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EXPANSION-COEFFICIENTS AS APPROXIMATE SOLUTION OF DIFFERENTIAL EQUATIONS

By

Nicholas M. KATZ

INTRODUCTION.

One of the fundamental themes of Dwork's work is the study of the variation with parameters of the zeta function in a parameterized family of varieties, and of the p-adic cohomology which gives rise to them. A basis example of such a family is the Legendre family of elliptic curves

$$E : y^2 = x(x-1)(x-\lambda)$$

over the ring

$$R = \mathbb{Z}[1/2,\lambda][1/\lambda(1-\lambda)] .$$

It was (implicitly) known to Gauss that the differential of the first kind $\omega = dx/y$, viewed as lying in $H^1_{DR}(E/R)$, is annihilated by the action of the hypergeometric differential operator with parameters $(1/2,1/2,1)$,

$$\mathcal{D} = \lambda(1-\lambda)(\frac{d}{d\lambda})^2 + (1-2\lambda)\frac{d}{d\lambda} - \frac{1}{4} ,$$

acting on $H^1_{DR}(E/R)$ via the Gauss-Manin connection.

One of the early indications of the possible existence of a p-adic theory was Igusa's discovery ([3]) that in any characteristic $p \neq 2$, the Hasse-invariant $A_p(\lambda) \in \mathbb{F}_p[\lambda]$ of the <u>same</u> Legendre curve, considered in characteristic p, provided an $\mathbb{F}_p$-polynomial solution of the same differential equation :

$$\mathcal{D}(A_p(\lambda)) = 0 \quad \text{in} \quad \mathbb{F}_p[\lambda] \ .$$

One knows that the $A_p(\lambda)$ for variable p may all be obtained simultaneously from the expansion coefficients of ω at the origin. Explicitly, the quantity

$$t = 1/\sqrt{x}$$

provides a formal parameter at the origin for E/R, in terms of which the expansion of ω is given by

$$\omega = \frac{dx}{y} = \frac{dx}{\sqrt{x(x-1)(x-\lambda)}}$$

$$= \frac{-2\,dt}{\sqrt{(1-t^2)(1-\lambda t^2)}}$$

$$= -2 \sum_{n\geq 0} P_{2n+1}(\lambda) t^{2n+1} \frac{dt}{t} \ ,$$

with expansion coefficients

$$P_{2n+1}(\lambda) = (-1)^n \sum_{a+b=n} \binom{-1/2}{a}\binom{-1/2}{b} \lambda^b \ .$$

Thus we have

$$P_{2n+1}(\lambda) \in \mathbb{Z}[1/2][\lambda] \ , \ \deg P_{2n+1} = n \ .$$

The relation of the Hasse invariant $A_p(\lambda)$ to the P_n's is simply

$$A_p(\lambda) \equiv P_p(\lambda) \quad \text{mod } p \ .$$

From this point of view, Igusa's observation becomes the statement that

$$\nabla(\mathcal{D})(\omega) = 0 \quad \text{in} \quad H^1_{DR}(E/R)$$

$$\mathcal{D}(P_p) \equiv 0 \text{ mod } pR, \text{ for all } p \neq 2 \ .$$

By explicit computation of the P_N in this case, the reader can convince himself that one has the more general congruence-differential equation

$$\mathcal{D}(P_N) \equiv 0 \quad \text{mod } N.R$$

for every integer N .

The purpose of this note is to point out that this is a completely general phenomenon.

THE MAIN RESULT.

Let A be a noetherian ring, R a smooth A-algebra, and X a smooth R-scheme, purely of relative dimension $N \geqslant 1$. We suppose given

- a marked R-valued point $O \in X(R)$;
- a set $(T_1,\ldots,T_N)$ of local coordinates on X at O ;
- a (possibly empty) finite set of R-smooth divisors $D_j \subset X$ which have normal crossings relative to R, and which satisfy the following condition : for each j, either D_j is disjoint from $O \in X(R)$, or, in a small enough Zariski neighborhood of O, D_j is defined by $T_i = O$ (for some $i = i(j)$).

On X we dispose of the following complexes, in increasing order of generality :

$\Omega^{\bullet}_{X/R}$, the de Rham complex of X/R

$\Omega^{\bullet}_{X/R}(\log \cup D_j)$, the logarithmic de Rham complex of X/R with respect to the D_j's

and for any collection of integers n_j, one for each D_j, the complex

$$\Omega^{\bullet}_{X/R}(\log \cup D_j) \otimes (\underset{j}{\otimes} I^{-1}(D_j)^{\otimes n_j}) ,$$

where $I^{-1}(D_j)$ denotes the inverse ideal sheaf of D_j in X.
The third of these includes the previous two as the specal case "no D_j's at all" and "all n_j's = O".

We denote by

$$H^p_{DR}(X(\log \cup D_j + \sum n_j D_j) / R)$$

the hypercohomology groups of X with coefficients in the complex

$$\Omega^{\bullet}_{X/R}(\log \cup D_j) \otimes (\otimes I^{-1}(D_j)^{\otimes n_j}) .$$

The Katz-Oda construction [4] of the Gauss-Manin connection carries over mutatis mutandis to this case (simply filter the X/Aanalogue,

$$\Omega^{\bullet}_{X/A}(\log \cup D_j) \otimes (\otimes I^{-1}(D_j)^{\otimes n_j}) ,$$

by the ideals generated by the pull-backs of the $\Omega^i_{R/A}$, and proceed as in [4]). Thus the cohomology groups

$$H^p_{DR}(X(\log \cup D_j + \sum n_j D_j) / R)$$

have the structure of R-modules (not necessarily of finite type unless X is assumed proper over R) endowed with the integrable Gauss-Manin connection ∇ relative to A. For any "P-D differential operator" (in the sense of [0]) $\mathcal{D}$ on R relative to A, we can thus speak of the A-linear endomorphism

$$\nabla(\mathcal{D}) : H^p_{DR}(X(\log \cup D_j + \sum n_j D_j)/ R) \circlearrowleft .$$

Let us denote by $\hat{X}$ the formal completion of X along O. In terms of the local coordinates $T_1,\ldots,T_N$, we have an isomorphism of pointed formal R-schemes

$$\mathrm{Spf}(R[[T_1,\ldots,T_N]]) \xleftarrow{\sim} \hat{X} .$$

Each divisor D_j is either disjoint from $\hat{X}$, or is defined in $\hat{X}$ by an equation $T_i = 0$. Therefore if we invert $T_1,\ldots,T_N$, we obtain a natural R-linear morphism of complexes

$$\Gamma(X,\Omega^{\bullet}_{X/R}(\log \cup D_j + \sum n_j D_j)) \longrightarrow (\Omega^{\bullet}_{R[[T_1,\ldots,T_N]]/R})[1/T_1,\ldots,1/T_N] ,$$

which we call "formal expansion at O".

THEOREM. Let $\mathcal{D}$ be a P-D differential operators on R relative to A, $p \geq 1$ an integer, and

$$\omega \in H^0(X , \Omega^p_{X/R}(\log \cup D_j + \sum n_j D_j))$$

a closed p-form on X with the allowed poles along the D_j. Let us denote by $a(\underline{k},\underline{w}) \in R$ the coefficients of the formal expansion at O of ω :

$$\omega \sim \sum_{1\leq k_1<\ldots<k_p\leq N} \sum_{w \in \mathbb{Z}^N} a(k,w) \prod_{i=1}^{N} T_i^{w_i} \prod_{v=1}^{p} \frac{dT_{k_v}}{T_{k_v}} .$$

If the cohomology class $\tilde{\omega}$ of ω is killed by $\mathcal{D}$, i.e., if

$$\nabla(\mathcal{D})(\tilde{\omega}) = 0 \quad \text{in} \quad H^p_{DR}(X(\log \cup D_j + \sum n_j D_j)/R) ,$$

then each expansion coefficient $a(\underline{k},\underline{w}) \in R$ satisfies the congruence differential equation

$$\mathcal{D}(a(\underline{k},\underline{w})) \equiv 0 \mod \sum_{v=1}^{p} w_{k_v} R .$$

Proof : By the functoriality of the Gauss-Manin connection we may replace X by an affine open neighborhood of 0 which is etale over $\mathbb{A}^N_R$ by $(T_1,\ldots,T_N)$, and in which the D_j's, if there are any, are defined by $T_i = 0$ for various i. Increasing the number of D's, we may suppose we have $D_1,\ldots,D_N$, with D_i defined by $T_i = 0$. Because X is etale over $\mathbb{A}^N_R$, any differential operator $\mathcal{D}$ on R has a unique extension $\tilde{\mathcal{D}}$ to X which on the subring $R[T_1,\ldots,T_N]$ is given by

$$\tilde{\mathcal{D}}(\sum a(\underline{w})\underline{T}^{\underline{w}}) = \sum \mathcal{D}(a(\underline{w})) \cdot \underline{T}^{\underline{w}} .$$

The $\mathcal{O}_X$-modules

$$\Omega^p_{X/R}(\log \cup D_j + \sum n_j D_j)$$

are $\mathcal{O}_X$-free, with basis

$$\frac{1}{\prod_{j=1}^{N} T_j^{n_j}} \frac{dT_{k_1}}{T_{k_1}} \wedge \ldots \wedge \frac{dT_{k_p}}{T_{k_p}} , \qquad 1 \leq k_1 < \ldots < k_p \leq N .$$

We extend $\tilde{\mathcal{D}}$ to the entire complex

$$\Omega^{\bullet}_{X/R}(\log \cup D_j + \sum n_j D_j)$$

by defining, for $f \in \mathcal{O}_X$,

$$\tilde{\mathcal{D}}(f \cdot \frac{1}{\prod_j T_j^{n_j}} \cdot \prod_v \frac{dT_{k_v}}{T_{k_v}}) = \tilde{\mathcal{D}}(f) \cdot \frac{1}{\prod_j T_j^{n_j}} \prod_v \frac{dT_{k_v}}{T_{k_v}} .$$

It is transparent from the definitions that this action of $\tilde{\mathcal{D}}$ induces $\nabla(\mathcal{D})$ on the cohomology. Therefore, if ω has formal expansion

$$\omega \sim \sum_{\underline{k},\underline{w}} a(\underline{k},\underline{w}) \prod_i T_i^{w_i} \prod_v \frac{dT_{k_v}}{T_{k_v}} ,$$

it is obvious by T-adic continuity that $\tilde{\mathcal{D}}(\omega)$ has formal expansion

$$\tilde{\mathcal{D}}(\omega) \sim \sum_{\underline{k},\underline{w}} \mathcal{D}(a(\underline{k},\underline{w})) \prod_i T_i^{w_i} \prod_v \frac{dT_{k_v}}{T_{k_v}} .$$

This being the case, the hypothesis

$$\nabla(\mathcal{D})(\tilde{\omega}) = 0 \quad \text{in} \quad H^p_{DR}(X(\log \cup D_j + \sum n_j D_j) / R)$$

guarantees that the formal differential form over R

$$\sum_{\underline{k},\underline{w}} \mathcal{D}(a(\underline{k},\underline{w})) \cdot \prod_i T_i^{w_i} \prod_v \frac{dT_{k_v}}{T_{k_v}}$$

is formally exact. Writing it as the exterior derivative of a formal (p-1)-form over R and equating coefficients yield the asserted congruences on the $\mathcal{D}(a(\underline{k},\underline{w}))$. Q.E.D.

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