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THE FREE BOUNDARY OF A SEMILINEAR ELLIPTIC EQUATION

by D. PHILLIPS (Purdue University)

1. INTRODUCTION.

We consider the Dirichlet problem

$$(1.1) \quad \Delta u = \lambda f(u) \quad \text{in } \Omega \quad (\lambda > 0) \\ u = 1 \quad \text{in } \partial\Omega$$

where Ω is a bounded domain in \mathbb{R}^n with $\partial\Omega \in C^{2+\bar{\alpha}}$, $f(s) = 0$ if $s \leq 0$, $f(s) = s^p \hat{f}(s)$ if $s > 0$ ($0 < p < 1$) with $\hat{f} \in C^2([0, \infty))$ and $0 < m \leq \hat{f} \leq M < \infty$.

A solution, u_λ , satisfies $0 \leq u_\lambda < 1$ in Ω . Our objective is to study the geometric structure and location of the level set $\{u_\lambda = 0\} \equiv N_\lambda$.

The motivation for this work stems from the theory of reaction-diffusion in porous pellets. The region Ω plays the role of the pellet which is partially comprised of a catalyst. The function, u , represents the concentration of a gaseous impurity normalized to 1 on the boundary. And the parameter λ is a modeling coefficient related to the reaction rate.

The pellet is a filter in the sense that the impurity reacts with the catalyst and is removed as the total fluid passes through. As a result N_λ is the portion of the pellet not utilized (see [1]).

One of our main results is the following:

Theorem 1. *If Ω is a two-dimensional convex domain and f satisfies*

$$(1.2) \quad f'(s) + \frac{f(s)}{1-s} > 0 \quad \text{for } 0 < s \leq 1$$

then (1.1) has a unique solution for each $\lambda \geq 0$. Moreover there is a $\lambda_* > 0$ so that

$$\begin{aligned} N_\lambda &= \phi && \text{if } \lambda < \lambda_* \\ N_\lambda &\text{ is a single point if } && \lambda = \lambda_* \\ N_\lambda &\text{ is a convex domain if } && \lambda > \lambda_* . \end{aligned}$$

The process is assumed to be isothermal. This allows us to consider one equation as opposed to a coupled system in u and a temperature field t .

A model found in the literature [1] for isothermal processes has

$$f(s) = s^p \exp\left(\frac{-v}{\beta + 1 - \beta s}\right) \quad (v, \beta \text{ constants, } v > 0, \beta > 0).$$

This equation for f satisfies (1.2) for a certain range of the parameters v and β .

The research described in this paper has been done jointly with Avner Friedman [3].

2. EXISTENCE AND UNIQUENESS RESULTS.

From the structure of $f(s)$ and the boundary conditions one can show that classical solutions to (1.1) exist ($C^{2+\alpha}$, $\alpha = \min(\bar{\alpha}, p)$). This can be done by minimizing the functional

$$(2.1) \quad J_\lambda(v) = \int_\Omega \left(\frac{1}{2} |\nabla v|^2 + \lambda F(v) \right) dx$$

$$F(s) = \int_0^s f(t) dt, \quad \text{subject to the b.c. } v = 1.$$

With $\Delta u \geq 0$, from the maximum principle we get $u < 1$ in Ω . And since $f(s) = 0$ for $s \leq 0$, $0 \leq u$. As $f(s)$ is only Hölder continuous near $s = 0$ (not Lipschitz), N may be nontrivial for a particular solution.

The function $f(s)$ is not assumed to be increasing and as a result (1.1) has in general more than one solution. Nonetheless (1.2) is a sufficient condition for

uniqueness to the n -dimensional problem. This was shown in [2] and in [3] using a different argument.

3. COMPARISON THEOREMS.

Although a number of our results pertain to an arbitrary solution of (1.1), we are mainly interested in solutions that are elements of a family for which we have comparison theorems. Such families are minimums of (2.1), maximal, and minimal solutions of (1.1).

Definition. A solution \bar{u}_λ of (1.1) is a *maximal solution* if for any other solution u_λ , $u_\lambda(x) \leq \bar{u}_\lambda(x)$ in Ω .

Theorem 2. Let $\lambda_1 < \lambda_2$ and $\bar{u}_{\lambda_1}(\underline{u}_{\lambda_1})$ be respective maximal (minimal) solutions.

Then

$$\bar{u}_{\lambda_2} < \bar{u}_{\lambda_1} \quad \text{on} \quad \{\bar{u}_{\lambda_2} > 0\} \quad ,$$

$$\underline{u}_{\lambda_2} < \underline{u}_{\lambda_1} \quad \text{on} \quad \{\underline{u}_{\lambda_2} > 0\} \quad ,$$

and $\{\bar{u}_{\lambda_1} = 0\} \cap \{\underline{u}_{\lambda_1} = 0\}$ is contained in the interior of

$$\{\bar{u}_{\lambda_2} = 0\} \cap \{\underline{u}_{\lambda_2} = 0\} \quad .$$

An analogous comparison result is true for minimizers of (2.1) as well.

The advantage of considering a branch of solutions with the monotonicity relations above is that if $\lambda_1 < \lambda < \lambda_2$ then

$$(3.1) \quad N_{\lambda_2} \subset\subset N_\lambda \subset\subset N_{\lambda_2} \quad .$$

Thus we are able to use N_{λ_2} and N_{λ_1} as barriers to obtain properties of ∂N_λ .

4. ASYMPTOTIC ESTIMATES.

For large values of the parameter λ the reaction occurs near $\partial\Omega$. We have the following theorem.

Theorem 3. *There are positive constants λ_0 , γ_0 , and c so that if $\lambda \geq \lambda_0$ and u_λ is any solution to (1.1) then*

$$(4.1) \quad \left\{ x \in \Omega \mid \text{dist}(x, \partial\Omega) > \frac{\gamma_0}{\sqrt{\lambda}} + \frac{c}{\lambda} \right\} \subset N_\lambda \subset \left\{ x \in \Omega \mid \text{dist}(x, \partial\Omega) > \frac{\gamma_0}{\sqrt{\lambda}} - \frac{c}{\lambda} \right\} .$$

Moreover if $x = h(t)$ is a local parameterization of $\partial\Omega$ and $\nu(t)$ is the normal pointing into Ω then there is a function $K(t)$ (depending on u_λ) so that ∂N_λ can be represented in the form

$$(4.2) \quad x = h(t) + K(t)\nu(t) ,$$

with

$$|K(t) - \frac{\gamma_0}{\sqrt{\lambda}}| < \frac{c}{\lambda} ,$$

$$|K(t)|_{C^{1+\delta}} < c .$$

5. CONVEXITY PROPERTIES.

The function u_λ grows away from the level set N_λ at a prescribed rate.

In particular if we set

$$g(u_\lambda(x)) = \int_0^{u_\lambda(x)} \frac{ds}{\sqrt{2F(s)}} , \quad F(s) = \lambda \int_0^s f(t) dt$$

then $g(u_\lambda(x))$ is Lipschitz in Ω and in a weak sense

$$|\nabla(g(u_\lambda))| = 1 \quad \text{on} \quad \partial N_\lambda .$$

Moreover if Ω is convex one can show that (see [4])

$$(5.1) \quad |\nabla g(u_\lambda)| \leq 1 \quad \text{in} \quad \Omega .$$

And with this it can be shown that

$$\Delta g(u_\lambda) \geq 0 \text{ in } \Omega .$$

Now let K be a smooth subdomain of Ω with $H^{n-1}(\partial K \cap \partial N_\lambda) = 0$.

We can apply Green's Theorem to $K \cap \{u_\lambda > 0\}$. We get

$$0 \leq \int_{K \cap \{u_\lambda > 0\}} \Delta g(u_\lambda) = - \int_{\partial N_\lambda \cap K} 1 dH^{n-1} + \int_{K \cap \{u_\lambda > 0\}} \nabla g(u_\lambda) \cdot \nu dH^{n-1} .$$

Using (5.1) we find that

$$H^{n-1}(\partial N_\lambda \cap K) \leq H^{n-1}(\partial K \cap \{u_\lambda > 0\}) .$$

That is ∂N_λ minimizes surface area subject to variations on one side. Such a boundary can be considered a surface of generalized positive mean curvature. When Ω is two-dimensional this becomes a statement of convexity.

We have the following theorem:

Theorem 4. *If Ω is convex, $\Omega \subset \mathbb{R}^2$, and u_λ any solution to (1.1) then each component of N_λ with nonempty interior is a convex domain.*

We can now describe the basic idea behind theorem 1. From section 1 (assuming (1.2)) the map, $\lambda \rightarrow u_\lambda(x)$, is well defined. Using (3.1) we see the sets, N_λ , are nested. One then must show that the map, $\lambda \rightarrow N_\lambda$, deforms continuously. Using this and the fact that for $\lambda \geq \lambda_0$, N_λ is a convex domain (theorems 3 and 4) one can show that N_λ is convex as long as $\text{int}(N_\lambda) \neq \emptyset$. Setting

$$\lambda_* = \inf\{\lambda \mid \text{int}(N_\lambda) \neq \emptyset \text{ for } \tilde{\lambda} > \lambda\}$$

one then proves that N_{λ_*} consists of one point.

Analogous theorems are shown for the Robins condition

$$\frac{\partial u}{\partial \nu} + \mu(u-1) = 0 \text{ on } \partial \Omega \quad (\mu > 0)$$

instead of $u = 1$ on $\partial \Omega$.

R E F E R E N C E S

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