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P-ADIC THETA SERIES WITH INTEGRAL COEFFICIENTS

Valentino CRISTANTE

0. INTRODUCTION.

Let R be the ring of the integers of a local field K, let k be its residue field, and assume k be perfect of characteristic $p \neq 0$. If A is an abelian variety over K with good reduction mod p, we will denote by A_0 its reduced variety, by e and e_0 the identity of A and A_0 respectively, by θ_0 the local ring of A at e_0 and by S its completion. So, if A has dimension n, $S = R[\![t_1,\dots,t_n]\!]$, where $(t_1,t_2,\dots,t_n)$ is a set of uniformizing parameters of A at e_0.

Now, if X is a divisor of A, rational over K, and if we denote by θ_X a theta of in $S_K = K[\![t_1,\dots,t_n]\!]$ (we are assuming that the polar part of X doesn't go through e), a natural question arises : is it possible to choose θ_X in S ? The answer, in general, is no. In fact, if $\theta_X \in S$, the image of θ_X in $S_0 = S\hat{\otimes}k$ would be a theta of the image X_0 of X in A_0. But, as shown in [7], if A_0 is not ordinary, or if $X_0 \not\equiv 0$, the thetas of X_0 live in a ring quite bigger than S_0. So, if we are looking for a positive answer to our former question, we must assume A_0 be ordinary. In fact, with this assumption, denoted by $\underset{\cdot}{D} = (D_1,\dots,D_n)$ a basis of the R-module of the invariant derivation of A, and by $(\eta_{1,X},\dots,\eta_{n,X})$ the n-uple of integrals of the second kind corresponding to the couple (X,D) (see section 3. for a precise definition), we will show that the system of differential equations

0.1. $$D_i\theta - \theta\eta_{i,X} = 0, \quad i = 1,2,\dots,n ,$$

has solutions in S. However, we will not use a direct approach to 0.1. In fact, if we denote by p_i, $i = 1,2,3$, the projections from A^3

to A, and if p_i^* are the corresponding applications from S to $S\hat{\otimes}S\hat{\otimes}S$, the system 0.1 is equivalent to the functional equation

$$0.2 \qquad \frac{((p_1+p_2+p_3)^*\theta)(p_1^*\theta)(p_2^*\theta)(p_3^*\theta)}{((p_1+p_2)^*\theta)((p_1+p_3)^*\theta)((p_2+p_3)^*\theta)} = F ,$$

where F is an equation of the divisor

$$Y = (p_1+p_2+p_3)^{-1}X + p_1^{-1}X + p_2^{-1}X + p_3^{-1}X - (p_1+p_2)^{-1}X - (p_1+p_3)^{-1}X - (p_2+p_3)^{-1}X$$

of A^3. Now, in wiew of the cohomological properties of F, the equation 0.2 is not only much more easier to solve than 0.1, but also allows to understand that 0.1 has solutions even in a more general situation.

After the construction of the solutions of 0.2, we will show how these are related to the canonical decomposition of $H^1_{DR}(A)$ (see [9] and [4]), and finally we'll give some explicit computation for the elliptic curves.

An analogous construction has been done by P. Norman using different techniques ; here I'd like to thank him for the useful conversations we had on these topics.

1. SPLITTING OF BI-MULTIPLICATIVE CO-CYCLES.

Let R be a commutative ring with identity, $t = (t_1,\ldots,t_n)$ a set of indeterminates over R, and let $S = R[\![t]\!]$ be a R-bi-algebra. For short, the image of t in $S\hat{\otimes}S$ given by the coproduct will be denoted by $t_1 \dot{+} t_2$.

1.1. DEFINITION. <u>An element</u> $H = H(t_1,t_2,t_3) \in S\hat{\otimes}S\hat{\otimes}S$ <u>is called a symmetric</u>, <u>bi-multiplicative (resp. bi-additive) co-cycle of</u> S <u>if</u>

i) $H(0,t_2,t_3) = 1$ (<u>resp</u>. $H(0,t_2,t_3) = 0$) ;

ii) $H(t_1,t_2,t_3) = H(t_{\sigma_1},t_{\sigma_2},t_{\sigma_3})$, <u>for each permutation</u> $\sigma \in \mathfrak{S}_3$;

iii) $H(t_1\dot{+}t_2,t_3,t_4)H(t_1,t_2,t_4) = H(t_1,t_2\dot{+}t_3,t_4)H(t_2,t_3,t_4)$ (resp. $H(t_1\dot{+}t_2,t_3,t_4)+H(t_1,t_2,t_4) = H(t_1,t_2\dot{+}t_3,t_4)+H(t_2,t_3,t_4)$) .

Moreover, if there exists an element $h \in S$ such that

1.2 $$\frac{h(t_1 \dot{+} t_2 \dot{+} t_3) h(t_1) h(t_2) h(t_3)}{h(t_1 \dot{+} t_2)\, h(t_1 \dot{+} t_3)\, h(t_2 \dot{+} t_3)} = H$$

(resp. $h(t_1 \dot{+} t_2 \dot{+} t_3) + h(t_1) + \ldots - h(t_2 \dot{+} t_3) = H$) the co-cycle H is called a co-boundary of S. Later on the left hand side of 1.2 will be denoted by $\mathfrak{D}_\mu^2 h$ (resp. $\mathfrak{D}_\alpha^2 h$).

For instance, if R and t have the same meaning as in the introduction, and if X is a divisor of A such that its reduced mod p doesn't go through e_0, one can choose for F (cfr. 02) a symmetric, bi-mult. co-cycle of S (see [7] and [5]). So our main goal in this section will be the proof of the following.

1.3. THEOREM. Let R be the ring of the Witt vectors with components in the algebraically closed field k of characteristic $p \neq 0$, and $S = R[[t]]$ be a multiplicative bi-algebra. Then each symmetric, bi-multiplicative co-cycle of S is a co-boundary of S.

The assumption about the algebraic closure of k seems necessary if we like results which can be applied to each divisor. Later on we will show that symmetric divisor possess theta series with integral coefficients even if k is only a perfect field.

In order to prove 1.3 we need some results which are given in theorem A.4 and section 2 of [5]. With our actual language they can be formulated in the following way :

1.4. THEOREM. If R is a $\mathbb{Q}$-algebra, each symmetric, bi-multiplicative (resp. bi-additive) co-cycle H of S is a co-boundary of S.

1.5. THEOREM. If R is an algebraically closed field of characteristic $p \neq 0$, and if S is a multiplicative bi-algebra, then each bi-multiplicative co-cycle H of S is a co-boundary of S.

If R is algebraically closed field of characteristic 0, and if A is an abelian variety over R, result 1.4, under the assumption that H be a rational function on A^3, was first proved in [2].

Since the symmetric, bi-additive co-cycles are more easy to use, we start with the following result :

1.6. PROPOSITION. Let S be as in 1.3 ; then each symmetric bi-additive co-cycle of $S_0 = S \hat{\otimes} k$ is a co-boundary of S_0.

In fact, as the following arguments will show, from 1.6 we deduce the following

1.7. PROPOSITION. Let S be as in 1.3 ; then each symmetric, bi-additive co-cycle of S is a co-boundary of S.

An finally, from 1.7 we can get 1.3.

Proof of $(1.6 \Longrightarrow 1.7)$. Let $H \in S \hat{\otimes} S \hat{\otimes} S$ be a symmetric, bi-additive co-cycle of S. Denote by H_0 the image of H in $S_0 \hat{\otimes} S_0 \hat{\otimes} S_0$; now as H_0 is a co-boundary of S_0, there exists an element $h_0 \in S_0$, such that $\mathcal{D}^2_\alpha h_0 = H_0$. If h is an element of S whose image in S_0 is h_0, and if $H_1 = \mathcal{D}^2_\alpha h$, we have $H \equiv H_1 \bmod p$; and then, since $\frac{1}{p}(H - H_1)$ is a symmetric, bi-additive co-cycle of S, our procedure may be repeated. As a consequence,

$$H = H_1 + pH_2 + p^2 H_3 + \dots ,$$

is a co-boundary of S, Q.E.D. .

Proof of $(1.7 \Longrightarrow 1.3)$. Let $F \in S \hat{\otimes} S \hat{\otimes} S$ be a symmetric, bi-multiplicative co-cycle of S, and denote by F_0 its canonical image in $S_0 \hat{\otimes} S_0 \hat{\otimes} S_0$. By 1.4 we know that F_0 is a co-boundary of S_0, so there exists θ_0 in S_0, s.t. $\mathcal{D}^2_\mu \theta_0 = F_0$. Now, denote by S^+ the kernel of the coidentity of S, and let θ' be an element of S, $\theta' \equiv 1 \bmod S^+$, whose image in S_0 is θ_0. If we denote by F_1 the co-boundary $\mathcal{D}^2_\mu \theta'$ of S, we have

$$F/F_1 \equiv \bmod p ,$$

and therefore

$$\log F = \log F_1 + pH ,$$

where H is a symmetric, bi-additive co-cycle of S. Now, by 1.7 there exists an element $h \in S$, s.t. $\mathcal{D}_\alpha^2 h = H$; and it is clear that $\theta = \theta'$ exp ph is an element of S which satisfies the equation $\mathcal{D}_\mu^2 \theta = F$,

Q.E.D. .

Now will give a lemma which will be used the proof of 1.6.

1.8. LEMMA. Let B be an integral domain of characteristic $p \neq 0$, $B[\![t_1,\dots,t_n]\!]$ a multiplicative bi-algebra ; then each symmetric additive co-cycle of $B[\![t]\!]$ is a co-boundary.

Proof. This result is probably well known ; nevertheless we'll give here a direct proof. Let g be such a co-cycle. Using the co-cycle property $g(t_1 \dot{+} t_2, t_3) + g(t_1,t_2) = g(t_1, t_2 \dot{+} t_3) + g(t_2,t_3)$, it is immediate to see that $(p\iota \hat{\otimes} p\iota)g$ is a co-boundary ($p\iota$ = multiplication by p), i.e. there exists an element $\tau \in B[\![t]\!]$ such that

$$\tau(t_1 \dot{+} t_2) - \tau(t_1) - \tau(t_2) = g(p\iota\, t_1 , p\iota\, t_2) .$$

From the last formula we deduce that

1.9. $$D\tau - \varepsilon(D\tau) = 0$$

for each invariant derivation D of $B[\![t]\!]$, where ε is the co-identity. But, as $B[\![t]\!]$ is multiplicative, 1.9 implies that $D\tau = 0$, and so $\tau = p\iota\sigma$, for $\sigma \in B[\![t]\!]$. In conclusion $\sigma(t_1 \dot{+} t_2) - \sigma(t_1) - \sigma(t_2) = g(t_1,t_2)$,

Q.E.D. .

Proof of 1.6. Let $H \in S_0 \hat{\otimes} S_0 \hat{\otimes} S_0$ be a symmetric bi-additive co-cycle of S_0 ; then by 1.8 there exists a (unique) element φ in $S_0 \hat{\otimes} S_0$, such that

1.10. $$\varphi(t_1,t_2+t_3) - \varphi(t_1,t_2) - \varphi(t_1,t_3) = H .$$

Now, if $\mu \in S_0 \hat{\otimes} S_0 \hat{\otimes} S_0$ is the element defined by

$$\mu(t_1,t_2,t_3) = \varphi(t_1,t_2+t_3) + \varphi(t_2,t_3) - \varphi(t_1+t_2,t_3) - \varphi(t_1,t_2) ,$$

as a consequence of the co-cycle properties of H (see def. 1.1) we have

$$\mu(t_1,t_2,t_3\dot{+}t_4) = \mu(t_1,t_2,t_3) + \mu(t_1,t_2,t_4) .$$

But, since $S_0 \hat{\otimes} k[\![t_1,t_2]\!]$ is a multiplicative $k[\![t_1,t_2]\!]$-bi-algebra, from the last formula we deduce that $\mu = 0$. As a consequence, recalling also point ii) of 1.1, we conclude that φ is a symmetric, additive co-cycle of S_0 ; so using 1.8 again we have

$$\varphi(t_1,t_2) = \tau(t_1\dot{+}t_2) - \tau(t_1) - \tau(t_2) ,$$

and finally $\mathcal{D}^2_\alpha\tau = H$, Q.E.D. .

1.11. <u>Remark</u>. Let S as in 1.3, and $u = (u_1,\ldots,u_n)$ be a basis of the integrals of the first kind of S, i.e. a basis of the R-module of the additive elements u of $S_K = S\hat{\otimes}K$ such that Du is in S for each invariant derivation D of S . If F is a symmetric, bi-multiplicative co-cycle of S, and if $\theta \in S$ is a solution of the equation

1.12. $$\mathcal{D}^2_\mu\theta = F ,$$

each solution of 1.12 in S_K is of the form $\theta\exp(L(u)+Q(u))$, where L(u) and Q(u) are linear and respectively quadratic forms of the u_i's. Now, since in S there is no element of the form $\exp Q(u)$ (see [MA]), we conclude that all solutions of 1.12 in S are of the form $\theta\exp L(u)$.

Now we'll show that, if 1.12 admits a even solution, i.e. invariant with respect to the inversion of S_K , it is sufficient to assume k perfect ; more precisely we have the following.

1.13. THEOREM. <u>Let</u> k <u>be a perfect field of characteristic</u> $p \neq 0,2$, <u>let</u> R <u>be the ring of the Witt vectors with components in</u> k , <u>and</u> $S = R[\![t_1,\ldots,t_n]\!]$ <u>be a bi-algebra of multiplicative type</u>. <u>Then if</u> F <u>is a symmetric</u>, <u>bi-multiplicative co-cycle of</u> S, <u>such that</u> $F(t_1,t_2,t_3) = F(\dot{-}t_1,\dot{-}t_2,\dot{-}t_3)$ ($\dot{-}t$ <u>is the image of</u> t <u>given by the inversion of</u> S), <u>the equation</u> 1.12 <u>has a unique solution</u> $\tilde{\theta} \in S$ <u>which satisfies the relation</u>

1.14. $$\tilde{\theta}(t) = \tilde{\theta}(-t) .$$

<u>Moreover if</u> $\theta \in S_K$ $(K = \mathrm{Frac}R)$ <u>is a solution of</u> 1.12 <u>which satisfies</u> 1.14 <u>we have</u>

1.15. $$\tilde{\theta} = \lim_{n\to\infty} \theta / (p\iota)^{-n} \theta^{p^{2n}} \;;$$

finally, the direct relation between $\tilde{\theta}$ and F is the following :

1.16. $$\tilde{\theta} = \lim_{n\to\infty} (p\iota)^{-n} (1 / \prod_{j=1}^{p^n-1} F(t, \dot{-}t, j\,\iota\, t)^{p^n - j})$$

where the limits 1.15 and 1.16 are considered in the topology of $\varinjlim (S \xrightarrow{p\iota} S \xrightarrow{p\iota} \ldots)$ given by the system $I_{m,n} = t^m S + p^n S$ of ideals of S .

Proof. Let $\bar{R}$ be the ring $W(\bar{k})$ of the Witt vectors with components in the algebraic closure $\bar{k}$ of k . By 1.3 we known that there exists a solution $\theta(t)$ of 1.12 in $\bar{R}[\![t]\!]$; but in wiew of the properties of F , also $\theta(-t)$ is a solution of 1.12, and so $\theta(t)^{1/2}\, \theta(-t)^{1/2}$ is a solution of 1.12 which satisfies 1.14. Now, each element of $\bar{R}[\![t]\!]$ which satisfies 1.14 is a even power series of u (see 1.11) ; therefore it can't be multiplied by an exponential of a linear form L(u) without loosing the property 1.14. As a consequence 1.12 has a unique solution $\tilde{\theta}$ in $\bar{R}[\![t]\!]$ such that $\tilde{\theta}(t) = \tilde{\theta}(-t)$. Now we'll show that $\tilde{\theta}$ is in $R[\![t]\!]$. In fact by 1.1, if θ is a solution of 1.12 in S_K which satisfies 1.14, we have

1.17. $$\theta(t)^{p^{2n}} / (p\iota)^n \theta(t) = \prod_{j=1}^{p^n-1} F(t, \dot{-}t, j\,\iota\, t)^{p^n - j} \;,$$

for each $n \geqslant 1$. So the remaining part of the theorem will be proved if we verify that

1.18. $$\tilde{\theta}(t) = \lim_{n\to\infty} \theta(t) / (p\iota)^{-n} \theta(t)^{p^{2n}} \;.$$

Now the relation between θ and $\tilde{\theta}$ must be $\theta = \tilde{\theta} \exp Q(u)$, where Q(u) is a quadratic form.

But $\lim_{n\to\infty} (p\iota)^{-n}\, \tilde{\theta}^{p^{2n}} = 1$, and $\lim_{n\to\infty} (p\iota)^{-n} (\exp Q(u))^{p^{2n}} = \exp W(u)$, Q.E.D. .

1.19. Remark. With the notation of 1.17 also the limit

1.20. $$\lim_{n\to\infty} \prod_{j=1}^{p^n-1} F(t, \dot{-}t, j\,\iota\, t)^{(p^n-1)/p^{2n}}$$

exists in S_K : it gives the (unique) solution θ_O of 1.12 in S_K which satisfies 1.14 and the initial condition

$$\varepsilon(D'D \log\theta) = 0 ,$$

for each couple (D,D') of invariant derivations of S_K. In fact, in order to show that 1.20 exists, we remark that by 1.14, $\theta = 1+Q(u)+\ldots$, where Q is a quadratic form ; as a consequence

$$\lim_{n\to\infty} \frac{1}{p^{2n}} \log(p\iota)^n\theta = Q(u) ,$$

and finally

$$\theta_O = \theta / \exp Q(u) .$$

This is the procedure used in [10] ; but in general θ_O isn't in S .

1.15. Remark. Let $\mathcal{S}$ be the completion of the perfect closure of $S_O = S \hat{\otimes} k$ and Biv($\mathcal{S}$) the completion of the ring of Witt bivectors with components in $\mathcal{S}$. Using the methods described in [12] (see in particular th. 8.1) on can define a canonical embedding j of a subring of S_K, containing all solutions of 1.12, in Biv($\mathcal{S}$) . In such situation $\tilde{\theta}$ is characterized by the property $j\tilde{\theta} \in W(\mathcal{S})$. Since 1.12 has solutions with this pecularity also when S_O is a affine algebra of a general B-T group, it would be interesting to describe the functions (series) which correspond to them.

2. THETA SERIES.

In this section we'll translate the previous results in a geometric language.

2.1. THEOREM. *If* k, K *and* R *have the same meaning as in* 1.13, *if* A *is an abelian variety over* K *with good reduction* mod p, *and if the reduced variety* A_O *is ordinary ; then each divisor* X *of* A, *rational over* K, *has a theta series in* $\overline{R}((t))$, *where* $\overline{R}$ *is the ring of the Witt vectors of the algebraic closure* $\overline{k}$ *of* k, *and* $t = (t_1,\ldots,t_n)$ *is a set of uniformizing parameters of* A *at the identity point* e_O *of* A_O. *Moreover if* X *is totally symmetric, i.e. if there exists* X' , *s.t.*

$X = X' + (-\iota)^{-1}X'$, X possesses a theta series in $R((t))$, $\tilde{\theta}_X$ which satisfies the relation $\tilde{\theta}_X(t) = \tilde{\theta}_X(-t)$. The series $\tilde{\theta}$ is determined up to a constant.

Proof. We'll assume that the support of the reduced divisor X_O doesn't intersect e_O ; in fact the result in general is an immediate consequence of this particular situation (see remark 2.2). If Y has the same meaning as in the introduction, as remarked in section 1, we can choose as an equation of Y a symmetric, bi-multiplicative co-cycle F of S. At this point it is clear that the first part of the theorem is a consequence of 1.3. Now if X is totally symmetric, $F(\dot{-}t_1,\dot{-}t_2,\dot{-}t_3)$ is, as F, an equation of Y, which satisfies i) of 0.1, and so $F(\dot{-}t_1,\dot{-}t_2,\dot{-}t_3) = F(t_1,t_2,t_3)$. As a consequence, the second part our theorem follows immediately by the first part of 1.13, Q.E.D. .

2.2. Remark. The assumption $\operatorname{supp} X_O \cap E_O = \emptyset$ used in the proof of 2.1 is not necessary. In fact each divisor X' of A rational over K can be written as $X' = X'' + (f)$, where X" satisfies the assumption and f is in $R((t))$. In this case we define $\theta_{X'} = \theta_{X''} f$. In particular, if $X = X' + (-\iota)^{-1}X$, $\tilde{\theta}_X(t) = \tilde{\theta}_{X''+(-\iota)^{-1}X''}(t)\, f(t) f(\dot{-}t)$ is determined up to a multiplicative constant. Finally, if the polar part X"' of X satisfies the previous assumption, i.e. $\operatorname{sup} X_O''' \cap e_O = \emptyset$, $\overline{R}((t))$ and $R((t))$ can be replaced by $\overline{R}[\![t]\!]$ and $R[\![t]\!]$ respectively.

3. THE CANONICAL SPLITTING OF $H^1_{DR}(A)$ ASSOCIATED TO $\tilde{\theta}$.

With the notations and assumptions of 2., we recall that to A and S are associated the free R-modules $H^1_{DR}(A)$ and $H^1_{DR}(S)$ of rank 2n and n respectively. For our purposes, the more convenient description of them is the following (see [3] and [4]) :

we start with two sub-R-modules of $S_K = K[\![t]\!]$: the first is

$$I_2(A) = \{ f \in S_K \mid df \text{ is a diff. of } S, \text{ and } f(t_1 \dot{+} t_2) - f(t_1) - f(t_2) \in K(A^2) \} ;$$

the second is

$I_2(S) = \{ f \in S_K \mid df$ is a diff. of S, and $f(t_1 \dot{+} t_2) - f(t_1) - f(t_2) \in S \hat{\otimes} S \}$.

Clearly $I_2(A)$ contains the local ring $\mathcal{O}_O$ of A at e_O, and $I_2(S)$ contains S. With these notations, we have :

$$H^1_{DR}(A) = I_2(A)/\mathcal{O}_O \quad \text{and} \quad H^1_{DR}(S) = I_2(S)/S .$$

Now let I_1 be the sub-R-module of $I_2(A)$ (and of $I_2(S)$) given by the additive elements :

$$I_1 = \{ f \in I_2(A) \mid f(t_1 \dot{+} t_2) - f(t_1) - f(t_2) = 0 \} .$$

It is well known that I_1 is a free R-module of rank n, and that $I_1 \cap S = \{0\}$. Therefore, by a comparison of the ranks, we conclude that the canonical map of $I_2(A)$ in $H^1_{DR}(S)$ is surjective, that $I_2(A) = I_1 \oplus (I_2(A) \cap S)$, and finally that

3.1. $$H^1_{DR}(A) = I_1 \oplus (I_2(A) \cap S) / \mathcal{O}_O :$$

this is the canonical splitting of $H^1_{DR}(A)$. Now we'll show how the sub-R-module $N = (I_2(A) \cap S)/\mathcal{O}_O$ of $H^1_{DR}(A)$ is related to the theta series.

3.2. THEOREM. Let A be an abelian variety as in 2.1 ; let $X > 0$ be a totally symmetric, ample divisor of A rational over K, and and $\tilde{\theta}$ (one of) its theta series in S. If Lie(S) denotes the R-module (dual of I_1) of the invariant derivations of S, then the image of the map $\lambda : D \longrightarrow D\log\tilde{\theta}$ of Lie(S) in S is contained in $I_2(A)$. Moreover, if $N_{\tilde{\theta}}$ denotes the image of $\lambda(\mathrm{Lie}(S))$ in $H^1_{DR}(A)$, we have

$$N = \{ f \mid f \in H^1_{DR}(A),\ p^n f \in N_{\tilde{\theta}} \text{ , for some } n \in \mathbb{N} \} .$$

Proof. As in the proof of 2.1 we'll assume, for simplicity, that $\mathrm{Supp}\, X_O \cap e_O = \emptyset$. So we can assume $\tilde{\theta} \equiv 1 \bmod S^+$, and therefore

3.3. $$F = \mathcal{D}^2_\mu \tilde{\theta}$$

is a symmetric, bi-multiplicative co-cycle of S which is in $K(A^3)$. Now, if we transform both therms of 3.3 by the operator $(\iota \hat{\otimes} \iota \hat{\otimes} \varepsilon D)\log$, and successively by $(\iota \hat{\otimes} \varepsilon D')$, where $D, D' \in Lie(S)$, we have

$$3.4.\quad (\iota \hat{\otimes} \iota \hat{\otimes} \varepsilon D) \log F(t_1,t_2,t_3) = (D\log\tilde{\theta})(t_1 \dot{+} t_2) - (D\log\tilde{\theta})(t_1) - (D\log\tilde{\theta})(t_2) + \varepsilon(D\log\tilde{\theta})\text{ , and}$$

$$3.5.\quad (\iota \hat{\otimes} \varepsilon D' \hat{\otimes} \varepsilon D) \log F(t_1,t_2,t_3) = (DD'\log\tilde{\theta})(t_1) - \varepsilon(DD'\log\tilde{\theta})\ ,$$

which say precisely that $D\log\tilde{\theta}$ is in $I_2(A)$. Since $\tilde{\theta} \in S$, $\lambda(Lie(S))$ is contained in S, and so $N_{\tilde{\theta}}$ is contained in N. Finally, since X is ample $\lambda(Lie(S))$ is a free R-module which doesn't intersect θ_0 (see [1] and [6]), so by comparing the ranks we conclude that $N_{\tilde{\theta}}$ is isogenous to N, Q.E.D. .

3.6. <u>Remark</u>. If Δ is the determinant of the map $Lie(S) \longrightarrow N$, $\|1/\Delta\|_p$ is the separable degree of the polarization associated to X_0 (cfr. [MA]). So, in particular, if A is principally polarized, one can choose X in such a way that $N = N_{\tilde{\theta}}$.

3.7. <u>Remark</u>. Let G_l and $G_{\acute{e}t}$ be the local an the étale component of the Barsotti-Tate group G of the reduced abelian variety A_0. By results on the crystalline cohomology (see [3] and [9_a]), $H^1_{DR}(A)$ is canonically isomorphic to the Dieudonné module D(G) and $H^1_{DR}(S)$ is canonically isomorphic to the Dieudonné module $D(G_l)$. Moreover the canonical map from $H^1_{DR}(A)$ onto $H^1_{DR}(S)$ corresponds to the projection $D(G) = D(G_l) \oplus D(G_{\acute{e}t}) \longrightarrow D(G_l)$; therefore N_θ as Dieudonné module, is isogenous to $D(G_{\acute{e}t})$. As a consequence, if V denotes the Verschiebung of $H^1_{DR}(A)$ and $\overline{D\log\theta}$ the image of $D\log\theta$ in $H^1_{DR}(A)$, we have $\lim_{i\to\infty} V^i(\overline{D\log\theta}) = 0$, for each $D \in Lie(S)$.

4. AN EXAMPLE.

Let $\mathbb{F}_p$ be the Galois field with p elements, $p \neq 2$, and let λ_0 be an intederminate over $\mathbb{F}_p$. We shall denote by k the perfect field $\mathbb{F}_p(\lambda_0, \lambda_0^{1/p}, \lambda_0^{1/p^2}, \dots)$, by $R = W(k)$ the ring of the Witt vectors with components in k, and by λ an element of R whose image in k is λ_0. Now we consider the cubic E_λ over $K = \mathrm{Frac}R$, whose affine equation is

i) $$y^2 = (1-x^2)(1-\lambda x) .$$

If we choose as identity the point e of coordinates x=0, y=1, (E_λ, e) is an abelian variety which satisfies the request of 3.2. Moreover 2e is a totally symmetric divisor which gives a principal polarization, and so the image of $D\log\tilde{\theta}_{2e}$ in $H^1_{DR}(E_\lambda)$ spans N (see th. 3.2). This, in wiew of 3.7, is equivalent to saying that the image $\overline{D\log\tilde{\theta}_{2e}}$ is an eigenvector of the Frobenius of $H^1_{DR}(E_\lambda)$ corresponding to a unit eigenvalue. So, as remarked also by Norman (see [11]), $\overline{D\log\tilde{\theta}_{2e}}$ spans the Dwork's sub-crystal of $H^1_{DR}(E_\lambda)$ (see [8] and [9]). The aim of this example is to give an explicit computation for $\overline{D\log\tilde{\theta}_{2e}}$.

Since x is a uniformizing parameter of E_λ at e_0, a basis of $H^1_{DR}(E_\lambda)$ is given by the canonical images of two series u and v of the following type :

$u = \sum_{i=1}^{\infty} (c_i/i)x^i$ and $v = \sum_{i=1}^{\infty} (b_i/i)x^i$, where c_i and b_i are in R.

In particular we can choose u and v in such a way that $du = dy/y$ and $dv = xdx/y$; with this choice $b_i = c_{i-1}$, if $i > 1$, and $b_1 = 0$.

Now let $\tilde{\theta}(x) \in R[\![x]\!]$ be a theta series of 2e (see th. 3.2) and let D be the derivation of S defined by $Du = 1$. By 3.2

ii) $$D\tilde{\theta} / \tilde{\theta} \equiv v + au , \qquad \mathrm{mod}\ R((x)) ,$$

where a is in R. Since $D\tilde{\theta}/\tilde{\theta} - 2Dx/x \in R[\![x]\!]$, we deduce that

iii) $$v + au \equiv 0 , \qquad \mathrm{mod}\ R[\![x]\!] .$$

The relation iii), as shown in [4], allows to compute a :

$$a = -\lim_{i\to\infty} c_{p^i-1} / c_{p^i} .$$

To finish, let us show how the image of $v + au$ may be recovered from each theta, θ, of 2e which satisfies the property $\theta(x) = \theta(-x)$. As we have shown in 1.13, there exists a constant $b \in K$, such that

$$\text{iv)} \qquad D\theta/\theta + bu \equiv \tilde{D}\tilde{\theta}/\tilde{\theta}\,, \qquad \text{mod } R((x))\,,$$

and so

$$\text{v)} \qquad D\theta/\theta + bu \equiv 0\,, \qquad \text{mod } R((x))\,.$$

The relation v) determines b. In fact if $z = \exp u - 1$, and if $\log(\theta/x^2) = \sum_{i=1}^{\infty} a_i z^i$, v) is equivalent to

$$\text{vi)} \qquad (1+z) \sum_{i=1}^{\infty} i a_i z^{i-1} + b \sum_{i=1}^{\infty} (-1)^{i-1} z^i/i \equiv 0\,, \qquad \text{mod } R[\![x]\!]\,;$$

and therefore

$$b = -\lim_{i\to\infty} p^i((p^i+1)a_{p^i+1} + p^i a_{p^i})\,.$$

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