{(1)} + J(1)), y(\omega_1^{(2)} + I(2) - i_1^{(2)}; \omega(2) - \omega_1^{(2)} + i_1^{(2)} + J(2)) \Big].$$

One checks by direct computations that χ and D defined above satisfy the trace formulas (2.3) and (2.4). Thus $(I^n \otimes L(\sigma^{-1}\tilde{H}^*(\hat{W}_n)'), \chi, D)$ is an algebraic fibration of mixed type. Its cochain complex $C^*_{I^n,\chi}(I^n \otimes L(\sigma^{-1}H^*(\hat{W}_n)'))$ is proved to be the minimal model for the fibration (3.3).

Now let M be an n-dimensional manifold as stated in the introduction, and $\Omega^*(M)$ be its de Rham algebra. A choice of Pontrjagin forms $\tilde{p}_i \in \Omega^{4i}(M)$ makes $\Omega^*(M)$ an I^n-algebra via the homomorphism defined by $\bar{c}_{2i} \to \tilde{p}_i, \bar{c}_{2i-1} \to 0$. By the scalor extension, we obtain a DG Lie algebra over $\Omega^*(M)$

(3.10) $$(\Omega^*(M) \otimes_{I^n} (I^n \otimes L(\sigma^{-1}H^*(\hat{W}_n)') \cong \Omega^*(M) \otimes L(\sigma^{-1}H^*(\hat{W}_n)'); 1 \otimes_{I^n} D)$$

whose cochain complex $C^*_R(\Omega^*(M) \otimes L(\sigma^{-1}H^*(\hat{W}_n)')$ over R is a model for the cross-section space $\Gamma((f_M^C) * (\hat{\gamma}_n))$. This is <u>the Haefliger model</u> for the Gelfand-Fuks cochain complex $C_c^*(L_M)$. Notice that $(f_M^C) * (\hat{\gamma}_n)$ admists a unique homotopy class of cross-sections since the fiber $EU_n^{(2n)}$ is $2n$-connected.

REMARK 3.11. : The minimal model for the algebraic fibration (3.3) is isomorphic to that of DGA $I_{(n)} = I^n/(\deg > 2n)$. So the minimal model above can also be regarded as the minimal model $M_{I_{(n)}}$. In fact, the modulo $(\bar{M}_{I_{(n)}})^3$-reduction of the formulas (3.6)-(3.9) gives rise to formulas (2.15)-(2.19) of Hurder-Kamber [5].

Since $R[p_1, p_2, \ldots, p_{n/2}] \cong I^n/(\bar{c}_{2i-1})$, we obtain the minimal model (= the Postnikov decomposition) for the algebraic fibration $P_n \to P_n \otimes \hat{W}_n$ by putting $\bar{c}_{2i-1}$ in the model above. This is a complete answer to the computational problem posed in [3].

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