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DUALITY THEOREMS FOR FOLIATIONS

by

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INTRODUCTION. In this paper we establish several results on the basic cohomology of foliations. The first main result, Theorem 1.4, establishes for any foliation $\mathfrak{F}$ a canonical isomorphism between the cohomology of $\mathfrak{F}$-basic forms and the homology of transveral and holonomy-invariant currents with respect to $\mathfrak{F}$.

For the further statements we need the concepts of tense and taut foliations. A Riemannian foliation $\mathfrak{F}$ on M is <u>tense</u>, if there exists a Riemannian metric on M such that the mean curvature of the leaves is covariant constant (i.e. parallel) along the leaves of $\mathfrak{F}$. If there exists a metric for which the mean curvature vanishes, then $\mathfrak{F}$ is said to be <u>taut</u> [14] [15]. The harmonic foliations of [5] to [10] are foliations with minimal leaves on a Riemannian manifold (M,g_M). Thus the harmonic foliations represent exactly the class of taut foliations.

The second main result, Theorem 2.14, establishes then for an oriented tense Riemannian foliation $\mathfrak{F}$ on a closed oriented manifold the finite-dimensionality of the cohomology of $\mathfrak{F}$-basic forms, and an isomorphism of certain basic cohomology groups in complementary dimensions. This isomorphism is a precursor of the Poincaré Duality Theorem 3.1, valid for the basic cohomology of taut Riemannian foliations. Thus the original assertion of Poincaré duality in [12] is correct (exactly) in the case of taut foliations. An example of a Riemannian foliation violating Poincaré duality was found in [2]. The results of the present paper answer several questions raised during the conference in Toulouse.

1. For a foliation $\mathfrak{F}$ on a manifold M, the complex of $\mathfrak{F}$-basic forms is the subcomplex $\Omega_B^{\cdot}(\mathfrak{F})$ of the DeRham complex $\Omega^{\cdot}(M)$ given by the forms ω satisfying $\theta(X)\omega = 0$, $i(X)\omega = 0$ for all $X \in \Gamma L$. Here $L \subset TM$ denotes the bundle of vectors tangent to $\mathfrak{F}$ with normal bundle $Q = TM/L$. Similarly we have the complex of $\mathfrak{F}$-basic forms with compact support $\Omega_{c,B}^{\cdot}(\mathfrak{F})$. The differential d_B is the restriction of d on $\Omega(M)$ to $\Omega_B(\mathfrak{F})$, and on $\Omega_c(M)$ to $\Omega_{c,B}(\mathfrak{F})$ respectively.

To explain the complex of transverse holonomy invariant currents with respect to $\mathfrak{F}$, we need the spectral sequence $E(\mathfrak{F})$ [5] [6]. It is associated to the following filtration of $\Omega^{\cdot}(M)$:

$$(1.1) \quad F^r\Omega^m = \{\omega \in \Omega^m \,|\, i(v)\omega = 0,\ v = X_1 \wedge \ldots \wedge X_{m-r+1},\ X_i \in \Gamma L\}\ .$$

Then $E_1^{r,s} = H^s(M, \Lambda^r \underline{Q}^{*L})$, where $\Lambda^{\cdot}\underline{Q}^{*L}$ is the sheaf complex of $\mathfrak{F}$-basic forms. Thus $E_1^{r,o} \cong \Omega_B^r(\mathfrak{F})$, $d_1 = d_B$ and $E_2^{r,o} \cong H^r(\Omega_B^{\cdot}(\mathfrak{F}))$. The spectral sequence convergences to the DeRham cohomology $H_{DR}^{\cdot}(M)$ of M.

The same construction applied to $\Omega_c^{\cdot}(M)$ yields a spectral sequence $E_c(\mathfrak{F})$ converging to $H(\Omega_c^{\cdot}(M))$.

Now Haefliger [4] (and Ruelle-Sullivan [13] for $r = 0$) introduced the transversal holonomy invariant forms with compact support $\Omega_c^{\cdot}(Tr\mathfrak{F})$. This complex satisfies $E_{c,1}^{\cdot,p} \cong \Omega_c^{\cdot}(Tr\mathfrak{F})$. For the dual space (of continuous linear functionals with respect to the C^∞-topology) we have canonically ($p = \dim\mathfrak{F}$)

$$(1.2) \quad E_{c,1}^{\cdot,p^*} \cong C_{\cdot}(Tr\mathfrak{F}) \text{ (transversal, holonomy-invariant currents).}$$

The canonical differential $\partial_1 = d_1^T$ on the LHS corresponds precisely to the boundary on currents.

Throughout this paper we assume M to be oriented, and $\mathfrak{F}$ a transverally oriented foliation of codimension q on $M^n (p+q = n)$. We consider the duality map sending a form α to the current $c(\omega) = \int_M \alpha \wedge \omega$. This induces homomorphisms $D_1 : E_1^{q-r,p-s} \to E_{c,1}^{r,s^*}$ and in particular

$$(1.3) \qquad D_1 : \Omega_B^{q-r}(\mathfrak{F}) \to E_{c,1}^{r,p^*}\ .$$

Our first result is then as follows.

1.4 DeRHAM DUALITY THEOREM. The homomorphisms D_1 are compatible with differentials and induce isomorphisms

$$D_{1*}: H^{q-r}(\Omega_B^{\cdot}(\mathfrak{F})) \xrightarrow{\cong} H_r(E_{c,1}^{\cdot,p^*},\partial_1)$$

for $r = 0,\dots,q$. The canonical map

$$\text{can}: H_r(E_{c,1}^{\cdot,p^*},\partial_1) \to E_{c,2}^{r,p^*}$$

to continuous linear functionals on $E_{c,2}^{r,p}$ is surjective with kernel $(\overline{0})$.

For the case of the point foliation ($q = n$) this reduces to DeRham's theorem identifying $H_{DR}^{\cdot}(M)$ with the homology of currents [3].

The proof consists in constructing diffusion operators

$$R_1: E_{c,1}^{r,p^*} \to \Omega_B^{q-r}(\mathfrak{F})$$

and homotopy operators

$$A_1: E_{c,1}^{r,p^*} \to E_{c,1}^{r+1,p^*}$$

satisfying the following properties:

(1.5) $$D_1R_1 - id_{E_1} = \partial_1A_1 + A_1\partial_1 \;;$$

(1.6) $$R_1\partial_1 = d_BR_1 \quad (d_B\text{: restriction of } d \text{ to } \Omega_B(\mathfrak{F}))\;;$$

(1.7) A_1 preserves diffuse currents (images under D_1) and induces

$$A_B: \Omega_B^{q-r}(\mathfrak{F}) \to \Omega_B^{q-r-1}(\mathfrak{F})\;;$$

(1.8) $$R_1D_1 - id_{\Omega_B} = d_BA_B + A_Bd_B\;;$$

The details of this construction will appear elsewhere. The surjectivity of the map can follows by an application of the Hahn-Banach theorem. ∎

2. In the remainder of this paper we discuss the case of tense and taut Riemannian foliations (see Introduction). Let $\chi_{\mathfrak{F}}$ denote the characteristic p-form of $\mathfrak{F}$ with respect to a bundle-like metric g_M on M [8] [10] [14]. Let ν be the transversal Riemannian volume on Q. Then $\nu \in \Omega_B^q(\mathfrak{F})$ and in fact $d\nu = 0$. We choose the orientation of $L \subset TM$ such that

$$(2.1) \qquad *\nu = \chi_{\mathfrak{F}} ,$$

i.e. $\mu = \nu \wedge \chi_{\mathfrak{F}}$ represents the Riemannian volume form of M. The characteristic form $\chi_{\mathfrak{F}}$ satisfies then the formula

$$(2.2) \qquad d\chi_{\mathfrak{F}} + \kappa \wedge \chi_{\mathfrak{F}} \equiv 0 \pmod{F^2\Omega^{p+1}(M)} ,$$

i.e. modulo $\mathfrak{F}$-trivial forms. Here $\kappa \in \Gamma Q^*$ is the mean-curvature form of $\mathfrak{F}$ with respect to g_M, given by $\kappa(s) = \mathrm{Tr}\, W(s)$, $s \in \Gamma Q$, i.e. as the trace of the Weingarten operator of $\mathfrak{F}$ [8],[10],[14]. $\mathfrak{F}$ is tense, if $\kappa \in \Omega_B^1(\mathfrak{F})$ for a suitable bundle-like metric g_M on M, and in this case one has $d_B\kappa = 0$.

For a bundle-like metric there is further a star operator $\bar{*}$ in $\Omega_B^{\cdot}(\mathfrak{F})$. The relationship between these operators is given by the formulas

$$(2.3) \qquad \bar{*}\alpha = (-1)^{p(q-r)}*(\alpha \wedge \chi_{\mathfrak{F}}) = (-1)^{p(q-r)}i(v)*\alpha$$

and

$$(2.4) \qquad *\alpha = \bar{*}\alpha \wedge \chi_{\mathfrak{F}} , \quad \text{for } \alpha \in \Omega_B^r(\mathfrak{F}) , \ v = \xi_1 \wedge \ldots \wedge \xi_p .$$

By (2.2) one has the following formula

$$(2.5) \qquad d(\alpha \wedge \chi_{\mathfrak{F}}) \equiv (d_B\alpha - \kappa \wedge \alpha) \wedge \chi_{\mathfrak{F}} , \quad \alpha \in \Omega_B^r(\mathfrak{F}) ,$$

or by (2.4)

$$d*\alpha \equiv (d_B\bar{*}\alpha - \kappa \wedge \bar{*}\alpha) \wedge \chi_{\mathfrak{F}} , \quad \alpha \in \Omega_B^{q-r}(\mathfrak{F}) ,$$

where the congruence again has to be taken modulo the $\mathfrak{F}$-trivial forms $\mathfrak{F}^{r+1}\Omega^{p+r}$. This motivates the following definitions.

For any $\gamma \in \Omega_B^1(\mathfrak{F})$ with $d_B\gamma = 0$ we define a new differential operator on $\Omega_B^{\cdot}(\mathfrak{F})$ of degree 1 by

(2.6) $$d_\gamma \alpha = d_B \alpha - \gamma \wedge \alpha \ , \quad d_\gamma^2 = 0$$

and an operator of degree -1 by

(2.7) $$\bar{d}_\gamma^* \alpha = (-1)^{q(r-1)+1} \bar{*} d_\gamma \bar{*} \alpha \quad \text{for} \quad \alpha \in \Omega_B^r(\mathfrak{F}) \ .$$

We define further a pairing $\Psi_1 : \Omega_B^\cdot \otimes \Omega_B^{q-\cdot} \to \mathbb{R}$ by $\Psi_1(\alpha,\beta) = \int_M \alpha \wedge \beta \wedge \chi_{\mathfrak{F}}$. This determines a scalar product on $\Omega_B^\cdot$ by

(2.8) $$\langle \alpha,\beta \rangle_B = \Psi_1(\alpha, \bar{*}\beta) \quad \text{for} \quad \alpha,\beta \in \Omega_B^r \ .$$

In view of (2.4) this coincides with the canonical scalar product on $\Omega^\cdot(M)$

(2.9) $$\langle \alpha,\beta \rangle = \int_M \alpha \wedge *\beta \ ,$$

and by (2.1) also

(2.10) $$\Psi_1(\alpha,\beta) = \langle \alpha \wedge \beta, \nu \rangle_B \ .$$

The operators d_B and d_κ for a tense foliation are related by

(2.11) $$\Psi_1(d_B\alpha,\beta) + (-1)^{r-1}\Psi_1(\alpha, d_\kappa \beta) = 0 \quad \text{for} \quad \alpha \in \Omega_B^{r-1}, \ \beta \in \Omega_B^{q-r} \ .$$

2.12 PROPOSITION. <u>For</u> $\alpha \in \Omega_B^{r-1}$, $\beta \in \Omega_B^r$

$$\langle d_B\alpha, \beta \rangle_B = \langle \alpha, \bar{d}_\kappa^* \beta \rangle_B$$

$$\langle d_\kappa\alpha, \beta \rangle_B = \langle \alpha, \bar{d}_B^* \beta \rangle_B$$

<u>i.e.</u> $\bar{d}_\kappa^*$ <u>is the adjoint</u> d_B^* <u>of</u> d_B, <u>and</u> $\bar{d}_B^*$ <u>the adjoint</u> d_κ^* <u>of</u> d_κ.

Note that this implies in particular for the transversal volume ν of $\mathfrak{F}$

(2.13) $$d_B^* \nu = \bar{*}\kappa \ .$$

We can now develop the harmonic theory for the transversally elliptic Laplacian $\Delta_B = (d_B + d_B^*)^2 = d_B d_B^* + d_B^* d_B$ as in [1] [12]. One obtains finite-dimensional spaces of harmonic forms $\mathcal{H}^r(\mathfrak{F})$ whose inclusions into $(\Omega_B^r(\mathfrak{F}), d_B)$ induce isomorphisms $\mathcal{H}^r(\mathfrak{F}) \xrightarrow{\cong} H^r(\Omega_B^\cdot(\mathfrak{F}))$ for $r = 0,\dots,q$. The finite-dimensionality of

$H^r(\Omega_B^\cdot, d_\kappa)$ follows in the same way by considering the Laplace operator $\Delta_\kappa = (d_\kappa + d_\kappa^*)^2$ on $\Omega_B^\cdot(\mathcal{F})$.

The second main result of this paper is then as follows.

2.14 THEOREM. *Let $\mathcal{F}$ be an oriented Riemannian foliation on a closed, connected and oriented manifold* M. *Then the following statements are equivalent:*

(i) $\mathcal{F}$ is tense,

(ii) there exists a bundle-like metric g_M, for which $d_B^*\nu \in \Omega_B^{q-1}(\mathcal{F})$.

If $\mathcal{F}$ is tense, the cohomology spaces $H(\Omega_B^\cdot, d_B)$ and $H(\Omega_B^\cdot, d_\kappa)$ are finite-dimensional and Ψ_1 induces a non-degenerate pairing

(2.15) $$\Psi_2\colon H^r(\Omega_B^\cdot, d_B) \otimes H^{q-r}(\Omega_B^\cdot, d_\kappa) \to \mathbb{R}$$

for $r = 0,\dots,q$.

The non-degeneracy of the pairing Ψ_2 gives rise to isomorphisms

(2.16) $$D_{B*}\colon H^r(\Omega_B^\cdot, d_B) \xrightarrow{\cong} H^{q-r}(\Omega_B^\cdot, d_\kappa)^*$$

(2.17) $$D_{B*}\colon H^{q-r}(\Omega_B^\cdot, d_\kappa) \xrightarrow{\cong} H^r(\Omega_B^\cdot, d_B)^* .$$

Let $\sigma(\alpha) = \alpha \wedge \chi_{\mathcal{F}}$. Then σ induces a chain map $\sigma_1\colon (\Omega_B^\cdot, d_\kappa) \to (E_1^{\cdot,p}, d_1)$ by (2.5), and hence a map $\sigma_2\colon H(\Omega_B^\cdot, d_\kappa) \to E_2^{\cdot,p}$ with (continuous) dual $\sigma_2^*\colon E_2^{\cdot,p^*} \to H(\Omega_B^\cdot, d_\kappa)^*$. The map (2.16) above is then related to the duality map D_{1*} by the following commutative diagram

(2.18) $$\begin{array}{ccc} H^{q-r}(\Omega_B^\cdot, d_B) & \xrightarrow[\cong]{D_{B*}} & H^r(\Omega_B^\cdot, d_\kappa)^* \\ \cong\ \downarrow D_{1*} & \searrow D_2 & \uparrow \sigma_2^* \\ H_r(E_1^{\cdot,p^*}, \partial_1) & \xrightarrow{\text{can}} & E_2^{r,p^*} \end{array}$$

in which all maps are isomorphisms as a consequence of our results. In particular D_2 is an isomorphism.

The map (2.17) appears similarly in the $\pm$ commutative diagram

(2.19)
$$\begin{array}{ccc} E_2^{q-r,p} & \xrightarrow{D_{1*}} & H_r(\Omega_B^{\cdot *}, \partial_B) \\ \uparrow \sigma_2 & \searrow D_2 & \downarrow \text{can} \\ H^{q-r}(\Omega_B, d_\kappa) & \xrightarrow[\cong]{D_{B*}} & H^r(\Omega_B^{\cdot}, d_B)^* \end{array}$$

Here $(\Omega_B^{\cdot *}, \partial_B)$ denotes the complex of basic currents.

2.20 PROPOSITION. *All maps in (2.19) are isomorphisms. In particular, the duality map D_1 induces isomorphisms D_2: $E_2^{q-r,p} \cong E_2^{r,o*}$, i.e.*

(2.21)
$$D_2\colon H^{q-r}(\Omega_c(\mathrm{Tr}\mathfrak{F})) \overset{\cong}{\to} H^r(\Omega_B, d_B)^*, \quad r = 0,\dots,q .$$

3. We consider finally the case of taut Riemannian foliations. In the notation of §2, they are characterized by $\kappa = 0$ and $d_\kappa = d_B$.

3.1 THEOREM. *Let $(M,\mathfrak{F})$ be given as in Theorem 2.14. Then the following statements are equivalent:*

(i) *$\mathfrak{F}$ is taut;*

(ii) *there exists a bundle-like metric Riemannian metric g_M such that the transversal invariant volume $\nu \in \Omega_B^q(\mathfrak{F})$ satisfies $d_B^*\nu = 0$ (or equivalently $\Delta_B\nu = 0$);*

(iii) *$\dim H^q(\Omega_B^{\cdot}) < \infty$, and there exists a volume form $\omega_0 \in \Gamma(\Lambda^p L^*)$, such that the associated pairing $\Psi_{\omega_0}\colon \Omega_B^{\cdot} \otimes \Omega_B^{q-\cdot} \to \mathbb{R}$ induces a mapping $H^q(\Omega_B, d_B) \to H^o(\Omega_B, d_B)^* \cong \mathbb{R}$.*

If $\mathfrak{F}$ is taut, the basic cohomology spaces $H(\Omega_B^{\cdot}, d_B)$ are finite-dimensional and Ψ_1 induces a non-degenerate pairing $H^r(\Omega_B^{\cdot}) \otimes H^{q-r}(\Omega_B^{\cdot}) \to \mathbb{R}$ for $r = 0,\dots,q$, i.e. the basic cohomology algebra satisfies Poincaré Duality.

A complete proof of this theorem will appear elsewhere. The pairing Ψ_{ω_0} in (iii) is defined by $\Psi_{\omega_0}(\alpha,\beta) = \int_M \alpha \wedge \beta \wedge \omega$, where $\omega \in \Omega^p(M)$ is a form representing $\omega_0 \in E_0^{o,p}$. Ψ_{ω_0} is independent of the choice of ω. We also note that the mapping $H^q(\Omega_B^{\cdot}) \to H^o(\Omega_B^{\cdot})^* \cong \mathbb{R}$ is then necessarily surjective. Namely the cohomology class of the transversal volume $\nu \in \Omega_B^q$ is mapped to a non-zero number, i.e. the volume of M. As a consequence of the theorem, one has in fact $H^q(\Omega_B, d_B) \cong \mathbb{R}$. In a paper appearing in these proceedings, Y. Carrière asserts that the examples of (non-taut) Riemannian flows of [2] are actually tense. Thus they satisfy $H^q(\Omega_B, d_B) = 0$, but, by Theorem 2.14, $H^q(\Omega_B, d_\kappa) \cong H^o(\Omega_B, d_B)^* \cong \mathbb{R}$ and $H^{q-1}(\Omega_B^{\bullet}, d_\kappa) \cong H^1(\Omega_B, d_B) \neq 0$.

Let $\mathfrak{F}$ be a compact foliation (all leaves compact). In view of Rummler's results [14] the conditions in Theorem 3.1 are satisfied exactly in the locally stable case. The leaf space B of $\mathfrak{F}$ is then a Satake manifold with a canonical submersion f: $M \to B$ and the DeRham complex of B is identified with the basic complex $\Omega_B^{\bullet}(\mathfrak{F})$ via f^*.

3.2 COROLLARY. <u>If $\mathfrak{F}$ is a locally stable compact foliation on</u> M, <u>then the DeRham cohomology of the leaf space</u> B <u>satisfies Poincaré duality</u>.

The isomorphism D_2: $E_2^{o,p} \xrightarrow{\cong} H^q(\Omega_B^{\bullet})^*$ of (2.21) for r = q was established by Rummler [14] for this particular case.

Combination with the preceding results now also yields Poincaré duality for the homology of transversal holonomy invariant currents. Some geometric applications were discussed in [9] [10].

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