PHILIPPE ROBBA

Index of p-adic differential operators III. Application to twisted exponential sums

Astérisque, tome 119-120 (1984), p. 191-266

<http://www.numdam.org/item?id=AST_1984__119-120__191_0>

Société Mathématique de France
Astérisque 119-120 (1984) p.191-266

INDEX OF p-ADIC DIFFERENTIAL OPERATORS
III. APPLICATION TO TWISTED EXPONENTIAL SUMS

by

Philippe ROBBA(*)

§ 1. INTRODUCTION.

1.1. Motivated by the recent work of Dwork, Sperber and Adolphson on p-adic cohomology associated with certain twisted exponential sums and their relations with the p-adic theory of special functions, we give in this article a systematic treatment of one-variable cohomology of p-adic analytic type and we apply this to the explicit calculation of one-variable twisted exponential sums and of the associated L-functions.

We shall in particular consider $H/\ell H$ where H is a space of analytic functions in one variable and ℓ is an ordinary linear differential operator of order n with rational coefficients. This analytic cohomology will be compared with the algebraic cohomology $\underline{L}/\ell\underline{L}$ where $\underline{L}$ is the ring of rational function whose poles lie in a fixed set chosen such that $\underline{L}$ is stable under ℓ. Finiteness of $\underline{L}/\ell\underline{L}$ is a well known consequence of partial fraction decomposition and of the theory of indicial polynomials of ordinary linear differential operators (cf. [Ad]). We are concerned with the 3 questions :

1.1.1. Is $H/\ell H$ finite ?

1.1.2. What is the dimension of $H/\ell H$?

1.1.3. Under what conditions does the imbedding of $\underline{L}$ into H lead to an isomorphism of $\underline{L}/\ell\underline{L}$ with $H/\ell H$?

(*) This work was done while the author was visiting Princeton University.

We answer these questions when ℓ is of first order.

We apply our present results to determine the L-functions associated with twisted exponential sums by means of the Reich-Monsky trace formula. Of course A. Weil determined the degree and verified the Riemann hypothesis for all L-functions on curves but our treatment is useful for investigations of the p-adic value of the roots and for studying the variation of roots with parameters.

1.2. We give now a brief historical survey.

For analytic theory the first results seem to have been those of Dwork [Dw 5] who in his work on zeta functions formulated an ad hoc p-adic cohomology naturally associated with the theory of exponential sums in several variables. He considered the space H_n of power series in n variables which converged and were bounded in a fixed open polydisk and studied the Kozul complex of commuting differential operators $D_1,\dots,D_n$ on H_n where $D_i = \exp f \circ x_i \frac{\partial}{\partial x_i} \circ \exp(-f)$ and where f is a fixed polynomial in several variables. The central fact was the hypothesis that $\exp f \notin H_n$, but in much of his work additional hypotheses were imposed upon f and upon the polydisks (defining H_n) so that an <u>effective</u> reduction theory could be deduced. By an effective reduction theory we mean that each polynomial ξ (in n-variables) can be represented as a sum $\xi = \eta + \sum_{i=1}^{n} D_i \xi_i$ where η is a polynomial of degree bounded independently of ξ and such that, in the sup norm on the polydisk, $|\eta|, |\xi_i|, \dots, |\xi_n|$ may all be bounded in terms of $|\xi|$.

For the one variable theory the situation in which H is the space of functions analytic on a disk $B = D(0,r)$ of radius r goes back to E. Lutz [Lu] who observed that if 0 is an ordinary point of ℓ then the solution of ℓ at the origin all have non-trivial p-adic radii of convergence. Although Lutz did not consider this question, it is obvious (since non-homogeneous linear equations can be solved by quadratures) that in this situation $H/\ell H$ is finite provided r is small enough. This was generalized by Clark [Cl] who showed that the same result holds if the origin is a singular point provided the exponents (i.e. the roots of the indicial polynomial at the origin) are p-adic non-Liouville numbers. The result of Clark was based upon a lower asymptotic bound for $|\alpha(\alpha+1)\dots(\alpha+s-1)|^{1/s}$ as

$s \to \infty$ for each zero α of the indicial polynomial. In effect Clark analyzed the recursion formulae involved in solving a non-homogeneous equation of the form $\ell\xi = \eta$.

In the work of Adolphson [Ad] the key point is that $H/\ell H$ may be determined if ℓ is <u>analytically equivalent</u> on $D(0,r^-)$ to a differential operator whose solution matrix is x^B where B is a constant matrix satisfying the following condition : (N) Each eigenvalue of B is p-adically non-Liouville. Adolphson globalized this result by means of the Mittag-Leffler theorem. Let A be the complement of a finite union of disks, on each of which ℓ is analytically equivalent to an operator whose solution matrix is $x_i^{B_i}$ where x_i is a suitable local variable and B_i is a constant matrix satisfying condition N . Adolphson determines $H_A/\ell H_A$ where H_A is the ring of <u>functions</u> analytic on A (this is not valid for the ring of analytic element on A) .

We show here (§ 9) that Adolphson's theory needs not be restricted to the case in which the excluded disks contain only regular singular points. We may exclude disks on which ℓ is analytically equivalent to a normalized differential operator whose solution matrix is $x^B \exp\Delta(x^{-1/k!})$ where x denotes the local variable and B is a constant matrix which commutes with the polynomial matrix Δ . It is known by the theorem of Turrittin as explained in Baldassarri's article [Ba] that subject to the hypothesis that the exponents diffferences of ℓ at the origin are non-Liouville, ℓ is surely analytically equivalent to such a normalized operator on a disk $D(0,r)$ provided r is small enough. Unfortunately, effective estimates for r are not available. Even in the ordinary case (i.e. $\Delta = 0$, B diagonal with $0,1,\dots,k-1$ as eigenvalues) for differential equations defined overs $\mathbb{Q}(x)$ we may have $r = 1$ for almost all primes while there are examples of the Lame equation with exponents in $2^{-1}\mathbb{Z}$ and four singular points such that $r = p^{-1/(p-1)}$ for an infinite set of p .

It is implicit in our reduction procedures that a good knowledge of the first order case is essential. While in this case the explicit representation of the kernel near the singular point is most evident we do not use this explicit representation but rather start with a knowledge of the radius of convergence of the kernel at a generic point on the boundary of each excluded disk. We believe that this is the best formulation for avoiding the problem of domain of analytic equivalence in the higher order case.

Even for the case $n = 1$ these questions are of interest. Ad hoc treatments of special cases may be found in [Boy] , [Dw 1], [Dw 2], [A-S] , [La] for Gauss sums, twisted Kloosterman sums, L-series of certain cyclic coverings of the sphere with four ramified points, together with the relation to the p-adic theory of gamma functions, Bessel functions and hypergeometric functions. The present work provides a common basis for these and many other special differential equations e.g. the confluent hypergeometric function and the Jordan-Pochhammer differential equations.

1.3. We now outline our article.

In sections 2 and 3 we give basic definitions and recall properties of analytic functions and of the index of liner operators.

In section 4 we determine the index in a disk of a first order scalar differential operator in terms of the radius of convergence of the solution near the generic point on the boundary. We then give some variations on this basic formula and compute an explicit example.

In section 5 we consider Dwork's p-adic cohomology and solve problem 1.1.1 - 1.1.3.

In section 6 we sketch Dwork's theory of L-functions in the one variable case. We show how we can recapture some of Weil's results for L-functions in the case of twisted exponential sums.

In section 7 we investigate the p-adic values of the zeros of L-functions associated with exponential sums in one variable. This section is closest in spirit to the work of Dwork in the 1960's.

In section 8 we explain the dual theory and the functional equation of these L-functions.

In section 9 we explain how Adolphson's theory can be generalized.

I thank Princeton University for its hospitality and in particular N. Katz and B. Dwork for their helpful suggestions.

For the convenience of the reader we include a list of the more frequently used symbols and indicate the paragraph where they are defined.

§ 2. NOTATIONS. ANALYTIC ELEMENTS. GENERIC DISKS.

2.1. Let K be an algebraically closed field of characteristic zero, complete under a non-archimedean valuation. Let Ω be an algebraically closed field, complete under a valuation extending that of K, and linearly disjoint form $K(x)$ over K. Assume further that the residue classe field $\overline{\Omega}$ of Ω is a transcendental extension of the residue class field $\overline{K}$ of K.

2.2. For each $c \in \Omega$ and each positive number r let

$$B(c,r^-) := \{x \in r\,,\ |x-c| < r\}$$

$$B(c,r^+) := \{x \in r\,,\ |x-c| \leqslant r\}$$

$$C(c,r) := \{x \in r\,,\ |x-c| = r\}\ .$$

In this article we shall only consider disks with radius $r \in |\Omega^*|$, and thus the circumference $C(c,r)$ is not empty.

2.3. For $f \in \Omega[[x-c\,,\frac{1}{x-c}]]$, $f = \sum_{\nu=-\infty}^{+\infty} b_\nu (x-c)^\nu$, analytic in the annulus $\Delta = B(c,R+) - B(c,r^-)$ (so $\lim_{\nu\to+\infty} |b_\nu| R^\nu = 0$ and $\lim_{\nu\to-\infty} |b_\nu| r^\nu = 0$), let for $r < \rho < R$

$$|f|_c(\rho) := \sup_\nu |b_\nu| \rho^\nu$$

$$\mathrm{ord}_c^+(f,\rho) := \{\sup \nu,\ |b_\nu|\rho^\nu = |f|_c(\rho)\}$$

$$\mathrm{ord}_c^-(f,\rho) := \{\inf \nu,\ |b_\nu|\rho^\nu = |f|_c(\rho)\}\ .$$

It is well known [Am] that

$$\mathrm{ord}_c^+(f,\rho) = \left(\frac{d \log |f|_c(\rho)}{d \log \rho}\right)^+$$

$$\mathrm{ord}_c^-(f,\rho) = \left(\frac{d \log |f|_c(\rho)}{d \log \rho}\right)^-$$

where $(\frac{du}{ds})^+$ (resp. $(\frac{du}{ds})^-$) denotes the right hand (resp. the left hand) derivative of u with respect to r.

It is also known that the function ord_c^+ and ord_c^- are increasing functions of ρ. From this one can deduce easily the following result.

SCHWARZ'S LEMMA : Let f be analytic in the annulus Δ then

$$|f|_c(R) \geq \left(\frac{R}{r}\right)^{\mathrm{ord}_c^+(f,r)} |f|_c(r)$$

$$|f|_c(r) \geq \left(\frac{r}{R}\right)^{\mathrm{ord}_c^-(f,R)} |f|_c(R) .$$

2.4. These definitions are extended to functions f, meromorphic in Δ, by writing, if $f = g/h$ where g and h are both analytic in Δ,
$|f|_c(\rho) = |g|_c(\rho)/|h|_c(\rho)$, $\mathrm{ord}_c^{\pm}(f,\rho) = \mathrm{ord}_c^{\pm}(g,\rho) - \mathrm{ord}_c^{\pm}(h,\rho)$.

If $f \in \Omega[[x-c]]$ is analytic in the disk $B(c,R^+)$, it is known that $\mathrm{ord}_c^+(f,\rho)$ (resp. $\mathrm{ord}_c^-(f,\rho)$) is the number of zeros of f in the disk $B(c,\rho^+)$ (resp. $B(c,\rho^-)$) .

For $f \in \Omega[x]$, we denote $\mathrm{ord}_c f$ the order of the zero of f at c, and write $\mathrm{ord}_\infty f = -\deg f$. These definitions are extended as usual to rational functions.

2.5. Let $\mathbb{P} = \mathbb{P}(\Omega) = \Omega \cup \{\infty\}$. For $A \subset \mathbb{P}$ let $A^c = \mathbb{P} - A$. Let $A \subset \mathbb{P}$ with $d(A,A^c) = \inf\{|x-y|,\ x \in A,\ y \in A^c\} > 0$. We denote by $R(A)$ the set of all rational functions with coefficients in K without poles in A . An analytic element on A with coefficients in K is the uniform limit on A of elements of $R(A)$. We shall denote by $H(A)$ the set of analytic elements on A with coefficients in K . For $f \in H(A)$ let $\|f\|_A = \sup_{x \in A} |f(x)|$. This defines a norm on $H(A)$ and with this norm $H(A)$ is a Banach space over K .

If $A = B(c,r^+)$ and $f \in H(A)$, then f is analytic on $B(c,r^+)$ and $\|f\|_A = |f|_c(r)$.

2.6. Let $r \in |\Omega^*|$ and let $t \in C(c,r)$. We shall say that t is generic on the circumference $C(c,r)$ if the disk $B(t,r^-)$ (which is contained in $C(c,r)$) contains no point in K .

By our hypothesis on Ω there is always a generic point on $C(c,r)$. The disk $B(t,r^-)$ will then be said to be generic in the circumference $C(c,r)$. If $f \in K(x)$, then f has neither poles nor zeros in $B(t,r^-)$ and $|f(t)| = \|f\|_{B(t,r^-)} = |f|_c(r)$.

§ 3. INDEX OF A LINEAR OPERATOR.

We recall some well-known properties of the index of a linear operator. The proofs can be found in [Ro 2, § 4].

3.1. Let U and V be two vector spaces over K. Let $L(U,V)$ denote the space of linear mappings from U into V. If U and V are topological vector spaces let $\mathcal{L}(U,V)$ denote the subspace of continuous linear mappings.

We say that $L \in L(U,V)$ has an index if it has a finite-dimensional kernel and a finite-dimensional cokernel. The index of L is then $\chi(L\,;U,V) = \dim \operatorname{Ker} L - \dim \operatorname{coker} L$.

We shall also write $\chi(L)$ when there is no ambiguity, or $\chi(L;U)$ if $U = V$. Thus $\chi(L)$ is the Euler-Poincaré characteristic of the complex

$$0 \longrightarrow U \xrightarrow{L} V \longrightarrow 0 .$$

If U and V are complete metric spaces and L is continuous and has an index, then L is a homomorphism onto its image and $\operatorname{Im} L$ is closed.

3.2. LEMMA. Let $L \in L(U,V)$ and $Q \in L(V,W)$. If two of the three operators Q, L and QL have indexes, then the third one also has an index and

$$\chi(QL) = \chi(Q) + \chi(L) .$$

3.3. LEMMA. Assume that the following diagram is commutative with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & U_1 & \longrightarrow & U_2 & \longrightarrow & U_3 & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle L_1} & & \downarrow{\scriptstyle L_2} & & \downarrow{\scriptstyle L_3} & & \\ 0 & \longrightarrow & V_1 & \longrightarrow & V_2 & \longrightarrow & V_3 & \longrightarrow & 0 \end{array}$$

and that two of the three operators $L_1\ L_2\ L_3$ have an index, then the third one also has an index and

$$\chi(L_1) - \chi(L_2) + \chi(L_3) = 0 .$$

3.4. COROLLARY. Let $Q_1 \in L(U_1,V_1)$ and $Q_2 \in L(U_2,V_2)$ have indexes. Then $Q_1 \oplus Q_2 \in L(U_1 \oplus V_1,\ U_2 \oplus V_2)$ has index and

$$\chi(Q_1 \oplus Q_2) = \chi(Q_1) + \chi(Q_2) .$$

Proof. Apply lemma 3.3 to the situation

$$\begin{array}{ccccccccc} 0 & \longrightarrow & U_1 & \longrightarrow & U_1 \oplus U_2 & \longrightarrow & U_2 & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle Q_1} & & \downarrow{\scriptstyle Q_1 \oplus Q_2} & & \downarrow{\scriptstyle Q_2} & & \\ 0 & \longrightarrow & V_1 & \longrightarrow & V_1 \oplus V_2 & \longrightarrow & V_2 & \longrightarrow & 0 \end{array}$$

3.5. COROLLARY. Let $U = U_1 \oplus H$ and $V = V_1 \oplus G$ with $\dim H < +\infty$, $\dim G < +\infty$. Let $L \in L(U,V)$. Assume that the restriction L_1 of L to U_1 maps U_1 into V_1. Then, if L_1 has an index, L also has an index and

$$\chi(L) = \chi(L_1) + \dim H - \dim G .$$

Proof. Apply lemma 3.3 to the situation

$$\begin{array}{ccccccccc} 0 & \longrightarrow & U_1 & \longrightarrow & U & \longrightarrow & H & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle L_1} & & \downarrow{\scriptstyle L} & & \downarrow{\scriptstyle \overline{L}} & & \\ 0 & \longrightarrow & V_1 & \longrightarrow & V & \longrightarrow & G & \longrightarrow & 0 \end{array}$$

where $\overline{L}$ is defined in order to make the diagram commutative. As H and G have finite dimension $\overline{L}$ has an index, $\chi(\overline{L}) = \dim H - \dim G$.

3.6. LEMMA. L *is injective and has an index if and only if* L *has a left inverse* L' *which has an index*. *If furthermore* U *and* V *are Banach spaces and* L *is continuous* L' *can be chosen continuous*.

3.7. LEMMA. *Let* U *and* V *be Banach spaces*, $L \in \mathcal{L}(U,V)$. *Suppose that* L *is injective and has an index and Let* L' *be a continuous left inverse of* L . *If* $Q \in \mathcal{L}(U,V)$ *and* $\|Q-L\| < 1/\|L'\|$, Q *is injective and has an index* $\chi(Q) = \chi(L)$.

3.8. LEMMA. *Let* U *and* V *be complete metric spaces*, *and let* U_1 (*resp*. V_1) *be a subspace dense in* U (*resp*. V) . *Let* $L \in L(U,V)$. *Assume that its restriction* L_1 *to* U_1 *belongs to* $L(U_1,V_1)$. *Assume further that* L *and* L_1 *have indexes*, *then*

$$\chi(L) \geqslant \chi(L_1) .$$

If we have equality : $\chi(L) = \chi(L_1)$, *then* $\operatorname{Ker} L = \operatorname{Ker} L_1$, $V/\operatorname{Im} L \simeq V_1/I_m L_1$, *and a complementary of* $\operatorname{Im} L_1$ *in* V_1 *is also a complementary of* $\operatorname{Im} L$ *in* V .

§ 4. INDEX OF A DIFFERENTIAL OPERATOR OF ORDER 1.

We establish a relation between the index of a differential operator of order 1, viewed as a linear operator on the space of analytic elements on a disk, and the radius of convergence of its solutions near the generic point on the boundary of the disk.

We make a conjecture for a similar relation in the case of differential operators of order > 1 and we give an example which supports this conjecture.

We show how we can also compute the index in $H(A)$ where A is a disk minus a finite union of disks. Another example will be given in the next paragraph with Dwork's cohomology.

4.1. Let $\underline{R} = K[x]\left[\frac{d}{dx}\right]$ be the ring of linear differential operators with polynomial coefficients, the multiplication being given by $\frac{d}{dx} \circ a = a\frac{d}{dx} + \frac{da}{dx}$.

Let A be a bounded subset of Ω (with $d(A,A^c) > 0$). Then $\underline{R}$ is identified naturally with a subring of continuous endomorphism of $H(A)$. Let $c \in \Omega$, $r \in |\Omega^*|$. For simplicity we shall use the notations

$$H_c(r^+) := H(B(c,r^+)) \quad \text{and} \quad H_c(r^-) := H(B(c,r^-)) .$$

Then if $L \in \underline{R}$, viewed as element of $\mathcal{L}(H_c(r^+),H_c(r^+))$ (resp. $\mathcal{L}(H_c(r^-),H_c(r^-))$, has an index, we shall denote this index by $\chi_c^+(L,r)$ (resp. $\chi_c^-(L,r)$).

4.2. We now consider a differential operator of first order, $L = a\frac{d}{dr} + b$ with a, $b \in K[x]$, together with $\ell = \frac{1}{a}L = \frac{d}{dx} + \frac{b}{a}$. Let t_r be a generic point on the circumference $C(c,r)$ and let u be a solution of $Lu = 0$ in a neighborhood of t_r. Denote by $\rho_c(L,r)$ the radius of convergence of u.

The main result of this paragraph is the following theorem.

THEOREM. <u>Assume that, for</u> $r = r_0$, $\rho_c(L,r_0) < r_0$. <u>Then for</u> r <u>close enough to</u> r_0, L <u>is injective and has an index in</u> $H_c(r^+)$ (<u>resp.</u> $H_c(r^-)$), $\rho_c(L,r)$ <u>is a continuous function of</u> r <u>and we have the relations</u>

$$(4.2+) \qquad \left(\frac{d \log \rho_c(L,r)}{d \log r}\right)^+ = \chi_c^+(L,r) + \mathrm{ord}_c^+(a,r)$$

$$(4.2-) \qquad \left(\frac{d \log \rho_c(L,r)}{d \log r}\right)^- = \chi_c^-(L,r) + \mathrm{ord}_c^-(a,r) .$$

<u>Remark</u> : The fact that L is injective and has an index in $H_c(r^\pm)$ is a special case of Theorem 6.16 of [Ro 2] which asserts a similar property for differential operators of any order, but we shall give a simple proof valid for operators of order one. The new feature in this theorem is formula (4.2) which permits us to compute the index. Some lemmas are needed for the proof.

4.3. LEMMA. <u>Let</u> $Q = \sum \frac{a_m}{m!}\frac{d^m}{dx^m} \in \underline{R}$. <u>Viewed as a continuous endomor-</u>

<u>phism of</u> $H_c(r^+)$ (<u>resp</u>. $H_c(r^-)$), Q <u>has operator norm</u>

$$||Q||_r = \max_m |a_m|_c(r)/r^m .$$

See [Ro 2] § 1.11.

4.4. LEMMA. <u>Let</u> $a \subset K[x]$. <u>Viewed as an endomorphism of</u> $H_c(r^+)$ (<u>resp</u>. $H_c(r^-)$), <u>multiplication by</u> a <u>is injective</u>, <u>has index</u> $\chi_c^+(a,r) = -\operatorname{ord}_c^+(a,r)$ (<u>resp</u>. $\chi_c^-(a,r) = -\operatorname{ord}_c^-(a,r)$) <u>and has a left inverse of norm</u> $1/|a|_c(r)$.

See [Ro 2] Theorem 4.15.

4.4. <u>Proof of Theorem 4.2</u>.

As c and L remain fixed in this paragraph, for simplicity we shall write ρ_r instead of $\rho_c(L,r)$.

4.4.1. Let u be analytic near t_r such that $Lu = 0$. Then $\ell = u \circ \frac{d}{dx} \circ u^{-1}$. Define

$$b_m = \frac{1}{m!} u(u^{-1})^{(m)} . \tag{4.4.1}$$

One has the recursion formula

$$b_0 = 1, \quad b_{m+1} = \frac{1}{m+1}(b' + \frac{b}{a} b_m) \tag{4.4.2}$$

and therefore $a^m b_m \in K[x]$.

On the other hand

$$\begin{aligned} \frac{1}{m!} \ell^m &= u \circ \frac{1}{m!}(\frac{d}{dx})^m \circ u^{-1} = \sum_{i=0}^{m} b_{m-i} \frac{1}{i!} (\frac{d}{dx})^i \\ \frac{a^m}{m!} \ell^m &= a^m b_m + R_m \end{aligned} \tag{4.4.3}$$

where $R_m = \sum_{i=1}^{m} a^m b_{m-i} \frac{1}{i!} (\frac{d}{dx})^i \in \underline{R}$.

By Taylor's formula

$$u^{-1}(x) = \sum_{m \geqslant 0} b_m(t_r)(x - t_r)^m .$$

As a connot be zero in the generic disk, u never vanishes in the generic disk and therefore if $\rho_r < r$, the radius of convergence of u^{-1} is also ρ_r. Therefore for all $\rho > \rho_r$

$$(4.4.4) \qquad \overline{\lim_{m\to\infty}} |b_m|_c(r)\rho^m = \overline{\lim_{m\to\infty}} |b_m(t_r)|\rho^m = +\infty.$$

We apply this to the case $r = r_0$ and $\rho = r_0$. So there exists $m > 1$ such that

$$(4.4.5) \qquad |b_m|_c(r_0)r_0^m > \max_{0\leqslant i\leqslant m-1} |b_i|_c(r_0)r_0^i$$

(observe that $|b_0|_c(r_0)r_0^0 = 1$).

We deduce from lemma 4.3

$$|a^m b_m|_c(r_0) > \max_{0\leqslant i\leqslant m-1} |a^m b_i|_c(r_0)r_0^{i-m}$$

$$= \max_{1\leqslant i\leqslant m} |a^m b_{m-i}|_c(r_0)r_0^{-i} = \|R_m\|_{r_0} .$$

As these functions are continuous functions of r, for r close to r_0 we have again

$$(4.4.6) \qquad |a^m b_m|_c(r) > \|R_m\|_r .$$

Then we deduce from (4.4.3), (4.4.6) and from lemmas 3.7 and 4.4 that $\frac{a^m}{m!}\ell^m$ is injective in $H_c(r^{\pm})$ and has index

$$(4.4.7) \qquad \chi_c^{\pm}(\frac{a^m}{m!}\ell^m, r) = -\operatorname{ord}_c^{\pm}(a^m b_m, r) .$$

Now ℓ is not and endomorphism of $H_c(r^{\pm})$, but for all $i \geqslant 0$, $\ell \in L(\frac{1}{a^i} H_c(r^{\pm}), \frac{1}{a^{i+1}} H_c(r^{\pm}))$. Considering the decomposition of the singular part around the zeros of a in $B(c,r^{\pm})$ one sees easily that

$$\frac{1}{a^{i+1}} H_c(r^{\pm}) = \frac{1}{a^i} H_c(r^{\pm}) \oplus G_i$$

with $\dim G_i = \mathrm{ord}_c^{\pm}(a,r)$ and thus $\dim G_i$ does not depend on i. Therefore one deduces from corollary 3.5 that if ℓ as an index as element of $L(\frac{1}{a^i} H_c(r^{\pm}), \frac{1}{a^{i+1}} H_c(r^{\pm}))$ for one i, then the same is true for all i and this index does not depend on i.

We have seen that ℓ^m has an index as element of $(H_c(r^{\pm}), \frac{1}{a^m}H_c(r^{\pm}))$ and the formula (4.4.7) can be rewritten

$$\chi(\ell^m ; H_c(r^{\pm}), \frac{1}{a^m} H_c(r^{\pm})) = -\mathrm{ord}_c^{\pm}(a^m b_m, r) . \tag{4.4.8}$$

Therefore $\ell \in L(\frac{1}{a^i} H_c(r^{\pm}), \frac{1}{a^{i+1}} H_c(r^{\pm}))$ has an index and using lemma 3.2 we obtain

$$\begin{aligned} m\chi(\ell;H_c(r^{\pm}), \tfrac{1}{a} H_c(r^{\pm})) &= \chi(\ell^m;H_c(r^{\pm}), \tfrac{1}{a^m} H_c(r^{\pm})) \\ &= -\mathrm{ord}_c^{\pm}(a^m b_m, r) \end{aligned} \tag{4.4.9}$$

and therefore $L = a\ell$ is injective and has an index in $H_c(r^{\pm})$ with

$$\begin{aligned} \chi_c^{\pm}(L,r) = \chi(\ell;H_c(r^{\pm}), \tfrac{1}{a} H_c(r^{\pm})) &= \tfrac{-1}{m} \mathrm{ord}_c^{\pm}(a^m b_m, r) \\ &= -\mathrm{ord}_c^{\pm}(a,r) - \tfrac{1}{m} \mathrm{ord}_c^{\pm}(b_m,r) \end{aligned} \tag{4.4.10}$$

One can find $\varepsilon > 0$ small enough such that

(4.4.11.1) for $r_0 - \varepsilon \leqslant r \leqslant r_0$: $\mathrm{ord}_c^-(a,r) = \mathrm{ord}_c^-$ and

$\mathrm{ord}_c^-(b_m,r) = \mathrm{ord}_c^-(b_m,r_0)$

(4.4.11.2) for $r_0 \leqslant r \leqslant r + \varepsilon$: $\mathrm{ord}_c^+(a,r) = \mathrm{ord}_c^+(a,r_0)$ and

$\mathrm{ord}_c^+(b_m,r) = \mathrm{ord}_c^+(b_m,r_0)$

(4.4.11.3) for $r_0 - \varepsilon \leqslant r \leqslant r_0 + \varepsilon$: $|b_m|_c(r) r^m > 1$.

and then one has

(4.4.12.1) for $r_0 - \varepsilon \leqslant r < r_0$: $\chi_c^-(L,r) = \chi_c^+(L,r) = \chi_c^-(L,r_0)$

(4.4.12.2) for $r_0 < r \leqslant r_0 + \varepsilon$: $\chi_c^-(L,r) = \chi_c^+(L,r) = \chi_c^+(L,r_0)$

(4.4.12.3) for $r_0 - \varepsilon \leqslant r \leqslant r_0 + \varepsilon : \rho_r < r$.

4.4.2. Now choose $r \in (r_0 - \varepsilon , r_0)$ and let

$$\rho := \rho_r \left(\frac{r_0}{r}\right)^{\chi_c^-(L,r) + \mathrm{ord}_c^-(a,r)} \qquad (4.4.13)$$

We shall prove that $\rho = \rho_{r_0}$. One can then deduce easily (4.2).

First we prove that one cannot have $\rho > \rho_{r_0}$. In fact assume that $\rho > \rho_{r_0}$. If $\rho \geqslant r_0$ choose m satisfying (4.4.5), then one has also

$$|b_m|_c(r_0)\ \rho^m > \sup_{0<i<m-1} |b_i|_r\ \rho^i . \qquad (4.4.14)$$

If $\rho < r_0$, choose m satisfying (4.4.14). Such an m exists because of (4.4.4) applied to $r = r_0$. Then one has also (4.4.5). Inequality (4.4.14) implies

$$|b_m|_c(r_0)\ \rho^m > 1 . \qquad (4.4.15)$$

As (4.4.5) is satisfied we deduce from equation 4.4.7 that

$$\mathrm{ord}_c^-(a^m b_m) = -m\,\chi_c^-(L,r_0)$$

and therefore, by Schwartz's lemma,

$$|a^m b_m|_c(r) \geqslant |a^m b_m|_c(r_0)\ \left(\frac{r}{r_0}\right)^{-m\chi_c^-(L,r_0)} . \qquad (4.4.16)$$

From (4.4.11.1) one can deduce

$$|a^m|_c(r) = |a^m|_c(r_0)\ \left(\frac{r}{r_0}\right)^{m\ \mathrm{ord}_c^-(a,r)} . \qquad (4.4.17)$$

From (4.4.13), (4.4.15), (4.4.16), and (4.4.17) we get

$$|b_m|_c(r)\rho_r^m \geqslant |b_m|_c(r_0)\left(\frac{r}{r_0}\right)^{-m(\chi_c^-(L,r)+\mathrm{ord}_c^-(a,r))}\rho_r^m = |b_m|_c(r_0)\rho^m > 1$$

which is not true.

We prove now that we cannot have $\rho < \rho_{r_0}$. In fact assume $\rho < \rho_{r_0}$. Define

$$\rho' := \rho_{r_0}\left(\frac{r}{r_0}\right)^{\chi_c^-(L,r) + \mathrm{ord}_c^-(a,r)} \quad (4.4.18)$$

Then one has $\rho' > \rho_r$. Therefore one can find $m \geqslant 1$ such that

$$|b_m|_c(r)\ \rho'^m > \max_{0 \leqslant i \leqslant m-1} |b_i|_c(r)\rho'^i \quad (4.4.19)$$

$$|b_m|_c(r)\ r^m > \max_{0 \leqslant i \leqslant m-1} |b_i|_c(r) r^i . \quad (4.4.20)$$

In particular one has

$$|b_m|_c(r)\rho'^m > 1 . \quad (4.4.21)$$

From (4.4.10) and paragraph 4.4.1 we deduce

$$\mathrm{ord}_c^+(a^m b_m, r) = - m\ \chi_c^+(L,r)$$

therefore using (4.4.12.1) we get

$$\mathrm{ord}_c^+(a^m b_m, r) = - m\chi_c^-(L,r_0) \quad (4.4.22)$$

and therefore by Schwarz's lemma

$$|a^m b_m|_c(r_0) > |a^m b_m|_c(r)\ \left(\frac{r_0}{r}\right)^{-m\chi_c^-(L,r_0)} . \quad (4.4.23)$$

As (4.4.17) is again valid, we get from (4.4.18), (4.4.21) and (4.4.23)

$$|b_m|_c(r_0)\rho_{r_0}^m \geqslant |b_m|_c(r)\rho_{r_0}^m\left(\frac{r_0}{r}\right)^{-m(\chi_c^-(L,r_0+\mathrm{ord}_c^-(a,r_0))} = |b_m|_c(r)\rho'^m > 1$$

which is not true.

4.4.3. Formula (4.2 +) is proven in the same way.

4.5. The formulas of theorem 4.2 are very similar to the formulas of § 2.3. The purpose of the next proposition is to show that $\chi_c^{\pm}(L,r)$ enjoys properties similar to that of $\mathrm{ord}_c^{\pm}(f,r)$.

PROPOSITION. Let $c \in r$. Let $L, Q \in \underline{R}$ and assume that for $r \in (r_0, r_1)$ $r \in |\Omega^*|$, L and Q have no solution in the generic disk of the circumference $C(c,r)$. Then L and Q have indexes in $H_c(r^{\pm})$ and further

i) For $r_0 < r < r' < r$.

$$0 > \chi_c^-(L,r) \geqslant \chi_c^+(L,r) \geqslant \chi_c^-(L,r') \geqslant \chi_c^+(L,r')$$

ii) $\chi_c^+(LQ,r) = \chi_c^+(L,r) + \chi_c^+(Q,r)$

$\chi_c^-(LQ,r) = \chi_c^-(L,r) + \chi_c^+(Q,r)$

iii) Let $r \in |\Omega^*|$, with $r \in (r_0, r_1)$. Consider the partition of the closed disk $B(c,r^+)$ into its residue classes, i.e. $B(c,r^+) = \bigcup_{i \in I} B(c_i, r^-)$ with $|c_i - c_j| = r$ for $i \neq j$. Then $\chi_{c_i}^-(L,r) = 0$ for almost all i and

$$\chi_c^+(L,r) = \sum_{i \in I} \chi_{c_i}^-(L,r) .$$

Proof. The fact that L and Q have indexes and are injectives results from theorem 4.16 of [Ro 2] .

i) $0 \geqslant \chi_c^-(L,r)$ comes from the fact that L is injective ; $\chi_c^-(L,r) \geqslant \chi_c^+(L,r)$ is a consequence of iii) together with the property that all $\chi_{c_i}^-(L,r)$ are $\leqslant 0$.
$\chi_c^+(L,r) \leqslant \chi_c^-(L,r')$ is a consequence of lemma 3.8 and of the fact that $H_c(r'^-)$ is a dense subspace of $H_c(r^+)$.

ii) Is a special case of lemma 3.2.

iii) Let J be a finite subset of I and let

$$A := B(c,r^+) - \bigcup_{i \in J} B(c_j, r^-) .$$

Then by lemma 4.10 one has

$$(L\,;\,H(A)) = \chi_c^+(L,r) - \bigcup_{i \in J} \chi_{c_i}^-(L,r) .$$

As L is injective in H(A) its index is < 0 and thus for all finite subsets J of I

$$\sum_{i \in J} \chi_{c_i}^-(L,r) > \chi_c^+(L,r) .$$

As all these numbers are negative or zero integers, we get that for almost all i , $\chi^-_{c_i}(L,r) = 0$.

Now in theorem 4.16 of [Ro 2] it is proven that there exist $P \in \underline{R}$ and $a \in K(x)$ such that

$$\|PL - a\|_r < 1/|a|_c(r) .$$

Therefore using lemma 3.7 and lemma 4.3 one obtains

$$\chi^+_c(PL,r) = \chi^+_c(P,r) + \chi^+_c(L,r) = \chi^+_c(a,r) = -\operatorname{ord}^+_c(a,r)$$

$$\chi^-_{c_i}(PL,r) = \chi^-_{c_i}(P,r) + \chi^-_{c_i}(L,r) = \chi^-_{c_i}(a,r) = -\operatorname{ord}^-_{c_i}(a,r) \quad \text{for all } i .$$

Therefore

$$[\chi^+_c(P,r) - \sum_{i\in I} \chi^-_{c_i}(P,r)] + [\chi^+_c(L,r) - \sum_{i\in I} \chi^-_{c_i}(L,r)]$$

$$= -[\operatorname{ord}^+_c(a,r) - \sum_{i\in I} \operatorname{ord}^-_{c_i}(a,r)] = 0$$

As

$\chi^+_c(P,r) - \sum_{i\in I} \chi^-_{c_i}(P,r)$ and $\chi^+_c(L,r) - \sum_{i\in I} \chi^-_{c_i}(L,r)$ are non-positive integers they must be zero, which ends the proof.

4.6. PROPOSITION. Let $L = a\frac{d}{dx} + b \in \underline{R}$. Assume that $\rho_c(L,r_0) < r_0$ and that $\operatorname{ord}^-_c(a,r_0) = 0$. Let ρ be the radius of convergence of the solution of L near c . Then $\rho > \rho_c(L,r_0)$ and $\rho = \rho_c(L,r_0)$ if and only if $\chi^-_c(L,r_0) = 0$.

Proof. If for some $r < r_0$, $\rho_c(L,r) = r$ then, as L has no singularity in the disk $B(c,r^-)$, by the transfer principle [DW 3] the solution of L near c converges in $B(c,r^-)$. If the radius of convergence ρ were greater than r, we would have also $\rho_c(L,r) = \rho > r$ which would contradict the hypothesis. Thus $\rho = r$.

Assume that $\chi^-_c(L,r_0) = 0$. For $r < r_0$ as long as $\rho_c(L,r) > r$ we have $\chi^-_c(L,r) = 0$ by proposition 3.5. By theorem 4.2 we obtain $\frac{d\log \rho_c(L,r)}{d\log r} = 0$ so $\rho_c(L,r) = \rho_c(L,r_0)$. Therefore we shall have $\rho_c(L,\rho) = \rho$ for $\rho = \rho_c(L,r_0)$.

Assume that $\chi_c^-(L,r_0) \neq 0$, then in some interval (r_1,r_0), $\chi_c^-(L,r)<0$ and otherwise $\chi_c^-(L,r) \leq 0$. From theorem 4.2 we deduce that $\rho_c(L,r)$ is a decreasing function of r, strictly decreasing in (r_1,r_0), so $\rho_c(L,r) > \rho_c(L,r_0)$. And therefore if $\rho_c(L,\rho) = \rho$ we have $\rho > \rho_c(L,r_0)$.

4.7. Example. Let $K = \mathbb{C}_p$. Let $L = \frac{d}{dx} - x^{p-1} = \exp(x^p/p) \circ \frac{d}{dx} \circ \exp(-x^p/p)$.

A solution $u(x)$ of L near t is given by $u(x) = \exp(\frac{x^p}{p} - \frac{t^p}{p})$.
Set $x = t+y$, $u(t+y) = \exp(t^{p-1}y + \frac{p-1}{2} t^{p-2}y^2 + \ldots + ty^{p-1} + \frac{y^p}{p}) =$ $\exp(t^{p-1}y)\exp(\frac{p-1}{2} t^{p-2}y^2) \ldots \exp(y^p/p)$.

If the functions $\exp \frac{1}{p}\binom{p}{i} t^{-i}y^i$, $1 \leq i \leq p$, have different radii of convergence, the radius of convergence of u near t will be the infinum of these radii. So for t generic with $|t| = r$, we get

$$\rho_0(L,r) = \inf\left(\inf_{1\leq i\leq p-1} (p^{-1/(p-1)}/r^{p-i})^{1/i}, p^{-1/(p-1)}\right) = \begin{cases} p^{-1/(p-1)} & r \leq 1 \\ p^{-1/(p-1)}/r^{p-1} & r \geq 1 \end{cases}$$

(The value for $r = 1$ is obtained by continuity).
By formula (4.2) we obtain

$$\chi_0^{\pm}(L,r) = 0 \qquad p^{-1/(p-1)} < r < 1$$

$$\chi_0^{\pm}(L,r) = -(p-1) \qquad 1 < r$$

$$\chi_0^{-}(L,1) = 0, \qquad \chi_0^{+}(L,1) = -(p-1).$$

Then by proposition 4.5 iii) we see that for some c with $|c| = 1$, we must have $\chi_c^-(L,1) \neq 0$.

The method of proof of theorem 4.2 permits us to find the residue classes where the index of L is not 0. We use the notations of § 4.4. We see easily that

$$|b_i|_0(1) = 1 \quad \text{for} \quad 0 \leq i \leq p-1$$

and a well known formula gives

$$p!\, b_p \equiv -(x^{p(p-1)} + (p-1)!) \mod p\, \mathbb{Z}_p[x]$$

$$b_p \equiv \frac{1}{p}(x^{p(p-1)}-1) \equiv \frac{1}{p}(x-1)^p(x-2)^p \ldots (x-(p-1))^p \mod \mathbb{Z}_p[x] .$$

Thus $|b_p|_0(1) > \sup_{0\leqslant i\leqslant p-1} |b_i|_0(1)$, and thus for $|c| = 1$

$$\chi_c^-(L,1) = -\frac{1}{p}\,\mathrm{ord}_c^-(b_p,1)$$

and $\chi_c^-(L,1) = -1$ for $c = 1,2\ldots(p-1)$.

Consider the situation in the disk $B(1,1^-)$, let ρ_r denote the radius of convergence of the solution near the generic point of the circumference $C(1,r)$. For $p^{-1/p} < r < 1$ one has

$$|b_p|_1(r) > 1 > \sup_{0\leqslant i\leqslant p-1} |b_i|_1(r)$$

and $\mathrm{ord}_1^{\pm}(b_p,r) = p$. So for $p^{-1/p} < r < 1$ if $\rho_r < r$ one has

$$\chi_1^{\pm}(L,r) = -1$$

and therefore

$$\rho_r = \rho_1/r = p^{-1/(p-1)}/r .$$

One has $\rho_r = r$ for $r = p^{-1/2(p-1)} > p^{-1/p}$. We can conclude that the solution near 1, $u(x) = \exp(\frac{x^p-1}{p})$ has radius of convergence $p^{-1/2(p-1)}$. But

$$u(1+y) = \exp(y+\frac{y^p}{p}) \exp \frac{p-1}{2}\, y^2 \ldots \exp y^{p-1} .$$

As $\exp \frac{1}{p}\binom{p}{i} y^i$ has radius of convergence $\geqslant p^{-1/2(p-1)}$ for $2 \leqslant i \leqslant p-1$, one deduce that $\exp(y+\frac{y^p}{p})$ has radius of convergence $\geqslant p^{-1/2(p-1)}$. This in ideed a well known fact which is usually proven using the Artin-Hasse exponential function.

Conversely one could have used the fact that the solution $\exp(\frac{x^p-1}{p})$ near 1 has radius of convergence greater than the solution near the generic point t (with $|t| = 1$) to conclude that $\chi_1^-(L,1) \neq 0$ and likewise for the residue classe $2\ldots p-1$. From the fact that

$$\chi_1^-(L,1) + \ldots + \chi_{p-1}^-(L,1) \geqslant \chi_0^+(L,1) = -(p-1)$$

one would have concluded that

$$\chi_1^-(L,1) = \ldots = \chi_{p-1}^-(L,1) = -1 .$$

4.8. <u>Case of a disk of center</u> ∞ .

Let $r \in |\Omega^*|$ denote $B_c(r^+) = \mathbb{P}(\Omega) - B(c,r^-)$

$$B_c(r^-) = \mathbb{P}(\Omega) - B(c,r^+) .$$

Let $L = a\frac{d}{dx} + b$ with $a,\ b \in K(x)$ without poles in $B_c(r^+)$, (in particular without pole at ∞) .

Let $\rho_c(L,r)$ be the radius of convergence of a solution of L near the generic point t_r on the circumference $C(c,r)$. Assume that $\rho_c(L,r) < r$. Let $R = 1/r$. The change of variable $y = \frac{1}{x-c}$ defines an isomorphism between $H(B_c(r^+))$ and $H_0(R^+)$ (resp. between $H(B_c(r^-))$ and $H_0(R^-)$). The differential operator L becomes

$$\tilde{L} = \tilde{a}\frac{d}{dy} + \tilde{b}$$

with $\tilde{a}(y) = -a(c+\frac{1}{y})y^2$ and $\tilde{b}(y) = b(c+\frac{1}{y})$.

Near the generic point $\tilde{t}_R = 1/(t_r - c)$ a solution of $\tilde{L}$ has radius of convergence $\rho_0(\tilde{L},R) = \rho_c(L,r)/r^2$.

Now $\tilde{L}$ need hot have polynomial coefficients, but there exists $\Delta \in K[x]$ without zeros in $B(0,R^+)$ such that $\Delta\tilde{L} \in \underline{R}$, and multiplication by Δ is an invertible linear endomorphism of $H_0(R^+)$ as well as $H_0(R^-)$. So by formula (4.2 +)

$$\left(\frac{d\log\rho_0(L,R)}{d\log R}\right)^+ = \chi_0^+(\Delta\tilde{L},R) + \mathrm{ord}_0^+(\Delta\tilde{a},R) = \chi_0^+(\tilde{L},R) + \mathrm{ord}_0^+(\tilde{a},R)$$

$$= -\left(\frac{d\log\rho_c(L,r)}{d\log r}\right)^- + 2 = \chi(L;H(B_c(r^+))) - \mathrm{ord}_c^-(a,r) + 2$$

Thus

$$-\left(\frac{d\log\rho_c(L,r)}{d\log r}\right)^- = \chi(L\,;H(B_c(r^+))) - \mathrm{ord}_c^-(a,r)$$

and one can prove in the same way

$$-\left(\frac{d\log\rho_c(L,r)}{d\log r}\right)^+ = \chi(L\,;H(B_c(r^-))) - \mathrm{ord}^+(a,r) .$$

These formulae will help us to understand the formula that we shall obtain in proposition 4.11. They can be interpreted in the following way : $-\operatorname{ord}_c^-(a,r)$ is the number of zeros of a in the disk $B_c(r^+)$, while $-\left(\frac{d \log \rho_c(L,r)^-}{d \log r}\right)$ can be viewed as the derivative of $\log \rho_c(L,r)$ with respect to the logarithm of the radius in the direction of the "exterior normal".

For an alternate proof of these formulae see the proof of lemme 4.10.

4.9. A set A of the form

$$A := B(c,r) - \bigcup_{i=1}^{n} B(c_i,r_i) ,$$

where the disk considered can be circumferenced or not, but have their radii in $|\Omega^*|$, will be called a <u>Laurent domain</u>.

We can assume that the $B(c_i,r_i)$ are all disjoint. The disks $B(c_i,r_i)$ will be called the holes of A . The circumference $C(c_i,r_i)$ will be called the boundary of $B(c_i,r_i)$ and will be denoted $\partial B(c_i,r_i)$.

We shall say that A is a <u>closed Laurent domain</u> if one has

$$A = B(c,r^+) - \bigcup_{i=1}^{n} B(c_i,r_i^-) .$$

4.10. LEMMA. <u>Let</u> A <u>be a Laurent domain and let</u> T <u>be a hole of</u> A . <u>Let</u> $L \in \underline{R}$ <u>and assume that</u> L <u>has an index as endomorphism of</u> $H(A \cup T)$ <u>and that</u> L <u>has no solution converging in the generic disk of the boundary</u> ∂T <u>of</u> T . <u>Then</u> L <u>has an index as endomorphism of</u> $H(A)$ <u>and</u>

$$(L ; H(A)) = \chi(L ; H(A \cup T)) - \chi(L ; H(T)) .$$

<u>Proof</u>. Denote by $H^0(T^c)$ the set of analytic elements on T^c which are 0 at ∞ . Let Δ be the generic disk of T. By theorem 4.16 of [Ro 2] one knows that, under our hypothesis, L has index as endomorphism of $H(T)$ as well as endomorphism of $H(\Delta)$ and that further

$$\chi(L ; H(\Delta)) = 0 .$$

By the Mittag-Leffler theorem

$$H(A) = H(A \cup T) \oplus H^{0}(T^{c})$$
$$H(\Delta) = H(T) \oplus H^{0}(T^{c}) \ .$$

Consider then the commutative diagrams

$$(4.10.1) \qquad \begin{array}{ccccccccc} 0 & \longrightarrow & H(A \cup T) & \longrightarrow & H(A) & \longrightarrow & H^{0}(T^{c}) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle L} & & \downarrow{\scriptstyle L} & & \downarrow{\scriptstyle \bar{L}} & & \\ 0 & \longrightarrow & H(A \cup T) & \longrightarrow & H(A) & \longrightarrow & H^{0}(T^{c}) & \longrightarrow & 0 \end{array}$$

$$(4.10.2) \qquad \begin{array}{ccccccccc} 0 & \longrightarrow & H(T) & \longrightarrow & H(\Delta) & \longrightarrow & H^{0}(T^{c}) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle L} & & \downarrow{\scriptstyle L} & & \downarrow{\scriptstyle \tilde{L}} & & \\ 0 & \longrightarrow & H(T) & \longrightarrow & H(\Delta) & \longrightarrow & H^{0}(T^{c}) & \longrightarrow & 0 \end{array}$$

where $\bar{L}$ and $\tilde{L}$ are defined by reduction. It turns out that $\tilde{L}$ and $\bar{L}$ are the same operator : for $u \in H^{0}(T^{c})$

$$\bar{L}u = \tilde{L}u = (Lu)_{T}$$

is the singular part of Lu associated to the hole T .

From (4.10.2) and lemma 3.3 one deduces that $\tilde{L}$ has an index

$$\chi(\tilde{L}) = \chi(L ; H(\Delta)) - \chi(L ; H(T)) = - \chi(L ; H(T)) \ .$$

Then from (4.10.1) and lemma 3.3 we deduce that L has an index as endomorphism of $H(A)$ and

$$\chi(L ; H(A)) = \chi(L ; H(A \cup T)) + \chi(\bar{L})$$
$$= \chi(L ; H(A \cup T)) - \chi(L ; H(T)) \ .$$

4.11. PROPOSITION. <u>Let</u> $A = B(c,r^{+}) - \bigcup_{i=1}^{n} B(c_{i},r_{i}^{-})$ <u>be a closed Laurent domain</u>. <u>Let</u> $L = a\frac{d}{dx} + b \in R$. <u>Assume that</u> $\rho_{c}(L,r) < r$ (<u>resp</u>. $\rho_{c_i}(L,r_{i}) < r$ <u>for all</u> i) <u>then</u> L <u>has an index as endomorphism of</u> H(A) <u>and</u>

$$\chi(L;H(A)) = \left(\frac{d \log \rho_{c}(L,r)^{+}}{d \log r}\right) - \sum_{i=1}^{n}\left(\frac{d \log \rho_{c_i}(L,r_{i})^{-}}{d \log r_{i}}\right) - \sum_{\alpha \in A} \mathrm{ord}_{\alpha}\, a \ .$$

Proof. Using theorem 4.2 and lemma 4.9 we prove by induction on n that

$$\chi(L\,;\,H(A)) = \left(\frac{d\log\rho_c(L,r)}{d\log r} + \right) - \sum_{i=1}^{n}\left(\frac{d\log\rho_{c_i}(L,r_i)}{d\log r_i} - \right) - $$

$$[\mathrm{ord}_c^+(a,r) - \sum_{i=1}^{n}\mathrm{ord}_c^-(a,r_i)]$$

and then observe that

$$\sum_{\alpha\in A}\mathrm{ord}_\alpha\, a = \#\text{ zeros of } a \text{ in } A = \mathrm{ord}_c^+(a,r) - \sum_{i=1}^{n}\mathrm{ord}_c^-(a,r_i)\,.$$

4.12. Generalization : operator with coefficients analytic elements.

It is easily seen that theorem 4.2 and proposition 4.3 are still true if we assume that $L = a\frac{d}{dx} + b$ has its coefficients a, b in $H_c^+(r+\varepsilon)$ for some $\varepsilon > 0$. (This case can be even reduced to the case where $a \in K[x]$) .

Lemme 4.10 is also true if we assume that the coefficients of L belong to $H(A \cup T)$.

Now in proposition 4.11 we could assume that the coefficients a and b of L belong to $H(A_\varepsilon)$, with $A_\varepsilon = B(c,(r+\varepsilon)^+) - \bigcup_{i=1}^{n} B(c_i,(r_i-\varepsilon)^-)$, for some $\varepsilon > 0$. But then the proof that we have given is no longer available.

Conjecture : Proposition 4.11 is still true under the hypothesis $a,\ b \in H(A_\varepsilon)$.

4.13. Generalization : operators of higher order.

Let t_r be the generic point of the circumference $C(c,r)$. Let $L \in \underline{R}$. We shall say that L has a zero-kernel at t_r if L has no solution converging in the generic disk $B(t_r,r^-)$. It is known (theorem 4.16 of [Ro 2]) that if L has zero-kernel at t_{r_0} then L has index as endomorphism of $H_c(r^+)$ and $H_c(r^-)$ for r close enough to r_0 , but no formula for that index is known. We conjecture that the following formula gives the index.

Let L be of order n, $L = a\frac{d^n}{dx^n} + b_1\frac{d^{n-1}}{dx^{n-1}} + \ldots + b_n$. We denote

$\rho_c(L,r) = \max \rho(u_1)\ldots\rho(u_n)$, the maximum being taken upon all the families of n linearly independent solutions $u_1\ldots u_n$ of L in a neighborhood of t_r , $\rho(u_i)$ being the radius of convergence of u_i .

<u>Conjecture</u> : <u>If</u> L <u>has a zero-kernel at</u> t_r , <u>then</u>

$$\left(\frac{d \log \rho_c(L,r)}{d \log r}\right)^{\pm} = \chi_c^{\pm}(L,r) + \mathrm{ord}_c^{\pm}(a) .$$

Then to compute the index one has to be able to determine $\rho_c(L,r)$. In the case of operators of order one this can be easily done using the formal solution as we have seen in example 4.7 and as we will see more generally in § 5. In the case of order greater than one, we expect that it will be possible to compute $\rho_c(L,r)$ at least if c is an irregular singular point and r is small enough using the formal solution of Turritin (see [Ka 1]) .

This conjecture is supported by the following example where $L = L_1 \circ L_2$ with L_1 and L_2 in $\underline{R}$ of first order.

4.13.1. LEMMA. <u>Let</u> $L = L_1 \circ L_2$ <u>with</u> L_1 , $L_2 \in \underline{R}$ <u>of first order</u>. <u>Then</u> L <u>has a zero kernel at</u> t_r <u>if and only if</u> L_1 <u>and</u> L_2 <u>have both a zero kernel at</u> t_r <u>and then</u>

$$\rho_c(L,r) = \rho_c(L_1,r) + \rho_c(L_2,r) .$$

<u>Proof</u>. Let us write $t := t_r$, $\rho_1 := \rho_c(L,r)$, $\rho_2 := \rho_c(L_2,r)$ and let $\mathrm{Ker}_t\, L$ (resp. $\mathrm{Ker}_t\, L_i$) be the kernel of L (resp. L_i) in a neighborhood of t .

Denote by W_t^{ρ} the space of bounded analytic funcitons in the disk $B(t,\rho^-)$ for $\rho \leqslant r$. It is known (theorem 4.16 [Ro 2]) that L is an isomorphism of W_t^{ρ} if and only if L is injective in W_t^{ρ} . It is also known (theorem 3.5 [Ro 2]) that L has a zero-kernel at t if and only if L is injective in W_t .

If L has a zero-kernel at t, as $\mathrm{Ker}_t\, L_2 \subset \mathrm{Ker}_t\, L$, L_2 has also a zero-kernel at t. So L and L_2 are isomorphisms of W_t^r and therefore L_1 also, which implies that L_1 has a zero-kernel at t .

If L_1 and L_2 have zero kernel at t, L_1 and L_2 are isomorphisms of W_t^r , therefore L also, which implies that L has a zero-kernel at t.

If $\rho > \max(\rho_2,\rho_1)$, L_1 and L_2 are isomorphisms of W_t^ρ, therefore L also, which implies that no $u \in \mathrm{Ker}_t L$ has radius of convergence $> \max(\rho_2,\rho_1)$. Let $u_1 \in \mathrm{Ker}_t L$, $u_2 \in \mathrm{Ker}_t L_2$, $u_1 \neq 0$ $u_2 \neq 0$. Then $u_2 \in \mathrm{Ker}_t L$. If $\rho_1 < \rho_2$, a second element of $\mathrm{Ker}_t L$ is

$$v = u_2 \int \frac{u_1}{u_2}\, dx$$

and so $\rho(v) \geqslant \rho(u_1) = \rho_1$.

If $\rho_1 = \rho_2$, then for all $u \in \mathrm{Ker}_t L$, $\rho(u) = \rho_1 = \rho_2$ and thus $\rho_c(L,r) = \rho_1\rho_2$. If $\rho_1 < \rho_2$, wronskian $(u_2,v) = u_1u_2$ and so $\rho(\text{wronskian}) = \rho_1 \geqslant \rho(v)$ so $\rho(v) = \rho_1$ and $\rho_r(L) = \rho_1\rho_2$.

If $\rho_2 < \rho_1$, L_2 is an isomorphism of $W_t^{\rho_1}$, but not L_1, and so neither is L, which implies that there exists $u \in \mathrm{Ker}_t L$ with $\rho(u) = \rho_1$. Therefore again $\rho_c(L,r) = \rho_1\rho_2$.

4.13.2 COROLLARY. <u>If</u> $L = L_1 \circ L_2$, <u>with</u> L_1, $L_2 \in \underline{R}$ <u>of first order</u>, <u>the conjecture is true</u>.

<u>Proof</u>. Let $L_1 = a_1 \frac{d}{dx} + b_1$, $L_2 = a_2 \frac{d}{dx} + b_2$, then $a = a_1a_2$, so $\mathrm{ord}_c^{\pm} a = \mathrm{ord}_c^{\pm} a_1a_2 = \mathrm{ord}_c^{\pm} a_1 + \mathrm{ord}_c^{\pm} a_2$.

Further by lemma 3.2

$$\chi_c^{\pm}(L,r) = \chi_c^{\pm}(L_1,r) + \chi_c^{\pm}(L_2,r)$$

and by lemma 4.12.1

$$\left(\frac{d \log \rho_r(L)}{d \log r}\right)^{\pm} = \left(\frac{d \log \rho_r(L_1)}{d \log r}\right)^{\pm} + \left(\frac{d \log \rho_r(L_2)}{d \log r}\right)^{\pm}$$

and so the conjecture is a consequence of theorem 4.2.

4.14. <u>Reduction to normal form</u>.

Let L_1, L_2 be two n-dimensional linear differential operators defined over $H_0(r^+)$, i.e. two $n\times n$ matrices whose entries are polynomials in d/dx with coefficients in $H_0(r^+)$. These then are endomorphisms of $H_0(r^+) \overset{\text{def}}{=\!=} G$. Let V be an $n\times n$ matrix with coefficients in $H_0(r^+)$ and non-trivial determinent. Clearly V acts on G and provides an endomorphism of G into itself which is injective and has an index.

Now assume that

$$L_1 \circ V = V \circ L_2 \, .$$

(We shall say that L_1 and L_2 are equivalent). It is then obvious that L_1 and L_2 have the same index if either exists. Indeed we have

$$\chi(L_1,G) + \chi(V,G) = \chi(L_1 \circ V,G) = \chi(V \circ L_2,G) = \chi(L_2,G) + \chi(V,G) \, .$$

The application of this remark is that it permits the replacement of a differential operator by a Turrittin normal form (see [Ba]) in the calculation of index.

§ 5. RATIONAL COHOMOLOGY AND ANALYTIC COHOMOLOGY (DWORK'S COHOMOLOGIES).

5.1. Let $K = \mathbb{C}_p$. Let S be a finite subset of $\mathbb{P}(\mathbb{C}_p)$ with $\infty \in S$. Let $f \in \mathbb{C}_p(x)$ with $|f|_{gauss} := |f|_0(1) < 1$, and assume that the poles of f belong to S. Let $\alpha_i \in \mathbb{Z}$, $a_i \in S$, $1 < i < s$. Write

$$F = \prod_{i=1}^{s} (x-a_i)^{\alpha_i} \exp \pi \, f(x)$$

where $\pi = (-p)^{1/(p-1)}$.

Let $\underline{L}$ be the space of elements in $\mathbb{C}_p(x)$ with poles only in S. Then the differentiation d sends $F\underline{L}$ into $F\,\underline{L}\,dx$ because $F'/F \in L$. We shall be interested in the rational cohomology $F\,\underline{L}\,dx/d(F\underline{L})$.

In order to define a Frobenius mapping, one cannot work with the rational cohomology but one must work with the analytic cohomology.

Let $A = B(0,1^+) - \bigcup_{j=1}^{m} B(c_j,1^-)$, with $|c_i - c_j| = 1$, $i \neq j$,

and where the disks $B(c_j,1^-)$ are precisely the residue classes containing points of S, $(\mathbb{P}(\overline{\mathbb{C}}_p) - \overline{A} = \overline{S})$. Let

$A_\varepsilon = B(0,(1+\varepsilon)^+) - \bigcup_{i=1}^{s} B(c_i,(1-\varepsilon)^-)$ for $\varepsilon > 0$. The Washnitzer-Monsky's dagger space $\mathcal{H}^\dagger(A)$ is

$$\mathcal{H}^{\dagger}(A) := \bigcup_{\varepsilon > 0} H(A_\varepsilon) .$$

We shall say that the element of $\mathcal{H}^{\dagger}(A)$ are overconvergent analytic elements on A .

We shall consider the analytic cohomology $F\mathcal{H}^{\dagger}(A)dx/d(F\mathcal{H}^{\dagger}(A))$.

These cohomologies are considered by B. Dwork in his study of Zeta function and of L-functions [DW 1] and [DW 2] . We shall give an example of Dwork's techniques in the next paragraph.

An important question is to prove that these cohomologies are finite and to determine if the rational cohomology and the analytic cohomology are isomorphic. We shall answer that question in § 5.6.

5.2. Let $\ell := F\circ\frac{d}{d}\circ F^{-1} = \frac{d}{dx} - F'/F$. Clearly

$$F\underline{L}dx/d(F\underline{L}) \simeq \underline{L}/\ell\underline{L} \quad \text{and} \quad F\mathcal{H}^{\dagger}(A)dx/d(F\mathcal{H}^{\dagger}(A)) \simeq \mathcal{H}^{\dagger}(A)/\ell\mathcal{H}^{\dagger}(A) .$$

As Ker ℓ has dimension at most 1, the finiteness of cohomology is equivalent to the existance of index of ℓ, and the dimension of the cohomology is determined by the index of ℓ.

We shall first compute the index of ℓ in $\mathcal{H}^{\dagger}(A)$ (§ 5.4) then in $\underline{L}$ (§ 5.5), we will then get easily a criterium asserting that the rational and the analytic cohomology are equal.

The method developped in § 4 allows us to compute the index of ℓ in $H(A_\varepsilon)$ (and therefore in $\mathcal{H}^{\dagger}(A)$) provided that ℓ has a zero-kernel at the generic point of the circumference $C(c_i,(1-\varepsilon))$ for all i (and likewise for $C(0,1+\varepsilon)$) . This means more or less that ℓ has an irregular singularity in the corresponding hole $B(c_i,(1-\varepsilon)^-)$ (or that f has poles in this disk). If this is not the case then the method fails to work and in fact we do not expect ℓ to have an index in $H(A_\varepsilon)$ in that case (see lemma 4.10 [Ro 2]). Fortunately A. Adolphson [Ad] shows how to compute the index of a differential operator ℓ when you remove disks containing regular singularities. But then you have an index only if you consider the space of unbounded analytic functions $A(A_\varepsilon)$ on A_ε . Since $\mathcal{H}^{\dagger}(A) = \bigcup_{\varepsilon>0} A(A_\varepsilon)$ this causes no difficulty and supports our use of $\mathcal{H}^{\dagger}(A)$.

5.3. The following lemma allows us to compute the index in the case of a regular singularity.

DEFINITION. We say that $\alpha \in \mathbb{C}_p$ is a non-Liouville number [Cℓ] if

$$\liminf_{n\to+\infty} |\alpha-n|^{1/n} = 1 \quad \text{and} \quad \liminf_{n\to+\infty} |\alpha+n|^{1/n} = 1 .$$

Remark. All algebraic numbers are non-Liouville numbers.

LEMMA. Let $\ell = (x-c)\frac{d}{dx} - \alpha + (x-c)b(x)$ where $c \in \mathbb{C}_p$, $\alpha \in \mathbb{C}_p$ is non Liouville and $b(x) \in \mathbb{C}_p(x)$ without poles in the disk $B(c,r_0^-)$. Assume that ℓ has a formal solution $(x-c)^\alpha y(x)$ with $y(c) \neq 0$ and y analytic in $B(c,r_0^-)$. For $r < r_0$ write $B = B(c,r^-)$ and $\Delta = B(c,r_0^-) - B(c,r^+)$. Then for $r < r_0$ ℓ, as endomorphism of the space functions analytic on Δ (resp. on B), has index 0.

This is a special case of a more general result of Adolphson [Ad]. We can give a simpler proof due to the fact that we deal with an operator of order 1.

Proof. We may assume that $c = 0$. Let η be analytic on Δ (resp. on B), we want to solve the equation $\ell\zeta = \eta$. It is known that ζ is given formally by the formula

$$\zeta(x) = x^\alpha\, y(x) \int x^{-\alpha-1}\, y^{-1}(x)\eta(x)dx$$

where by definition

$$\int \sum_n a_n x^{n-\alpha-1}dx = \sum_n \frac{a_n}{n-\alpha}\, x^{n-\alpha} \quad \text{if } \alpha \text{ is not an integer.}$$

$$= \sum_{n\neq\alpha} \frac{a_n}{n-\alpha}\, x^{n-\alpha} + a_\alpha \log x \quad \text{otherwise} .$$

Note that y never vanishes in its domain of convergence, thus y^{-1} is analytic on $B(0,r_0^-)$ and $y^{-1}\eta$ is analytic on Δ (resp. on B). The hypothesis α non-Liouville implies then that the formal solution is also analytic on Δ (resp. on B) provided there is no logarithmic term. So if $\alpha \notin \mathbb{Z}$ (resp. $\alpha \notin \mathbb{N}$), ℓ is injective and surjective thus $\chi(\ell) = 0$.

If $\alpha \in \mathbb{Z}$ (resp. $\alpha \in \mathbb{N}$) our equation has a solution if and only if $a_\alpha = 0$, and so dim coker $\ell = 1$, but then dim Ker $\ell = 1$ and so $\chi(\ell) = 0$ again.

5.4. Computation of the analytic index.

We do not want to assume that the singularities of the differential operator all lie in A^c. So let us state again our hypothesis.

Let $c_1 \dots c_m \in \mathbb{C}_p$ with $|c_j| < 1$ for all i and $|c_i - c_j| = 1$ for $i \neq j$, and let $A = B(0,1^+) - \bigcup_{j=1}^{m} B(c_j, 1^-)$.

Let $f \in \mathbb{C}_p(x)$ with $|f|_{gauss} < 1$, let $a_i \in \mathbb{C}_p$, $\alpha_i \in \mathbb{Z}$, $1 < i < s$. Define $F := \prod_{i=1}^{s} (x-a_i)^{\alpha_i} \exp \pi f(x)$ and $\ell := F \circ \frac{d}{dx} \circ F^{-1}$. Let $P \in \mathbb{C}_p[x]$ be such that $L = P\ell$ has coefficients without poles in A. Without loss of generality one may assume that all the zeroes of P lie in A.

We define the exponent of L in the residue class $B(c,1^-)$ (resp. the residue class ∞, $\mathbb{P}(\mathbb{C}_p) - B(0,1^+)$) to be $\sum_{|c-a_i|<1} \alpha_i$ (resp. $\sum_{|a_i|<1} \alpha_i$). We shall assume that the exponents of L in the residue classes $B(c_j, 1^-)$, $1 < j < m$, and in the residue class ∞ are all non-Liouville numbers.

The object of this section is to show that, as endomorphism of $\mathcal{H}^\dagger(A)$, L has an index, and to compute $\chi(L ; \mathcal{H}^\dagger(A))$.

We shall first reduce the computation to the case where our differential operator L satisfies the supplementary condition :

Condition (*)

i) In the residue class $B(c_j, 1^-)$, $1 \leqslant j \leqslant m$, L has at most a singularity at c_j and, in the residue class ∞, L has at most a singularity at ∞. This means that if $a_i \in B(c_j, 1^-)$ then $a_i = c_j$, that in $B(c_j, 1^-)$ f has at most a pole at c_j, and that no a_i and no pole of f have absolute value > 1.

ii) Define $d_j := -\min(0, \mathrm{ord}_{\bar{c}_j} \bar{f})$, $1 \leqslant j \leqslant m$, and $d := -\min(0, \mathrm{ord}_\infty \bar{f})$. Where $\bar{f} \in \mathbb{F}_p^{alg}(x)$ is the reduction of f. Then for $1 \leqslant j \leqslant m$

$$d_j = -\min(0, \mathrm{ord}_c f) \quad \text{and} \quad p \nmid d_j \quad \text{if} \quad d_j \neq 0$$

$$d = -\min(0, \mathrm{ord}_\infty f) \quad \text{and} \quad p \nmid d \quad \text{if} \quad d \neq 0 .$$

We shall then compute the index $\chi(L\,;\mathcal{H}^{\dagger}(A))$ for an operator L satisfying condition (*).

5.4.1. <u>Reduction to the case of an operator satisfying condition</u> (*).

Let $B = B(c,1^-)$ be a residue class not contained in A .

a) If $a_i \in B(c,1^-)$, as $\alpha_i \in \mathbb{Z}_p, (\frac{x-a_i}{x-c})^{\alpha_i} = (1+\frac{c-a_i}{x-c})^{\alpha_i}$ defines a function analytic for $|x-c| > |c-a_i|$ which never vanishes. As $|c-a_i| < 1$, $(\frac{x-a_i}{x-c})^{\alpha_i}$ as well as $(\frac{x-a_i}{x-c})^{-\alpha_i}$ belongs to $\mathcal{H}^{\dagger}(A)$ and so multiplication by these functions is an invertible endomorphism of $\mathcal{H}^{\dagger}(A)$. Thus L has index if and only if $(\frac{x-a_i}{x-c})^{-\alpha_i}\circ L\circ(\frac{x-a_i}{x-c})^{\alpha_i}$ has index and their indexes are equal. Therefore we do not change the index of L if we replace $\prod_{|a_i-c|<1}(x-a_i)^{\alpha_i}$ by $(x-c)^{\alpha}$ with $\alpha = \sum_{|a_i-c|<1}\alpha_i$.

Now if $|a_i| > 1$, $(\frac{x}{a_i}-1)^{\alpha_i}$ again is an invertible element of $\mathcal{H}^{\dagger}$. As we do not change L if we replace $(x-a_i)^{\alpha_i}$ by $(\frac{x}{a_i}-1)^{\alpha_i}$, we see as previously that if we suppress the term $(x-a_i)^{\alpha_i}$ inf F we do not change the index of L .

b) Let f_B be the singular part of f corresponding to the residue class B, i.e. f_B is the sum of the singular parts of f corresponding to the poles of f in B . Consider the Laurent expansion of f_B .

$$f_B = \sum_{n=1}^{\infty}\frac{u_n}{(x-c)^n}\ .$$

Let $d_B = 0$ if $|u_n| < 1$ for all n
$= \sup\{n, |u_n| = 1\}$ otherwise.

Define $h_B = \sum_{n=1}^{d_B}\frac{u_n}{(x-c)^n}$. One has $|f_B-h_B|_{gauss} < 1$; and therefore $g_B = \exp\pi(f_B-h_B)$ belong to $\mathcal{H}^{\dagger}(A)$ and is invertible, so again L has the same index as $g_B^{-1}\circ L\circ g_B = P\circ(F/g_B)\circ\frac{d}{dx}\circ(F/g_B)^{-1}$ and F/g_B

has in B only a pole at c, with $\mathrm{ord}_c f = \mathrm{ord}_{\bar{c}} \bar{f}$ if $\mathrm{ord}_{\bar{c}} \bar{f} < 0$ and no pole in B if $\mathrm{ord}_{\bar{c}} \bar{f} \geqslant 0$.

The case of the residue class ∞ can be treated in the same way.

c) Suppose now that $d_B \neq 0$ and $p | d_B$, $d_B = p\delta$. Choose $v \in \mathbb{C}_p$ such that $v^p = u_{p\delta}$ (with u_n as defined in § b). Then it is well known that $\phi = \exp\pi(\frac{u_{p\delta}}{(x-c)^{p\delta}} - \frac{v}{(x-c)^{\delta}})$ belongs to $\mathcal{H}^{\dagger}$ and is invertible. So considering $\phi^{-1} \circ L \circ \phi$ we will have an operator with same index as L but we will have reduced $-\mathrm{ord}_{\bar{c}} \bar{f}$. Repeating this procedure we can obtain eventually either $\mathrm{ord}_{\bar{c}} \bar{f} > 0$ or $p \nmid \mathrm{ord}_{\bar{c}} \bar{f}$.

5.4.2. <u>Computation of the index when L satisfies condition</u> (*).

For simplicity write $B^+ := B(0,1^+)$, $B_j := B(c_j,1^-)$ $1 \leqslant j \leqslant m$.

Let $Q(x) = \prod_{j=1}^{m} (x-c_j)^{d_j+1}$.

Then QL has coefficients without poles in B^+. For $0 \leqslant s \leqslant m$ define $A_s := B^+ - \bigcup_{j=1}^{s} B_j$. Thus $A_0 = B^+$ and $A_m = A$. We shall prove by induction on s that QL has an index in $\mathcal{H}^{\dagger}(A_s)$ and we shall compute that index.

a) Computation of $\chi(QL, \mathcal{H}^{\dagger}(B^+))$.

If $d \neq 0$, let t be a generic point on the circumference $C(0,r)$ with $r > 1$. Let $x = t + y$. Near t QL has a solution

$$\prod_{j=1}^{m} (1 + \frac{y}{t-c_j})^{\alpha_j} \exp\pi(f(t+y) - f(t)) .$$

As $\alpha_j \in \mathbb{Z}_p$, $(1 + \frac{y}{t-c_j})^{\alpha_j}$ has radius of convergence $|t-c_j| = r$.

$\exp\pi(f(t+y)-f(t)) = \exp\pi \sum_{\nu=1}^{\infty} ((f^{(\nu)}(t)/\nu!)y^{\nu})$. As t is generic, for all ν

$$|f^{(\nu)}(t)/\nu!| = |f^{(\nu)}/\nu!|_0(r) \leqslant r^{d-\nu}$$

and as $p \nmid d$

$$|f'(t)| = |f'|_0(r) = r^{d-1},$$

therefore the radius of convergence of $\exp(f(t+y)-f(t))$ is $1/r^{d-1}$.

We deduce from theorem 4.2 that for all $r > 1$

$$\chi(QL, H(B(0,r^+))) = -(d-1) - \mathrm{ord}_0^+(QP,r)$$
$$= -(d-1) - \sum_{j=1}^{m} (d_j+1) - \deg P$$

($\mathrm{ord}_0(P,r) = \deg P$ because all the zeros of P lie in A).

This index does not depend on r. Choose $R > 1$, and let E be a complementary subspace of $QL(H(B(0,R^+))$ in $H(B(0,R^+))$. For $1 < r < R$, $H(B(0,R^+))$ is dense in $H(B(0,r^+))$. Applying lemma 3.8, we see that for all $r \in (1,R)$,

$$\mathrm{Ker}(QL, H(B(0,r^+)) = \mathrm{Ker}(QL, H(B(0,R^+)) \ (= \{0\})$$

and that E is complementary to $\mathrm{Im}(QL, H(B(0,r^+))$ in $H(B(0,r^+))$.

Therefore

$$\mathrm{Ker}(QL, \mathcal{H}^\dagger(B^+)) = \mathrm{Ker}(QL, H(B(0,R^+))$$

and E is complementary to $\mathrm{Im}(QL, \mathcal{H}^\dagger(B^+))$ in $\mathcal{H}^\dagger(B^+)$, which shows that QL has an index in $\mathcal{H}^\dagger(B^+)$ with

$$\chi(QL, \mathcal{H}^\dagger(B^+)) = -(d-1) - \sum_{j=1}^{m} (d_j+1) - \deg P .$$

If $d = 0$, then $x\ell = x\frac{1}{P}L$ has near infinity the formal solution $x^\alpha y$ where $\alpha = \sum_{j=1}^{m} \alpha_j$ is a non-Liouvielle number and $y(x) = \prod_{j=1}^{m} (1 - \frac{c_j}{x})^{\alpha_j} \exp\pi(f(x) - f(\infty))$ is analytic for $|x| > 1$ with $y(\infty) = 1$. The version of lemma 5.3 for the residue class ∞ tells us that for $1 < r < R$, if Δ denotes the annuls $\Delta = B(0,R^-) - B(0,r^+)$, and $A(\Delta)$ (resp. $A_0(B^c(0,r^+))$) denotes the space of functions analytic on Δ (resp. analytic on $B^c(0,r^+)$ and zero at ∞),

$$\chi(x\ell, A(\Delta)) = 0, \quad \chi(x\ell, A_0(B^c(0,r^+)) = 0 .$$

As $\frac{1}{x}QP$ is an invertible element of $A(\Delta)$,

$$(QL, A(\Delta)) = \chi(\frac{QP}{x}x\ell, A(\Delta)) = \chi(x\ell, A(\Delta)) = 0 .$$

Let $N = \deg \frac{1}{x} QP = -1 + \sum_{j=1}^{m} (d_j+1) + \deg P$, then

$$\chi(QL\,;\, A_0(B^c(0,r^+)),\ x^N A_0(B^c(0,r^+)) = 0\,.$$

Let $\gamma : x^N A_0(B^c(0,r^+) \longrightarrow A_0(B^c(0,r^+))$ be the cut-off operator

$$\sum_{n=-\infty}^{N-1} a_n x^n \longmapsto \sum_{n=-\infty}^{-1} a_n x^n\,.$$

It is clear that $\chi(\gamma) = N$ and then

$$\chi(\gamma\circ QL\,;\, A_0(B^c(0,r^+)) = N\,.$$

By the Mittag-Leffler theorem

$$A(\Delta) = A(B(0,R^-)) \oplus A_0(B^c(0,r^+))\,.$$

Consider the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & A(B(0,R^-)) \arrow[r] \arrow[d, "QL"] & A(\Delta) \arrow[r] \arrow[d, "QL"] & A_0(B^c(0,r^+)) \arrow[r] \arrow[d, "\overline{QL}"] & 0 \\
0 \arrow[r] & A(B(0,R^-)) \arrow[r] & A(\Delta) \arrow[r] & A_0(B^c(0,r^+)) \arrow[r] & 0
\end{tikzcd}$$

where $\overline{QL}$ is defined by reduction. One sees that $\overline{QL} = \gamma\circ QL$.
Therefore, by lemma 3.3

$$\chi(QL, A(B(0,r^-)) = \chi(QL, A(\Delta)) - \chi(\gamma\circ QL, A_0(B^c(0,r^+)) = -N\,.$$

This index does not depend on r. The same proof used previously shows that QL has index in $\mathcal{H}^\dagger(B^+)$ with

$$\chi(QL,\mathcal{H}^\dagger(B^+)) = -N = 1 - \sum_{j=1}^{m}(d_j+1) - \deg P = -(d-1) - \sum_{j=1}^{m}(d_j+1) - \deg P\,.$$

b) Computation of $\chi(QL,\mathcal{H}^\dagger(A_s))$, $1 \leqslant s \leqslant m$.

Denote by γ_j the cut-off operator

$$\gamma_j : \sum_{n=-\infty}^{+\infty} a_n(x-c_j)^n \longmapsto \sum_{n=-\infty}^{-1} a_n(x-c_j)^n\,.$$

By the Mittag-Leffler theorem

$$\mathcal{H}^\dagger(A_s) = \mathcal{H}^\dagger(A_{s-1}) \oplus \mathcal{H}_0^\dagger(B_s^c)\,.$$

Consider the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{H}^\dagger(A_{s-1}) & \longrightarrow & \mathcal{H}^\dagger(A_s) & \longrightarrow & \mathcal{H}_0^\dagger(B_s^c) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle QL} & & \downarrow{\scriptstyle QL} & & \downarrow{\scriptstyle \overline{QL}} & & \\ 0 & \longrightarrow & \mathcal{H}^\dagger(A_{s-1}) & \longrightarrow & \mathcal{H}^\dagger(A_s) & \longrightarrow & \mathcal{H}_0^\dagger(B_s^c) & \longrightarrow & 0 \end{array}$$

where $\overline{QL}$ is defined by reduction. One sees that $\overline{QL} = \gamma_s \circ QL$. By lemma 3.3, if $\gamma_s \circ QL$ has an index then

$$\chi(QL, \mathcal{H}^\dagger(A_s)) = \chi(QL, \mathcal{H}^\dagger(A_{s-1})) + \chi(\gamma_s \circ QL, \mathcal{H}_0^\dagger(B_s^c)) .$$

We shall compute $\chi(\gamma_s \circ QL, \mathcal{H}_0^\dagger(B_s^c))$.

If $d_s \neq 0$ the same proof as in the case of the residue class ∞ shows that near the generic point r of the circumference $C(c_s, r)$ with $r < 1$, the solution of QL has radius of convergence $1/|f'(t)|_c(r) = r^{d_s+1}$, and by theorem 4.2 we have

$$\chi(QL ; H(B(c_s, r^-))) = d_s + 1 - \mathrm{ord}^-_{c_s}(QP, r) = 0 .$$

As in proof of lemma 4.10, we prove that

$$\chi(\gamma_s \circ QL ; H(B^c(c_s, r^-))) = -\chi(QL ; H(B(c_s, r^-))) = 0 .$$

This index does not depend on r, so as previously we deduce that

$$\chi(\gamma_s \circ QL ; \mathcal{H}^\dagger(B_s^c)) = 0 .$$

If $d_s = 0$, observe taht $QP = (x - c_s) \times$ polynomial with no zero in B_s. Further QL has near c_s the formal solution $(x-c_s)^{\alpha_s} y$ with α_s non-Liouville number, and $y = \prod_{j \neq s} \left(1 + \frac{x - c_s}{c_s - c_j}\right)^{\alpha_j} \exp\pi(f(x) - f(c_s))$ converges in B_s because $\alpha_j \in \mathbb{Z}_p$ and f has no poles in B_s.

By lemma 5.3 we have for all $0 < r < R < 1$, with

$$\Delta = B(c_s, R^-) - B(c_s, r^+),$$

$$\chi(QL ; A(\Delta)) = 0 \quad \text{and} \quad \chi(QL ; A(B,(c_s, R^-))) = 0 ,$$

from which we deduce as previously that

$$\chi(\gamma_s \circ QL ; A_0(B^c(c_s, r^+))) = 0 .$$

This index being independent of r we conclude again that

$$\chi(\gamma_s \circ QL ; \mathcal{H}^\dagger(B_s^c)) = 0$$

and therefore for all $s \in [1,m]$ $\chi(QL ; \mathcal{H}^\dagger(A_s)) = \chi(QL ; \mathcal{H}^\dagger(A_{s-1}))$ and so, using the fact that Q is an invertible element of $\mathcal{H}^\dagger(A)$,

$$\chi(L ; \mathcal{H}^\dagger(A)) = \chi(QL ; \mathcal{H}^\dagger(A)) = \chi(QL ; \mathcal{H}^\dagger(B^+)) = -(d-1) - \sum_{j=1}^{m} (d_{j+1}) - \deg P .$$

This formula was obtained under the hypothesis that P has all its zeros in A. If we drop this hypothesis then $\deg P$ must be replaced by

$$\mathrm{ord}_A P = (\text{number of zeros of } P \text{ in } A) = \sum_{a \in A} \mathrm{ord}_a P .$$

5.4.3. <u>Expression of the analytic index</u> (<u>the general case</u>).

We use the notation of the beginning of § 5.4.

Consider $\overline{f} \in \mathbb{F}_p^{alg}(x)$. For $c \notin \overline{A}$ (this includes the case $c = \infty$) define

$$\overline{n}_c := 1 - \inf(0, \sup(\mathrm{ord}_c(f - \phi^p + \phi) ; \phi \in \mathbb{F}_p^{alg}(x))) .$$

PROPOSITION. <u>The differential operator</u> L <u>defined in</u> § 5.4 <u>has index in</u> $\mathcal{H}^\dagger(A)$

$$\chi(L ; \mathcal{H}^\dagger(A)) = 2 - \sum_{c \notin \overline{A}} \overline{n}_c - \sum_{a \in A} \mathrm{ord}_a P .$$

5.5. <u>Computation of the algebraic index</u>.

Let S be a finite subset of $\mathbb{P}(\mathbb{C}_p)$ with $\infty \in S$. Assume that the poles of the coefficients of $L = P\ell$ lie in S (with ℓ and L as defined in § 5.4). Again let $\underline{L}$ denote the space of elements in $\mathbb{C}_p(x)$ with poles only in S.

Define for $c \in S$

$$n_c := 1 - \inf(0, \mathrm{ord}_c f) .$$

PROPOSITION. <u>The differential operator</u> L <u>defined in</u> § 5.4 <u>has index in</u> $\underline{L}$

$$\chi(L ; \underline{L}) = 2 - \sum_{c \in S} n_c - \sum_{a \notin S} \mathrm{ord}_a P .$$

Proof : We may assume that P has no zeros on S . Then
$\deg P = \sum_{a \notin S} \mathrm{ord}_a P$.

Let $S = \{\infty, c_1, \ldots, c_s\}$, then $\underline{L} = \mathbb{C}\,[x, \frac{1}{x-c_1}, \ldots, \frac{1}{x-c_s}]$. For $\bar{m} = (m, m_1, \ldots, m_s) \in \mathbb{N}^{s+1}$ let $\underline{L}(\bar{m})$ deonote the subspace of $\underline{L}$ spanned by $\{x^i\}_{i=0}^{m} \cup (\cup_{j=1}^{s} \{(x-c_j)^{-i}\}_{i=1}^{m_j})$. If $\bar{m}' \in \mathbb{N}^{s+1}$ we say that $\bar{m}' \geqslant \bar{m}$ if $m' \geqslant m$ and $m_i' \geqslant m_i$, $1 \leqslant i \leqslant s$. Definie $\bar{n} \in \mathbb{N}^{s+1}$ as

$$n = n_\infty - 2 + \deg P,\ n_i = n_{c_i} \quad 1 \leqslant i \leqslant s . \tag{5.5.1}$$

It is clear that for $\bar{m} \leqslant \bar{m}'$, $\underline{L}(\bar{m}) \subset \underline{L}(\bar{m}')$, that L maps $\underline{L}(\bar{m})$ into $\underline{L}(\bar{m} + \bar{n})$ and that $\underline{L} = \bigcup_{\bar{m}} \underline{L}(\bar{m})$.

Choose $\bar{m}$ so large that $\mathrm{Ker}(L,\underline{L}) \in \mathrm{Ker}(L,\underline{L}(\bar{m}))$ (hence they are equal) and such that for $\bar{m}' \geqslant \bar{m}$

$$\mathrm{ord}_\infty L x^{m'} = -m' - n,\ \mathrm{ord}_{c_i} L(x-c_i)^{-m_i'} = -m_i' - n_i,\quad 1 \leqslant i \leqslant s . \tag{5.5.2}$$

(This is possible because (5.5.2) is true except for a finite number of m', m_i') .

A consequence of (5.5.2) is that

$$L(\underline{L}(\bar{m})) = L(\underline{L}(\bar{m}')) \cap \underline{L}(\bar{m} + \bar{n}) . \tag{5.5.3}$$

Let G be a complementary subspace of $L(\underline{L}(\bar{m}))$ in $\underline{L}(\bar{m} + \bar{n})$. We claim that G is a complementary subspace of $L(\underline{L}(\bar{m}'))$ in $\underline{L}(\bar{m}' + \bar{n})$. In fact from (5.5.3) we deduce that $G \cap L(\underline{L}, \bar{m}')) = 0$. Further as we are dealing with finite dimensional spaces, it is well known that

$$\chi(L\,;\,\underline{L}(\bar{m}'), \underline{L}(\bar{m}'+\bar{n})) = \dim \underline{L}(\bar{m}') - \dim \underline{L}(\bar{m}'+\bar{n}) = -\,|\bar{n}| = -n - \sum_{j=1}^{s} n_j .$$

So this index does not depend on $\bar{m}'$, and we have the same index for $\bar{m}' = \bar{m}$. As $\mathrm{Ker}(L,\underline{L}(\bar{m})) = \mathrm{Ker}(L,\underline{L}(\bar{m}'))$, we see that

codim of $L(\underline{L}(\bar{m}'))$ in $\underline{L}(\bar{m}'+\bar{n})$ = codim of $\underline{L}(L(\bar{m}))$ in $\underline{L}(\bar{m}+\bar{n}) = \dim G$,

and this proves the claim.

Then it is clear that G is complementary to $L(\underline{L})$ in $L(\underline{L})$ and thus again

$$\chi(L\,;L) = -n - \sum_{j=1}^{s} n_j$$

which, taking into account definition 5.5.1, is the formula to be proved.

Note : This proposition could also be deduced from proposition 1 of [Ad] .

5.6. Comparison of the analytic and the algebraic cohomology.

Consider the situation decribed in § 5.1. We use the notations of § 5.1. If $F \in \mathcal{H}^\dagger(A)$, then the analytic cohomology is trivial as it is $\mathcal{H}^\dagger(A)dx / d\mathcal{H}^\dagger(A)$. So we assume that $F \notin \mathcal{H}^\dagger(A)$ (and therefore if $\ell = F \circ \frac{d}{dx} \circ F^{-1}$, $\mathrm{Ker}(L,\mathcal{H}^\dagger(A)) = 0)$. We assume also that the exponents α_i are non-Liouville numbers (for example algebraic numbers).

For each $c \in S$ define

$$d_c \;:= -\inf(0,\mathrm{ord}_c\, f)$$

and likewise for $c^* \notin \overline{A}$ define

$$\overline{d}_{c^*} \;:= -\inf(0,\mathrm{ord}_{c^*}\overline{f}) \;.$$

THEOREM. Under these hypotheses, the analytic cohomology is finite. The analytic cohomology and the algebraic cohomology are isomorphic if and only if S has exactly one point in each residue class not contained on A (i.e. card S = card $\overline{S}$), for each $c \in S$ $d_c = \overline{d}_{\overline{c}}$ and $p \nmid d_c$. Then a complementary submodule of $d(F\mathcal{H}^\dagger(A))$ in $F\mathcal{H}^\dagger(A)dx$ can be chosen in $F\,\underline{L}\,dx$.

Proof : The finiteness of the analytic cohomology results from the fact that ℓ has index in $\mathcal{H}^\dagger(A)$ (Proposition 5.4.3).

Now we want to know under which conditions $\mathcal{H}^\dagger(A)/\ell\mathcal{H}^\dagger(A) \simeq L/\ell L$. By our hypothesis $\mathrm{Ker}(\ell,\mathcal{H}^\dagger(A)) = \mathrm{Ker}(\ell,\underline{L}) = \{0\}$. So we want to know if

$$\chi(\ell\,;\mathcal{H}^\dagger(A)) = \chi(\ell\,;\underline{L}) \;.$$

For $c^* \in \overline{A}^c$ let n_{c^*} be as defined in § 5.4.3. Then $\overline{n}_{c^*} = 1 + d_{c^*}$ if $p \nmid \overline{d}_{c^*}$ and $\overline{n}_{c^*} < 1 + \overline{d}_{c^*}$ if $p \mid \overline{d}_{c^*}$. Now if $c \in S$ and $\overline{c} = c^*$, one has

$\bar{d}_{c^*} < d_c$, thus

$$\sum_{c \in S, \bar{c}=c^*} n_c = \sum_{c \in S, \bar{c}=c^*} (1+d_c) > 1 + \bar{d}_{c^*},$$

and we have equality if and only if there is only one $c \in S$ such that $\bar{c} = c^*$ and further $d_c = \bar{d}_{c^*}$ and $p \nmid d_c$.

As $\chi(\ell ; \mathcal{H}^\dagger(A)) = 2 - \sum_{c^* \in \bar{A}} \bar{n}_{c^*}$ and $\chi(\ell ; \underline{L}) = 2 - \sum_{c \in S} n_c =$

$2 - \sum_{c^* \notin \bar{A}} \left(\sum_{c \in S, \bar{c}=c^*} n_c \right)$ one can have equality if and only if

- for each $c^* \notin \bar{A}$, $\operatorname{card}(c \in S, \bar{c} = c^*) = 1$
- for each $c \in S$, $d_c = \bar{d}_{\bar{c}}$ and $p \nmid d_c$.

The last statement of the theorem is a consequence of lemme 3.8.

(Strictly we cannot apply lemma 3.8 as $\mathcal{H}^\dagger(A)$ is not a complete metric space, it is only an inductive limit of complete metric spaces. But all we need in the proof of lemma 3.8 is Banach's open mapping theorem which is true in $\mathcal{H}^\dagger(A)$).

§ 6. APPLICATION TO L-FUNCTIONS.

6.1. Notations.

We follow Serre's report [Se] (In order to comply with the tradition, in this § 6.1 only, χ will denote a multiplicative character and not an index, and ψ an additive character, later we will introduce the ψ-mapping of Dwork which is absolutely unrelated). Let p be a prime. Let ζ be a primitive p^{th} root of unity in $\mathbb{C}_p$. For $x \in \mathbb{F}_p$, let $e(x) = \zeta^x$. Denote by k the field of $q = p^m$ elements an by k_r the extension of k of degree r. Let $Tr_r : k_r \longmapsto \mathbb{F}_p$ be the absolute trace. If we put $\psi_r(x) = e(Tr_r(x))$ for $x \in K_r$, then

$$\psi_r : k_r \longrightarrow \mathbb{C}_p^*$$

is an additive character.

For $x \in \mathbb{F}_p^{alg}$ let $\operatorname{Teich}(x) \in Q^{alg} \subset \mathbb{C}_p$ be the Teichmuller representative of x. If we put $\chi(x) := \operatorname{Teich}(x)$ for $x \in k$, then

$\chi|k^*$ is a multiplicative character of k and

$$\chi_r : k_r^* \longrightarrow \mathbb{C}_p^*$$

is a multiplicative character of k_r^* where for $x \in k_r$ we define $\chi_r(x) = \chi(N_{k_r/k}(x)) = \text{Teich } x^{(q^r-1)/(q-1)}$.

Let $g \in k[x]$, $g \neq 0$. Let f, $h \in k(x)$ such that the pole of f and the zeros and poles of h are zeros of g . We define the twisted exponential sum, (TES), $S_r(g;f,h)$ by

$$S_r(g;f,h) := \sum_{x \in k_r, g(x) \neq 0} \psi_r(f(x))\chi_r(h(x)) .$$

To these TES is associated an L-function

$$L(g;f,h;t) = \exp\left(\sum_{r=1}^{\infty} S_r(g;f,h)t^r/r\right) .$$

It is a formal series in t with coefficients in $\mathbb{Q}(\mu_{(q-1)p})$ (where for $n \in \mathbb{N}$, μ_n represents the group of n^{th} roots of unity).
Let $Z := \{\infty\} \cup \{x \in \mathbb{F}_p^{alg} ; g(x) = 0\}$. For $x \in Z$ define

$$n_x := 1 - \inf(0, \sup(\text{ord}_x(f - \phi^r + \phi) ; \phi \in \mathbb{F}_p^{alg}(x)) .$$

6.2. THEOREM. <u>The function</u> $L(g;f,h;t)$ <u>is a polynomial in t of degree</u> $\sum_{x \in Z} n_x - 2$, <u>except if</u> $n_x = 1$ <u>for all</u> $x \in Z$ <u>and</u> $h(x) = ch_1(x)^{q-1}$ <u>with</u> $h_1 \in k[x]$ <u>where it is the trivial situation where for all</u> $r \in \mathbb{N}$ <u>and all</u> $x \in k_r$ <u>with</u> $g(x) \neq 0$, $\psi_r(f(x))\chi_r(h(x)) = \omega^r$ <u>with</u> ω <u>independant of</u> r <u>and</u> x <u>and then</u> $(1-q\omega t)L(g;f,h;t)$ <u>is a polynomial of degree</u> $\text{card}\cdot Z - 1$.

This result is due to A. Weil [We]. (See also [Se]).

We want to explain how this result is related to Dwork's cohomology considered in the previous paragraph. We give a rapid sketch of Dwork's theory.

6.3. <u>Dwork's theory</u>.

We consider first the case where $g(0) = 0$. We shall explain later how the general situation can be reduced to that case.

Let π denote a solution of

$$\pi^{p-1} = -p .$$

It is a well-known result of Dwork (cf. [La] for example) that there is a bijective correspondence between the primitive p^{th} roots of 1 and the solutions π of $\pi^{p-1} = -p$ under which ζ corresponds to π if and only if $\zeta \equiv 1 + \pi \bmod \pi^2$ (in $\mathbb{C}_p$). Further the fonction $\theta(x) := \exp\pi(x-x^p)$, defined near 0, has radius of convergence > 1 and $\theta(1) = \zeta$. Therefore for a suitable choice of π we shall have for $x \in \mathbb{F}_p$: $e(x) = \theta(\mathrm{Teich}(x))$.

Denote $\theta_{p^s}(x) := \theta(x)\theta(x^p)\ldots\theta(x^{p^{s-1}})$; near 0 $\theta_{p^s}(x) = \exp\pi(x-x^{p^s})$.

Then one can express the TES

$$(6.3.1) \quad S_r(g;f,h) = \sum_{x\in k_r,\, g(x)\neq 0} (\mathrm{Teich}\ h(x))^{(q^r-1/(q-1)}\theta_{q^r}(\mathrm{Teich}\ f(x)) .$$

We want now to express this sum as the trace of a linear operator.

Let $\underline{T}$ be the maximal unramified extension of $\mathbb{Q}_p$. Let $g^*\in\underline{T}[x]$, f^*, $h^*\in \underline{T}(x)$ be liftings of g,f,h respectively. Let $A = \{x \in \mathbb{C}_p ; |g^*(x)| = 1\}$, thus $\overline{A} = \{x \in \mathbb{F}_p^{alg} ; g(x) \neq 0\}$. We may assume that the poles of f^* and h^* do not belong to A . Consider the function

$$(6.3.2) \quad G(x) := h^*(x)\left(\frac{h^*(x^q)}{h^*(x)^q}\right)^{1/(q-1)} \exp\pi(f^*(x)^q - f^*(x^q))\,\theta_q(f^*(x)) .$$

$$(\text{Formally} \quad G(x) := \left(\frac{h^*(x^q)}{h^*(x)}\right)^{1/(q-1)} \exp\pi(f^*(x) - f^*(x^q))) .$$ We claim that G belongs to the Washnitzer-Monsky's dagger space $\mathcal{H}^\dagger(A)$. As $h\in K(x)$, $h(x)^q = h(x^q)$ so $|1-h^*(x^q)/h^*(x)^q|_{gauss} < 1$ and further the poles of $h^*(x^q)/h^*(x)^q$ do not belong to A . This proves that $(h^*(x^q)/h^*(x)^q)^{1/(q-1)}$ belongs to $\mathcal{H}^\dagger(A)$, because $(1+u)^{1/(q-1)}$ has radius of convergence 1.

As $f \in k(x)$, $|f^*(x)^q - f^*(x^q)|_{gauss} < 1$ and further the poles of $f^*(x)^q - f^*(x^q)$ do not belong to A, this proves that $\exp\pi(f^*(x)^q - f^*(x^q)) \in \mathcal{H}^\dagger(A)$ because exp u has radius of convergence 1. As θ_q

has radius of convergence > 1 and as $|f^*|_{gauss} = 1$ with the poles of f^* outside A, $\theta_q(f^*(x)) \in \mathcal{H}^\dagger(A)$.

<u>The ψ_q mapping</u>. As $g \in k[x]$, if $\xi \in \mathcal{H}^\dagger(A)$, the mapping

$$x \longmapsto (\psi_q\xi)(x) = \sum_{z^q=x} \frac{1}{q}\, \xi(z) \tag{6.3.3}$$

is defined on a neighborhood A_ε of A and defines an element $\psi_q\xi$ of $\mathcal{H}^\dagger(A)$. (cf. [Dw 1]), and if $\xi(x) = y(x^q)$ then $\psi_q\xi = y$. So ψ_q is a left inverse of the q-th power map $\phi_q : \xi(x) \longmapsto \xi(x^q)$.

<u>The Monsky-Reich trace formula</u>. Consider the endomorphism of $\mathcal{H}^\dagger(A)$

$$\psi_q \circ G : \xi \longmapsto \psi_q(G\xi) .$$

This is a nuclear operator (cf. [Mo]) and by the Monsky-Reich trace formula ([Mo] and [Re]) we have

$$(q^r-1)\mathrm{Tr}(\psi_q \circ G)^r = \sum_{x^{q^r}=x,\, x\in A} G(x)G(x^q)\ldots G(x^{q^{r-1}}) \tag{6.3.4}$$

(Here we use the fact that $0 \notin A$). (Remark : Reich proves the trace formula in the case of many variables under the assumption that the term of highest degree of g is square-free, which means that in the case of one variable one can consider only a polynomial g of degree 1. But it is easily seen that, in the case of one variable, the proof of Reich is still valid under the assumption that the zeroes of g are simple and this is obviously not a restriction).

<u>Dwork's expression of an L-function as a determinant</u>.

$$S_r(g;f,h;t) = (q^r - 1)\mathrm{Tr}(\psi_q \circ G)^r . \tag{6.3.5}$$

For $r \in \mathbb{N}$ let $G_r(x) = G(x)G(x^q)\ldots G(x^{q^{r-1}})$, clearly an element of $\mathcal{H}^\dagger(A)$. We assert that if $x \in A$, $x^{q^r} = x$ then

$$G_r(x) = (\mathrm{Teich}\ h(\bar{x}))^{(q^{r-1})/(q-1)} . \theta_{q^r}(\mathrm{Teich}\ f(\bar{x})) . \tag{6.3.6}$$

Since we are dealing with a product we way consider two basic cases separately.

Case 1. $h^* = 1$ (so $h = 1$).

Formally,

(6.3.7) $$G_r(x) = \exp\pi(f^*(x) - f^*(x^{q^r})) ,$$

a formula which may not be used for the calculation of $G_r(x)$ by composition of the function $u \longmapsto \exp\pi u$ with the function $x \longmapsto f^*(x) - f^*(x^{q^r})$. On the other hand $G_r(x)^p = \exp p\pi(f^*(x)-f^*(x^{q^r}))$ and there we may use composition . Thus if $x \in A$, $x = x^{q^r}$ then $G_r(x)^p = 1$, i.e. $G_r(x)$ is a p^{th} root of unity. On the other hand we may rewrite our formal expression in the form

(6.3.7.1) $$G_r(x) = \theta_{q^r}(f^*(x))\exp\pi(f^*(x)^{q^r} - f^*(x^{q^r})) .$$

Indeed observe that for x and y near 0, one has

$$\theta_q(x)/\theta_q(y) = \exp\pi(x-y)\ \exp\pi(y^q - x^q) .$$

Let R be the radius of convergence of θ_q $(R > 1)$. The functions on both sides of this equation are analytic functions in x and y in the set $\{(x,y) \in \mathbb{C}_p^2 ;\ |x| < R\ \ |y| < R\ \ |x-y| < 1\}$. Therefore by analytic continuation the previous relation is true everywhere in this set.

We use this property of the function θ_q to rewrite $G_r(x)$.

$$G_r(x) = \prod_{i=0}^{r-1} \exp\pi(f^*(x^{q^i})^q - f^*(x^{q^{i+1}})) \prod_{i=0}^{r-1} \theta_q(f^*(x^{q^i}))$$

$$= \exp\pi(\sum_{i=0}^{r-1}(f^*(x^{q^i})^q - f^*(x^{q^{i+1}})))\theta_{q^r}(f^*(x)) \prod_{i=0}^{r-1} \theta_q(f^*(x^{q^i}))/\theta_q(f^*(x)^{q^i})$$

$$= \theta_{q^r}(f^*(x))\ \exp\pi(\sum_{i=0}^{r-1}(f^*(x^{q^i})^q - f^*(x^{q^{i+1}})) + \sum_{i=0}^{r-1}(f^*(x^{q^i}) - f^*(x)^{q^i})$$

$$+ \sum_{i=0}^{r-1}(f^*(x)^{q^{i+1}} - f^*(x^{q^i})^q)$$

$$= \theta_{q^r}(f^*(x))\exp\pi(f^*(x)^{q^r} - f^*(x^{q^r})) .$$

From (6.3.7.1) we may compute for all $x \in A$

$$G_r(x) \equiv 1 + \pi(f^*(x) + \ldots + f^*(x)^{p^{m-1}}) \mod \pi^2 . \tag{6.3.8}$$

In particular then for $x^{q^r} = x$, $x \in A$, $G_r(x)$ is a p^{th} root of unity whose congruence class mod π^2 is known. This suffices to verify (6.3.6) in this case.

Case 2. $f^* = 0$ (so $f = 0$).

In this case we have formally

$$G_r(x) = (h^*(x^{q^r})/h^*(x))^{1/(q-1)} \tag{6.3.9}$$

which means that for $x \in A$,

$$G_r(x)^{q-1} = h^*(x^{q^r})/h^*(x) \tag{6.3.9.1}$$

$$G_r(x) \equiv h^*(x)^{(q^r-1)/(q-1)} \mod p . \tag{6.3.9.2}$$

These two equations show that if $x \in A$, $x^{q^r} = x$ then $G_r(x)$ is a $q-1^{th}$ root of unity whose congruence class is precisely that given by (6.3.6).

From (6.3.1), (6.3.4), (6.3.6) we deduce (6.3.5).

For the nuclear operator $\psi_q \circ G$ one can define the determinant $\det(1-t\psi_q \circ G)$ and one has the relation

$$\det(1-t\psi_q \circ G) = \exp(-\sum_{r\geqslant 1} \mathrm{Tr}(\psi_q \circ G)^r t^r/r) . \tag{6.3.10}$$

Relations (6.7.5) and (6.3.10) together with the definition of the L-function give

$$L(g;f,h;t) = \det(1-t\psi_q \circ G)/\det(1-qt\psi_q \circ G) . \tag{6.3.11}$$

Cohomological interpretation.

Let $F := h^*(x)^{-1/(q-1)} \exp \pi f^*(x)$ and define

$$D := F^{-1} \circ x \frac{d}{dx} \circ F . \tag{6.3.12}$$

Formally $\alpha := \psi_q \circ G = F^{-1} \circ \psi_q \circ F$. As operators on $\mathcal{H}^\dagger(A)$, D and α commute up to a factor q :

(6.3.13) $$\alpha \circ D = qd \circ \alpha \ .$$

We have te commutative diagram with exact rows

(6.3.14)
$$\begin{tikzcd} 0 \arrow[r] & \mathrm{Ker}\, D \arrow[r] \arrow[d, "q^{\alpha}"'] & \mathcal{H}^{\dagger}(A) \arrow[r, "D"] \arrow[d, "q^{\alpha}"'] & \mathcal{H}^{\dagger}(A) \arrow[r] \arrow[d, "\alpha"'] & \mathcal{H}^{\dagger}(A)/D\mathcal{H}^{\dagger}(A) \arrow[r] \arrow[d, "\overline{\alpha}"'] & 0 \\ 0 \arrow[r] & \mathrm{Ker}\, D \arrow[r] & \mathcal{H}^{\dagger}(A) \arrow[r, "D"] & \mathcal{H}^{\dagger}(A) \arrow[r] & \mathcal{H}^{\dagger}(A)/D\mathcal{H}^{\dagger}(A) \arrow[r] & 0 \end{tikzcd}$$

But $\mathrm{Ker}\ D \neq \{0\}$ only if $n_x = 1$ for all $x \in Z$ and $h(x) = h_1(x)^{q-1}$ with $h_1 \in \mathbb{F}_p^{alg}(x)$.

When $\mathrm{Ker}\ D = \{0\}$ then by 5.4.3

$$\dim \mathcal{H}^{\dagger}(A)/D\mathcal{H}^{\dagger}(A) = -\chi(D;\mathcal{H}^{\dagger}(A)) = \sum_{x \in Z} n_x - 2$$

and then from (6.3.11) and (6.3.12) we deduce

$$L(g;f,h;t) = \det(1-t\alpha)/\det(1-tq\alpha) = \det(1-t\overline{\alpha}) \ .$$

As α is a surjective map on $\mathcal{H}^{\dagger}(A)$, because it has a right inverse $G^{-1}\circ\phi_q$, the quotient map $\overline{\alpha}$ is surjective and hence invertible, so $\det(1-t\overline{\alpha})$ is a polynomial of degree $\sum_{x\in Z} n_x - 2$ and this proves the theorem 6.2.

If $\mathrm{Ker}\ D \neq \{0\}$, let u be a basis of Ker D . Then one has $G(x) = \omega u(x^q)/u(x)$ with ω as defined in theorem 6.2, and thus $\alpha u = \omega u$. So if α_1 is the restriction of α to Ker D, one has

$$L(g;f,h,t) = \det(1-t\alpha)/\det(1-tq\alpha) = \det(1-t\overline{\alpha})/\det(1-t\alpha_1) =$$
$$= \det(1-t\overline{\alpha})/(1-q\omega t) \ .$$

Further, as previously, $\overline{\alpha}$ is an invertible map and

$$\dim \mathcal{H}^{\dagger}(A)/D\mathcal{H}^{\dagger}(A) = -\chi(D;\mathcal{H}^{\dagger}(A)) + 1 = \Big(\sum_{x\in Z} 1\Big) - 2 + 1 = \mathrm{Card}\ Z - 1 \ .$$

This ends the verification of theorem 6.2 when $g(0) = 0$.

We now explain how the general case reduces to the case where $g(0) = 0$. Let $c \in k$. It follows directly from the definition of § 6.1 that

$$L(g_c;f_c,h_c;t) = L(g;f,h;t)$$

where $g_c(x) = g(x+c)$, $f(x) = f(x+c)$ and $h_c(x) = h(x+c)$. So, if

$g(c) = 0$ then $g_c(0) = 0$ and the theorem 6.2 is verified. Now let $c \in k$ such that $g(c) \neq 0$. One deduces easily form the definition that

$$L((x-c)g;f,h;t) = (1-\omega(c)t)L(g;f,h;t)$$

where

$$\omega(c) := \mathrm{Teich}(h(c))\,\theta_q(\mathrm{Teich}\, f(c))$$

(with notations of § 6.1, $\omega(c) = \chi(c)\psi_0(c))$.

Further notice that $n_c = 1$ because f is regular at c . We know that $L((x-c)g;f,h;t)$ satisfies theorem 6.2. To prove that $L(g;f,h;t)$ also satisfies theorem 6.2 it is enough to show that $\omega(c)^{-1}$ is a root if $L(g;f,h;t)$.

Suppose that there exists $c' \in k$ such that $\omega(c) \neq \omega(c')$. Then one has

$$L(g;f,h;t) = L((x-c)g;f,h;t)/(1-\omega(c)t)$$
$$= L((x-c')g;f,h;t)/(1-\omega(c')t)$$

and therefore $\omega(c)$ cannot be a pole of $L(g;f,h;t)$.

If may happen that g is never 0 in k (even if g is not a constant) and $\omega(c)$ is constant for all $c \in k$ (even if $n_x \neq 1$ for some $x \in Z$ or $h(x)$ is not of the form $ch_1(x)^{q-1}$) . So consider k_r the extension of k of degree r and let for $c \in k_r$ define

$$\omega_r(c) := \mathrm{Teich}(h(c))^{(q^r-1)/(q-1)}\,\theta_{q^r}(\mathrm{Teich}\, f(c)) .$$

Consider the L-function associated to k_r

$$L_r(g;f,h;t) := \exp\Big(\sum_{s=1}^{\infty} S_{rs}(g;f,h;t)t^{rs}/rs\Big) .$$

It follows from the definition that

$$L_r(g;f,h;t) = \prod_{\nu^r=1} L(g;f,h;\ t^{1/r}),$$

the product being over all r^{th} roots of unity, .

If we have not : $n_x = 1$ for all $x \in Z$ and $h(x) = c\ h_1(x)^{q-1}$ with $h_1 \in k(x)$, then for r big enough, $r > r_0$, ω_r is not constant on k_r, therefore it follows from the previous discussion that $L_r(g;f,h;t)$ is a polynomial of predicted degree. If $L(g;f,h;t)$ is not a polynomial, it is a rational function with one pole

$$L(g;f,h;t) = \prod_{i=1}^{\delta} (w-b_i t)/(1-\omega(0)t)$$

and $b_i \neq \omega(0)$ for all i. If L_r is a polynomial, then there must be an r^{th} root of unity, ν, such that $\omega(0)\nu = b_i$ for some i, $1 \leq i \leq \delta$. Let r run through $\delta+1$ distinct primes each greater than r_0. By the pigeon-hole principle there exists an integer i such that $\omega(0)\nu' = b_i\ \omega(0)\nu''$, where ν'(resp. ν'') is a r'-th (resp. r''-th) root of unity r' and r'' being distinct prime numbers. It is clear that $\nu' = \nu'' = 1$ and $\omega(0) = b_i$, contrary to hypothesis.

Consider now the case $n_x = 1$ for all $x \in Z$ and $h(x) = c\ h_1(x)^{q-1}$ for some $h_1 \in k(x)$. Let $g_0 := xg$. We are in the situation where $\operatorname{Ker} D \neq \{0\}$, and we have seen that $G(x) = \omega u(x^q)/u(x)$ where u is a basis of $\operatorname{Ker} D$. Therefore $\alpha(1) = \psi(G) = \omega$ which shows that ω is an eigenvalue of α and therefore of $\bar{\alpha}$. As

$$L(g_0;f,h,t) = \det(1-t\bar{\alpha})/\det(1-q\omega t)$$

we see that $(1-\omega t)$ divides $L(g_0;f,h;t)$ and therefore

$$\begin{aligned}(1-q\omega t)L(g;f,h;t) &= (1-q\omega t)L(g_0;f,h;t)/(1-q\omega t)\\ &= \det(1-t\bar{\alpha})/(1-\omega t)\end{aligned}$$

is a polynomial.

§ 7. ESTIMATE OF THE p-ADIC MAGNITUDE OF THE ROOTS OF THE L-FUNCTIONS.

We have seen in the previous paragraph that, thanks to Dwork's theory,

$$L(g;f,h;t) = \det(1-t\bar{\alpha})$$

where $\bar{\alpha}$ was a linear mapping in some finite dimensional space. If we write $L(g;f,h;t) = \Pi(1-\omega_i t)$, then the ω_i are the eigenvalues of our mapping $\bar{\alpha}$. The purpose of this paragraph is to obtain an estimate of the magnitude of the coefficient of the matrix of $\bar{\alpha}$ in a suitable basis. From this estimate we shall deduce, by a well-known procedure, an estimate of the p-adic magnitude of the ω_i. With the notation of § 6 we shall consider only the case :

$$g(x) = x,\ h(x) = 1,\ f(x) = \sum_{i=-d}^{d'} u_i x^i \quad \text{with } d,\ d' \geq 1,\ p \nmid d,\ p \nmid d',$$

$u_{-d} \neq 0$, $u_d \neq 0$.

7.1. Notations. (We change slightly the notations of § 6).

We consider $f(x) = \sum_{i=-d}^{d'} \bar{u}_i x^i \in \mathbb{F}_{p^m}[x, \frac{1}{x}]$ with $d, d' \geqslant 1$, $p \nmid d$, $p \nmid d'$ and $\bar{u}_{-d} \neq 0$, $\bar{u}_{d'} \neq 0$. We consider a lifting $f^*(x) = \sum_{i=-d}^{d'} u_i x^i$ with $u_i = \mathrm{Teich}(\bar{u}_i)$. We shall write $f^{*\sigma}(x) = \sum_{i=-d}^{d'} u_i^p x^i$.

Let A be the unit circumference $C(0,1)$. Then $\mathcal{H}^\dagger(A)$ can be identified with the space of Laurent series $\in \mathbb{C}_p[[x, \frac{1}{x}]]$ which converge in an unspecified annulus $\varepsilon < |x| < 1/\varepsilon$ with $\varepsilon < 1$. Then for $\xi = \sum_{n=-\infty}^{+\infty} a_n x^n \in \mathcal{H}^\dagger(A)$

$$(\psi\xi)(x) = \frac{1}{p} \sum_{z^p = x} \xi(z) = \sum_n a_{np} x^n .$$

Let $F := \exp \pi f^*(x)$ and define $D := F^{-1} \circ x \frac{d}{dx} \circ F = x \frac{d}{dx} + \pi x f^+(x)$. It is clear that D is injective in $\mathcal{H}^\dagger(A)$ and therefore by proposition 5.4.3 $D\mathcal{H}^\dagger(A)$ has codimension $d+d'$ in $\mathcal{H}^\dagger(A)$. By theorem 5.6 a complementary subspace of $D\mathbb{C}_p[x, \frac{1}{x}]$ in $\mathbb{C}_p[x, \frac{1}{x}]$ will be a complementary subspace of $D\mathcal{H}^\dagger(A)$. One sees easily that $B = \{x^i\}_{-d \leqslant i \leqslant d'-1}$ form a basis of such a complementary subspace.

Let $G(x) := \exp \pi (f^*(x) - f^*(x^p))$, $G \in \mathcal{H}^\dagger(A)$. Consider the mapping $\alpha := \psi \circ G$ of $\mathcal{H}^\dagger(A)$ into itself. We define the matrix $\Gamma = (\gamma_{ij})$, $-d \leqslant i \leqslant d'-1$, $-d \leqslant j \leqslant d'-1$, of the quotient mapping $\bar{\alpha}$ in $\mathcal{H}^\dagger(A)/D\mathcal{H}^\dagger(A)$ related to the basis B by

$$\alpha(x^i) = \sum_{j=-d}^{d'-1} \gamma_{ij} x^j \mod D\mathcal{H}^\dagger(A), \quad -d \leqslant i \leqslant d'-1 . \tag{7.1.1}$$

The main purpose of this section is to give estimates for the p-adic magnitude of the coefficients γ_{ij}.

We shall use the usual additive valuation on $\mathbb{Q}_p$ ord, normalized by the condition $\mathrm{ord}\, p = 1$. [ ] will denote the "integral part".

7.2. THEOREM. Assume that $p \geqslant 5$. Then

$$(7.2.1)\quad \begin{cases} \operatorname{ord} \gamma_{00} = 0 & \operatorname{ord} \gamma_{i,0} > 0 \quad \text{if} \quad i \neq 0 \qquad -d \leqslant i \leqslant d'-1 \\ \operatorname{ord} \gamma_{-d,-d} = 1 & \operatorname{ord} \gamma_{i,-d} > 1 \qquad\qquad\qquad -d+1 \leqslant i \leqslant d'-1 . \end{cases}$$

For $1 < k < d'-1$, define $pk = s_k' d' + q_k'$, $0 \leqslant s_k'$, $1 \leqslant q_k' \leqslant d'-1$. Then

$$(7.2.2)\quad \begin{cases} \operatorname{ord} \gamma_{q_k',k} = \frac{s_k'}{p-1} = [k \frac{p}{d'}] \frac{1}{p-1} \\ \operatorname{ord} \gamma_{i,k} > \frac{s_k'}{p-1} & -d \leqslant i < q_k' \\ \operatorname{ord} \gamma_{i,k} \geqslant \frac{s_k'}{p-1} & q_k' < i \leqslant d'-1 . \end{cases}$$

For $1 \leqslant k \leqslant d-1$ define $pk = s_k d + q_k$, $0 \leqslant s_k$, $1 \leqslant q_k \leqslant d-1$. Then

$$(7.2.3)\quad \begin{cases} \operatorname{ord} \gamma_{-q_k,-k} = \frac{s_k}{p-1} = [k \frac{p}{d}] \frac{1}{p-1} \\ \operatorname{ord} \gamma_{i,-k} > \frac{s_k}{p-1} & -q_k < i \leqslant d'-1 \\ \operatorname{ord} \gamma_{i,-k} \geqslant \frac{s_k}{p-1} & -d \leqslant i < -q_k . \end{cases}$$

7.3. Before proving this theorem we give some consequences.

As $(p,d') = 1$, we see that $q_k' \neq q_j'$ for $k \neq j$. (and likewise $q_k \neq q_j$ for $k \neq j$). Therefore in the expansion of $\det(\Gamma)$, the dominant term is $\gamma_{0,0}\, \gamma_{-d,-d} \prod_{k=1}^{d'-1} \gamma_{q_k',k} \prod_{k=1}^{d-1} \gamma_{-q_k,-k}$, the other terms having greater valuation. Therefore using lemma 7.11 one gets

$$\operatorname{ord}(\det \Gamma) = \sum_{k=1}^{d'-1} [k\tfrac{p}{d'}]\frac{1}{p-1} + \sum_{k=1}^{d-1} [k\tfrac{p}{d}]\frac{1}{p-1} + 1 =$$

$$= \frac{(d'-1)(p-1)}{2(p-1)} + \frac{(d-1)(p-1)}{2(p-1)} + 1 = \frac{d+d'}{2} .$$

7.3.1. It is well known that there is a functional equation relating the inverse roots of $L(g;f,h;t)$ with those of $L(g;-f,h^{-1},t)$. Specifically these inverse roots may be paired so that the products of each pair is q. (In § 8 we give a demonstration of this functional equation based upon Dwork's dual theory). Letting e (resp. e') denote the coefficient of the term of highest degree of the first (resp. 2^{nd}) L-function, we conclude that the product ee' is a certain power of q. Now e lies in $\mathbb{Q}(\mu_{(q-1)p})$ and e' is its image under complex conjugation, i.e. e and e' are conjugate over $\mathbb{Q}$ but not necessarily over $\mathbb{Q}_p$. If however m divides q-1 and h is the (q-1)/m power of an element of $\mathbb{F}_q(x)$ then e lies in $\mathbb{Q}(\mu_{pm})$ and so in particular if h=1 then m=1 and e and e' are elements of $\mathbb{Q}(\mu_p)$ conjugate over $\mathbb{Q}$. This field has precisely one p-adic valuation and so regardless of our imbedding into $\mathbb{C}_p$, $|e|$ and $|e'|$ coincide if $h=1$, and so $|e|$ is known. In the case of gauss sums the L-functions are polynomials of degree 1. In this case $h \neq 1$ and $|e|$ needs not coincide with $|e'|$.

7.3.2. For the application to L-series we are interested in fact in the mapping $\alpha^{\sigma^{m-1}} \circ \dots \circ \alpha$ whose quotient mapping has matrix $\Gamma^{\sigma^{m-1}} \dots \Gamma$ in the basis B. Then the estimates of theorem 7.2 together with the results of [Ma] and [Ka 2] on semi-linear mappings yield the following result :

7.4. THEOREM. <u>The Newtón polygon of the eigenvalue of</u> $\Gamma^{\sigma^{m-1}} \dots \Gamma$ <u>is above the Newtion polygon with slopes</u>

$$0, \quad [k\tfrac{p}{d'}]\frac{m}{p-1} \quad \text{for} \quad 1 \leqslant k \leqslant d'-1, \qquad [k\tfrac{p}{d}]\frac{m}{p-1} \quad \text{for} \quad 1 \leqslant k \leqslant d-1, \ m$$

<u>and their endpoints meet</u>.

(The fact that the endpoints meet is just the result on the estimates of $\det \Gamma$ mentioned previously).

7.5. If $p \equiv 1 \bmod d'$ (or $d' = 1$) and $p \equiv 1 \bmod d$ (or $d = 1$), then

$$q'_k = k, \quad s'_k = [k\tfrac{p}{d'}] = k\,[\tfrac{p-1}{d'}], \quad q_k = k, \quad s_k = [k\tfrac{p}{d}] = k\,\frac{p-1}{d}$$

so in that case we can show using the result of Sp, that the two polygons coincide. Precisely :

THEOREM. <u>If</u> $p \equiv 1 \bmod d'$ (<u>or</u> $d' = 1$) <u>and</u> $p \equiv 1 \bmod d$ (<u>or</u> $d = 1$) (<u>and</u> $p \geqslant 5$) <u>then the eigenvalues</u> (ω_i), $-d \leqslant i \leqslant d'-1$, <u>of</u> $\Gamma^{\sigma^{m-1}} \dots \Gamma$ <u>can be arranged so that</u>

$$\operatorname{ord}(\omega_i) = m\, i/d' \qquad 0 \leqslant i \leqslant d'-1$$
$$\operatorname{ord}(\omega_{-i}) = m\, i/d \qquad 1 \leqslant i \leqslant d\ .$$

7.6. We note that in the case $f^*(x) = x + x^{-d}$, Sperber has obtained more precise results (unpublished). He shows that for $p \equiv 1$ mod, $p > 2(d+1)$ the Newton polygon of the eigenvalues lies above the Newton polygon with slopes $\{0, \frac{1}{d}, \dots, \frac{d-1}{d}, 1\}$ which is precisely the Newton polygon for the eigenvalues when $p \equiv 1 \bmod d$.

The fact that for $p \not\equiv 1 \bmod d$ the Newton polygon of the eigenvalues lies above the other Newton polygon is illustrated by the example $f(x) = x + x^{-3}$. Then for $p \equiv 2 \bmod 3$, $p >> 0$, Sperber shows (unpublished) that the eigenvalues have valuations

$$0\ ,\ \left(\frac{p-2}{3} + 3\right)\frac{1}{p-1}\ ,\quad \left(\frac{2p-1}{3} - 3\right)\frac{1}{p-1}\ ,\ 1\ .$$

7.7. We turn now to the proof of theorem 7.2.

Define the coefficients $a_{n,j}$ by the relations

$$x^n = \sum_{j=-d}^{d'-1} a_{n,j} x^j \mod D\mathcal{H}^\dagger(A) \qquad n \in \mathbb{Z}\ . \tag{7.7.1}$$

LEMMA. <u>For</u> $n \geqslant d'$

$$\begin{aligned} \operatorname{ord} a_{n,j} &\geqslant -\left(\left[\tfrac{n}{d'}\right] - 1\right)/(p-1) && -d \leqslant j < 0 \\ &\geqslant -\left[\tfrac{n-j}{d'}\right]/(p-1) && 0 \leqslant j \leqslant d'-1\ . \end{aligned} \tag{7.7.2}$$

<u>For</u> $n > d$

$$\begin{aligned} \operatorname{ord} a_{-n,j} &\geqslant -\left[\tfrac{n-1+i}{d}\right]/(p-1) && -d \leqslant j < 0 \\ &\geqslant -\left(\left[\tfrac{n-1}{d}\right] - 1\right)/(p-1) && 0 \leqslant j \leqslant d'-1\ . \end{aligned} \tag{7.7.3}$$

<u>Proof</u> : For $n \geqslant d'$

$$D(x^{n-d'}) = (n-d')x^{n-d'} + \pi \sum_{j=-d}^{d'} j u_j x^{j+n-d'}$$

and thus

$$x^n = \frac{*}{\pi}\, x^{n-d'} + \sum_{i=1}^{d+d'} *\, x^{n-i} \quad \mod D\mathcal{H}^\dagger(A) \tag{7.7.4}$$

where $*$ is used to indicate that we have a coefficient which is a p-adic integer. Therefore

$$a_{n,j} = \frac{*}{\pi} a_{n-d',j} + \sum_{i=1}^{d+d'} * \, a_{n-i,j} \, ,$$

and

$$(7.7.5) \quad \operatorname{ord} a_{n,j} \geqslant \min(\operatorname{ord}(a_{n-d',j}) - 1/(p-1), \min_{1\leqslant i\leqslant d+d'} \operatorname{ord}(a_{n-i,j})) \, .$$

So the inequalities (7.7.2) are easily proved by induction once they have been proven for $d' \leqslant n < 2d'$. But then they are easily deduced from (7.7.5) noting that for $-d \leqslant i,j < d'$ $a_{i,j} = 1$ if $i = j$ and $a_{i,j} = 0$ if $i \neq j$.

For $n > d$

$$D(x^{-n+d}) = (-n+d)x^{-n+d} + \pi \sum_{j=-d}^{d'} j u_j x^{j-n+d}$$

and thus

$$(7.7.6) \qquad x^{-n} = \frac{*}{\pi} x^{-n+d} + \sum_{i=1}^{d+d'-1} * \, x^{-n+i}$$

therefore

$$a_{-n,j} = \frac{*}{\pi} a_{-n+d,j} + \sum_{i=1}^{d+d'-1} * \, a_{-n+i,j}$$

$$(7.7.7) \quad \operatorname{ord} a_{-n,j} \geqslant \min(\operatorname{ord}(a_{-n+d,j}) - 1/(p-1), \min_{1\leqslant i\leqslant d+d'-1} \operatorname{ord}(a_{-n+i,j}))$$

and then inequalities (7.7.3) are proven in the same way as inequalities (7.7.2).

7.8. We shall write

$$G(x) = \exp\pi(f^*(x) - f^{*\sigma}(x^p)) = \prod_{i=-d}^{d'} \exp\pi(u_i x_i - u_i^p x^{ip}) = \sum_{n=-\infty}^{+\infty} h_n x^n \, .$$

LEMMA. <u>For</u> $n \geqslant 0$

$$\operatorname{ord} h_n \geqslant \frac{n}{d'} \frac{p-1}{p^2} \, , \quad \operatorname{ord} h_{-n} \geqslant \frac{n}{d} \frac{p-1}{p^2} \, .$$

<u>Proof</u> : We recall ([DW 1] § 21) that the function $\theta(x) := \exp\pi(x-x^p)$ converges for $\operatorname{ord} x > -(p-1)/p^2$ and satisfies $\operatorname{ord} \theta(x) = 0$. There-

fore for $i > 0$, $\exp\pi(u_i x^i - u_i^p x^{ip}) = \theta(u_i x^i)$ converges for $\operatorname{ord} x > -(p-1)/ip^2$, while $\theta(u_{-i}x^{-i})$ converges for $\operatorname{ord} x < (p-1)/ip^2$. Therefore $G(x)$ converges for $-(p-1)/d'p^2 < \operatorname{ord} x < (p-1)/dp^2$ and besides $\operatorname{ord} G(x) = 0$. This implies, by Cauchy's inequalities,

$$\inf_{n\geqslant 0}(\operatorname{ord} h_n - n(p-1)/d'p^2) \geqslant 0,\ \inf_{n<0}(\operatorname{ord} h_n + n(p-1)/dp^2) \geqslant 0$$

which proves the lemma.

7.9. We shall need more precise estimates.

For the rational number a, let $]a[$ denote the smallest integer $\geqslant a$. Thus $]a[= [a]$ if a is an integer, otherwise $]a[= [a] + 1$.

LEMMA. <u>Assume</u> $p \geqslant 5$.

<u>For</u> $0 \leqslant n \leqslant 3pd'$ $\quad \operatorname{ord} h_n \geqslant]\frac{n}{d'}[\ \frac{1}{p-1}$.

<u>For</u> $0 \leqslant n \leqslant 3pd$ $\quad \operatorname{ord} h_{-n} \geqslant]\frac{n}{d}[\ \frac{1}{p-1}$.

<u>If</u> $n = sd'$, $0 \leqslant s \leqslant p-1$, $\operatorname{ord} h_n = \frac{s}{p-1} = \frac{n}{d'}\frac{1}{p-1}$.

<u>If</u> $n = sd$, $0 \leqslant s \leqslant p-1$, $\operatorname{ord} h_{-n} = \frac{s}{p-1} = \frac{n}{d}\frac{1}{p-1}$.

<u>Proof</u> : Recall that the Artin-Hasse series

$$E(x) = \exp(\sum_{s=0}^{\infty} x^{p^s}/p^s)$$

belongs to $\mathbb{Z}_p[[x]]$. Therefore

$$\theta(x) = E(\pi x) \prod_{s=2}^{\infty} \exp(-(\pi x)^{p^s}/p^s) = E(\pi x) \mod x^{p^2}.$$

If we write $\theta(x) = \sum_{j\geqslant 0} c_j x^j$, we obtain

$$\operatorname{ord} c_j \geqslant j/(p-1),\quad j \leqslant p^2-1$$
$$\operatorname{ord} c_j = j/(p-1),\quad j \leqslant p-1$$

and we have seen in the previous lemma $\operatorname{ord} c_j \geqslant j(p-1)/p^2$ for all j. We have

$$\sum_{n=-\infty}^{+\infty} h_n x^n = \prod_{\substack{k=-d\\k\neq 0}}^{d'} \sum_{j\geqslant 0} c_j(u_k x^k)^j\ \theta(u_0)$$

and so

$$h_n = \theta(u_0) \sum c_{j_{-d}} \dots c_{j_{d'}} u_{-d}^{j_{-d}} \dots u_{d'}^{j_{d'}}$$

the sum being taken over all families $(j_k)_{\substack{-d\leq k\leq d' \\ k\neq 0}}$ such that $\sum_{k=-d}^{d'} kj_k = n$. Let us write when $0 \leq n \leq 3pd'$

$$\sum_{k=-d}^{-1} kj_k = -s \,, \quad \sum_{k=1}^{d'} kj_k = n+s \,.$$

Therefore $\sum_{k=1}^{d'} j_k \geq n/d'$ and we have $=$ only if $d'j_{d'} = n$, $s=0$ and $i_k = 0$ for $k < d'$. The hypothesis $p \geq 5$ implies $p-1 > 3p/(p-1)$ and so if $j_k \geq p^2$ one has $\operatorname{ord} c_{j_k} \geq j_k(p-1)/p^2 \geq (p-1) > 3p(p-1) \geq \left]\frac{n}{d'}\right[\frac{1}{p-1}$.

If for all $k \leq 1$ $j_k \geq p^2 - 1$ then

$$\operatorname{ord}\left(\prod_{k=1}^{d'} c_{j_k} u_k^{j_k}\right) \geq \sum_{k=1}^{d'} \operatorname{ord} c_{j_k} \geq \sum_{k=1}^{d'} j_k/(p-1) \geq \left]\frac{n}{d'}\right[\frac{1}{p-1}$$

so $\operatorname{ord} h_n \geq \frac{n}{d'} \frac{1}{p-1}$.

If $n = sd'$,

$$\operatorname{ord}(c_s u_{d'}^s) = s/(p-1) = \frac{n}{d'} \frac{1}{p-1} \,,$$

while if $(j_{-d}, \dots, j_{d'}) \neq (0, \dots 0, s)$

$$\operatorname{ord}(c_{j_{-d}} \dots c_{j_{d'}} u_{-d}^{j_{-d}} \dots u_{d'}^{j_{d'}}) \geq \left(\sum_{k=1}^{d'} j_k\right)/(p-1) > \frac{n}{d'} \frac{1}{p-1}$$

which implies

$$\operatorname{ord} h_n = \frac{n}{d'} \frac{1}{p-1} \,.$$

The case of h_{-n} with $0 \leq n \leq 3pd$ is treated in the same way.

7.10. Proof of theorem 7.2.

$$\alpha(x^i) = \psi(x^i G(x)) = \sum_{n=-\infty}^{+\infty} h_{np-i} x^n = \sum_{n=-\infty}^{+\infty} h_{np-i} \sum_{k=-d}^{d'-1} a_{n,k} x^k \bmod D\mathcal{H}^\dagger(A)$$

and so

$$\gamma_{i,k} = \sum_{n=-\infty}^{+\infty} h_{np-i} a_{n,k} = h_{kp-i} + \sum_{n=d'}^{+\infty} h_{np-i}\, a_{n,k} + \sum_{n=d+1}^{+\infty} h_{-np-i}\, a_{-n,k} \,. \tag{7.10.1}$$

We consider first the case $0 \leqslant k \leqslant d'-1$. We first estimate $\operatorname{ord} h_{kp-i}$. Let

$$pk = s'_k d' + q'_k \qquad 0 \leqslant s'_k ,\ 0 \leqslant q'_k \leqslant d'-1 .$$

$$(7.10.2)\quad \begin{cases} \text{if } i = q'_k,\ kp-i = s'_k d' \text{ and } \operatorname{ord} h_{kp-i} = s'_k/(p-1) & (\text{lemma } 7.9) \\ \text{if } q'k < i < d'-1,\ ]\frac{kp-i}{d'}[= s'_k \text{ and } \operatorname{ord} h_{kp-i} \geqslant s'_k/(p-1) \\ \text{if } -d \leqslant i < q'_{k'},\ ]\frac{kp-i}{d'}[\geqslant s'_k + 1 \text{ and } \operatorname{ord} h_{kp-i} > s'_k/(p-1) . \end{cases}$$

We shall now show that for $n \geqslant d'$ or $n < -d$ $\operatorname{ord}(h_{np-i} a_{n,k}) \geqslant s'_k/(p-1)$ if $i > q'_k$ and $\operatorname{ord}(h_{np-i} a_{n,k}) > s'_k/(p-1)$ if $i \leqslant q'_k$ and this will end the proof of (7.2.2)

i) consider the case $d' \leqslant n \leqslant 3d'$ and $np-i \leqslant 3d'p$. By lemma 7.7 and 7.9, recalling that $k \geqslant 0$,

$$(7.10.3)\qquad \operatorname{ord}(h_{np-i} a_{n,k}) \geqslant (]\frac{np-i}{d'}[- [\frac{n-k}{d'}])\frac{1}{p-1} .$$

Observe that for two rational number u and v

$$(7.10.4)\qquad]u[- [v] \geqslant]u-v[$$

$$(7.10.5)\qquad]u+v[\geqslant]u[+ [v] .$$

Thus we deduce from (7.10.3)

$$\operatorname{ord}(h_{np-i} a_{n,k}) \geqslant]\frac{n(p-1)+k-i}{d'}[\frac{1}{p-1} \geqslant (]\frac{(n-k)(p-1)-i}{d'}[+ [\frac{kp}{d'}])\frac{1}{p-1} .$$

Now observe that $]\frac{(n-k)(p-1)-i}{d'}[\geqslant 0$ is equivalent to $(n-k)(p-1) - i > -d'$ and this is satisfied because $i \leqslant d'$ and $n-k > 0$. So for all i

$$\operatorname{ord}(h_{np-i} a_{n,k}) \geqslant [\frac{kp}{d'}]\frac{1}{p-1} .$$

Now observe that $]\frac{n(p-1)+k-i}{d'}[> [\frac{kp}{d'}] = \frac{kp-q'_k}{d'}$ is equivalent to $n(p-1) + k - i > kp-q'_k$ and this is satisfied if $i \leqslant q'_k$ because then $(n-k)(p-1) > 0 \geqslant i - q'_k$. Thus for $i < q'_k$

$$\operatorname{ord}(h_{np-i} a_{n,k}) > [\frac{kp}{d'}]\frac{1}{p-1} .$$

ii) Consider the case $d' \leqslant n \leqslant 3d'$ and $np - i > 3pd'$. By lemmas 7.7 and 7.8

$$\operatorname{ord}(h_{np-i}a_{nk}) \geqslant \frac{np-i}{d'}\frac{p-1}{p^2} - [\frac{n-k}{d'}]\frac{1}{p-1}$$

$$> 3\frac{p-1}{p} - [\frac{3d'-k}{d'}]\frac{1}{p-1} \geqslant 3\frac{p-1}{p} - \frac{3}{p-1} = \frac{3(p^2-3p+1)}{p^2-p}$$

and for $p \geqslant 5$ one has $\frac{3(p^2-3p+1)}{p^2-p} > 1 \geqslant [\frac{kp}{d'}]\frac{1}{p-1}$. So for all i $\operatorname{ord}(h_{np-i}a_{nk}) > [\frac{kp}{d'}]\frac{1}{p-1}$.

iii) Consider the case $n > 3d'$. By lemmas 7.7 and 7.8

$$\operatorname{ord}(h_{np-i}a_{n,k}) \geqslant \frac{np-i}{d'}\frac{p-1}{p^2} - \frac{n-k}{d'}\frac{1}{p-1}$$

$$\geqslant \frac{np-d'}{d'}\frac{p-1}{p^2} - \frac{n}{d'}\frac{1}{p-1} \geqslant \frac{n(p(p-1)^2-p^2)-d'(p-1)^2}{d'p^2(p-1)}$$

$$\geqslant \frac{3(p^3-3p^2+p)-(p-1)^2}{p^2(p-1)} = \frac{3p^3-10p^2+5p-1}{p^2(p-1)}$$

and for $p \geqslant 5$ one has $\frac{3p^3-10p^2+5p-1}{p^2(p-1)} > 1 \geqslant [\frac{kp}{d'}]\frac{1}{p-1}$.

iv) Consider the case $n < -d$. We write $-n$ instead of n. For $-n < -2d$

$$\operatorname{ord}(h_{-np-i}a_{-n,k}) \geqslant \frac{np+i}{d}\frac{p-1}{p^2} - ([\frac{n-1}{d}]-1)\frac{1}{p-1}$$

$$\geqslant \frac{np-d}{d}\frac{p-1}{p^2} - (\frac{n-1}{p-1}-1)\frac{1}{p-1} = \frac{n(n(p-1)^2-p^2+d(p^2-(p-1)^2)+p^2}{dp^2(p-1)}$$

$$> \frac{2(p(p-1)^2-p^2)+p^2-(p-1)^2}{p^2(p-1)} = \frac{2p^3-6p^2+4p-1}{p^2(p-1)}$$

and for $p \geqslant 5$ one has $\frac{2p^3-6p^2+4p-1}{p^2(p-1)} > 1$.

For $-2d < -n < -d$ and $np + i < 3\,dp$, by lemmas 7.7 and 7.9 and (7.10.4)

$$\operatorname{ord}(h_{-np-i}a_{-n,h}) \geqslant]\frac{np+i}{d}[\frac{1}{p-1} - ([\frac{n-1}{d}]-1)\frac{1}{p-1}$$

$$>]\frac{n(p-1)+1}{d}[\frac{1}{p-1} > 1 .$$

For $-2d < -n < -d$ and $np + i > 3dp$, by lemmas 7.7 and 7.8

$$\operatorname{ord}(h_{-np-i}a_{-n,k}) \geqslant \frac{np+i}{d}\,\frac{p-1}{p^2} - \left(\left[\frac{n-1}{d}\right]-1\right)\frac{1}{p-1}$$

$$> \frac{3dp}{d}\,\frac{p-1}{p^2} - \frac{2}{p-1} = 3 - \frac{3}{p} - \frac{3}{p-1}$$

and $3 - (3/p) - 2/(p-1) > 1$ for $p \geqslant 5$.

This ends the verification of (7.2.2). In the same way we prove (7.2.3) for $0 \leqslant k \leqslant d-1$. Observe that (7.2.2) for $k = 0$ together with (7.2.3) for $k = 0$ gives the first half of (7.2.1).

It remains to estimate $\operatorname{ord}(\gamma_{i,-d})$. One has by lemma 7.9

$$\operatorname{ord}(h_{-dp-i}) \geqslant \left[\frac{dp+i}{d}\right]\frac{1}{p-1} > \frac{1}{p-1} > 1 \quad \text{if} \quad -d+1 \leqslant i \leqslant d'-1$$

and

$$\operatorname{ord}(h_{-dp-d}) = \frac{d(p-1)}{d}\,\frac{1}{p-1} = 1 .$$

And then the same computations as previously shown will show that $\operatorname{ord}(h_{np-i}a_{n,-d}) \geqslant 1$ for $n \geqslant d'$ or $n < -d$ and all $i \in [-d, d'-1]$, ending the verification of the second half of (7.2.1).

7.11. LEMMA. Let $p,\ d \in \mathbb{N}^*$ be such that $(p,d) = 1$. Then

$$S_{p,d} = \sum_{k=1}^{d-1} \left[k\,\frac{p}{d}\right] = \frac{(d-1)(p-1)}{2} . \tag{7.11.1}$$

Proof : We prove the formula by induction on d . Observe that trivially for $d = 1$, $S_{p,1} = 0$.

Write $p = sd + q$ with $s \geqslant 0$, $0 < q < d$. We have $(q,d) = 1$ and

$$S_{p,d} = \sum_{k=1}^{d-1} \left[k\,\frac{p}{d}\right] = \sum_{k=1}^{d-1} ks + \sum_{k=1}^{d-1} \left[k\,\frac{q}{d}\right] = \frac{d-1}{2}\,ds + S_{q,d} .$$

So we consider the case $1 \leqslant q < d$ and $(q,d) = 1$. For $1 \leqslant k \leqslant d-1$, one has $[k\,\frac{q}{d}] = j$ for

$$j\,\frac{d}{q} \leqslant k < (j+1)\frac{d}{q} . \tag{7.11.2}$$

But as $(q,d) = 1$ and $j < q$, $j\,\frac{q}{d}$ is not an integer and $(j+1)\frac{d}{q}$ is not an integer either except for $j = q-1$. Therefore (7.11.2) can be written

$$(7.11.3) \qquad [j\tfrac{d}{q}] < k \leqslant [(j+1)\tfrac{d}{q}]$$

except for $j = q-1$ where we have $k < d-1$. We obtain

$$s_{q,d} = \sum_{k=1}^{d-1} [k\tfrac{d}{q}] = \sum_{j=1}^{q-2} j([(j+1)\tfrac{d}{q}] - [j\tfrac{d}{q}] + (q-1)((d-1) - (q-1)\tfrac{d}{q}) =$$

$$= (q-1)(d-1) - \sum_{j=1}^{q-1} j\tfrac{d}{q} = (q-1)(d-1) - s_{d,q} .$$

As $q < d$ we can use our induction hypothesis for $s_{d,q}$ and thus we obtain

$$s_{q,d} = (q-1)(d-1) - \frac{(d-1)(q-1)}{2} = \frac{(d-1)(p-1)}{2}$$

and therefore

$$s_{q,d} = \frac{d-1}{2}(ds+q-1) = \frac{(d-1)(p-1)}{2} .$$

§ 8. DUAL THEORY.

We shall give the proof of the functional equation of $L(g;f,h;t)$ (cf. 7.3.1). We follow the exposition of the functional equation of a special case given by Dwork in [Dw 1]. For details we refer to that article.

8.1. The Frobenius map.

Let $\underline{T}$ be the maximal unramified extension of $\mathbb{Q}_p$. Let σ be the unique automorphism of $\underline{T}$ which lifts the Frobenius automorphism of $\mathbb{F}_p^{alg}$, $x \longmapsto x^p$. If $f \in \underline{T}(x)$, then f^σ is the rational function obtained by applying σ to the coefficients of f.

Let S be a finite subset of $\underline{T}$ with $0, \infty \in S$ and such that for all $c, c' \in S - \{\infty\}$, $c \neq c'$ implies $|c-c'| = 1$. Let $f \in \underline{T}(x)$ with its poles in S and $|f|_{gauss} = 1$ such that for all $c \in S$ if $d_c := -\mathrm{ord}_c f$ then $d_c = -\mathrm{ord}_c \bar{f}$, $d_c > 1$ and $p \nmid d_c$. Let $h \in \underline{T}(x)$ with its poles and zeros in S. We shall write $F(x) := h(x)^{-1/(q-1)} \exp\pi f(x)$. Let A be the union of residue classes not containing points in S.

We introduce the differential operator

$D_F = F^{-1} \circ x\frac{d}{dx} \circ F = x\frac{d}{dx} + \pi x f' - \frac{x}{q-1}\frac{h'(x)}{h(x)}$. By 5.4 we know that

$\underline{W}_F = \mathcal{H}^{\dagger}(A)/D_F\mathcal{H}^{\dagger}(A)$ is a vector space of dimension $\delta = \sum_{c\in S}(d_c+1) - 2$. By 5.6 we know that we can choose representants of $\underline{W}_F$ in $\underline{L}(c)$. We shall choose as basis of $\underline{W}_F$:

(8.1.1) $B = \{x^i, 0\leqslant i\leqslant d_\infty - 1\ ;\ x^{-i}, 1\leqslant i\leqslant d_0\ ;\ (x-c)^{-i}, 1\leqslant i\leqslant d_c+1,\ c\in S-\{0,\infty\}\}$.

Recall that $q = p^m$. For $1\leqslant j\leqslant m$ define

$$F^{\sigma^j}(x) := h^{\sigma^j}(x)^{-q/p^j(q-1)} \exp f^{\sigma^j}(x) ,$$

and $F^{\sigma^0} = F$. Let $G_j(x) := F^{\sigma^j}(x)/F^{\sigma^{j+1}}(x^p)$, $0\leqslant j\leqslant m-1$. By writing

$$G_0(x) = h(x)\left(h^{\sigma}(x^p)^{p^{m-1}}/h(x)^{p^m}\right)^{1/(q-1)} \theta(f)\ \exp\pi(f(x)^p - f^{\theta}(x^p)), \text{ and}$$

$$G_j(x) = \left(\frac{h^{\sigma^j}(x^p)^{p^{m-j}}}{h^{\sigma^{j-1}}(x)^{p^{m-j+1}}}\right)^{1/(q-1)} \theta(f^{\sigma^{j-1}})\ \exp\pi(f^{\sigma^{j-1}}(x)^p - f^{\sigma^j}(x^p)))$$

for $j\geqslant 1$, one sees that $G_j\in\mathcal{H}^{\dagger}(A^{\sigma^j})$ and thus one can define $\psi\circ G_j$ acting from $\mathcal{H}^{\dagger}(A^{\sigma^j})$ into $\mathcal{H}^{\dagger}(A^{\sigma^{j+1}})$. (Formally $\psi\circ G_j = (F^{\sigma^{j+1}})^{-1}\circ\psi\circ F^{\sigma^j}$.

If $D_{F^{\sigma^j}}$ is the differential operator associated to F^{σ^j} :

$D_{F^{\sigma^j}} = x\frac{d}{dx} + (F^{\sigma^j})'/F^{\sigma^j}$ one sees that

$$(\psi\circ G_j)\circ D_{F^{\sigma^j}} = pD_{F^{\sigma^j}}\circ(\psi\circ G_j) .$$

Therefore one can define a mapping on the quotient space (the Frobenius map)

$$\alpha_{F^{\sigma^j}} : \underline{W}_{F^{\sigma^j}} \longrightarrow \underline{W}_{F^{\sigma^{j+1}}}$$

where $\alpha_{F^{\sigma^j}}(\xi) = (\psi\circ G_j)\xi \mod D_{F^{\sigma^j}}\mathcal{H}^{\dagger}(A^{\sigma^j})$.

In §§ 8.2 - 8.5, for simplicity of notation, we restrict ourselve to the case $j = 0$ (and write G instead of G_0), but it is clear that everything remains true in the general case.

8.2. Analytic dual theory.

We introduce the notation for $c\in S-\{\infty\}$

$$T_c = x - c$$

and for $c = \infty$

$$T_\infty = 1/x .$$

For $c \in S$ we define

$\hat{R}'_c$ = ring of Laurent series in T_c with coefficients in $\mathbb{C}_p$ which represent functions analytic on an annulus $0 < \varepsilon_c < |T_c| < 1$ with unspecified ε_c.

$\hat{R}_c$ = subring of Taylor series in T_c converging for $|T_c| < 1$, with the further condition, when $c = \infty$, that these series have non constant term (they are 0 at ∞).

We set

$$\hat{R}(S) = \bigoplus_{c \in S} \hat{R}_c \longrightarrow \bigoplus_{c \in S} \hat{R}'_c = \hat{R}'(s) .$$

We note that for each $c \in S$, $\mathcal{H}^\dagger(A) \subset \hat{R}'_c$, and we imbed $\mathcal{H}^\dagger(A)$ diagonally in $\hat{R}'$

$$\mathcal{H}^\dagger(A) \ni \eta \longmapsto (\eta, \eta, \ldots) \in \hat{R}' .$$

We define P_c (principal part at c) on $\hat{R}'_c$ by

$$P_c : \hat{R}'_c \longrightarrow \mathcal{H}^\dagger(A)$$

$$\sum_{j=-\infty}^{\infty} a_j T_c^j \longmapsto \begin{cases} \sum_{j=-\infty}^{-1} a_j T_c^j & c \neq \infty \\ \sum_{j=-\infty}^{0} a_j T_\infty^j & c = \infty \end{cases}$$

We have a natural projection of $\hat{R}'$ into $\mathcal{H}^\dagger(A)$

$$\gamma_+ : \hat{R}' \longrightarrow \mathcal{H}^\dagger(A)$$

$$\xi = (\xi_c)_{c \in S} \longmapsto \sum_{c \in S} P_c \xi_c .$$

We set

$$\gamma_- = I - \gamma_+$$

a mapping of $\hat{R}'$ into $\hat{R}$. The mapping γ_- annihilates $\mathcal{H}^\dagger(A)$ and so

$$\hat{R}' = \hat{R} \oplus \mathcal{H}^\dagger(A)$$

with γ_- (resp. γ_+) as the projection on the first (resp. the second) summand.

We define the residue map for $\hat{R}'_c$ $(c \in S)$

$$\mathrm{Res}_c : \hat{R}'_c \longrightarrow \mathbb{C}_p$$

$$\sum_{j=-\infty}^{+\infty} a_n T_c^j \qquad \begin{cases} -a_1 & \text{if} \quad c = \infty \\ a_{-1} & \text{if} \quad c \neq \infty . \end{cases}$$

We pair $\hat{R}'$ with itself by

$$\langle \xi, \eta \rangle = \sum_{c \in S} \mathrm{Res}_c \, \xi_c \, \eta_c .$$

This pairing restricts to a perfect pairing of $\hat{R}$ with $\mathcal{H}^{\dagger}(A)$. If L is an endomorphism of $\hat{R}'$ stable on $\mathcal{H}^{\dagger}(A)$ and if L^{*} is an endomorphism of $\hat{R}'$ dual to L under $\langle , \rangle$, then $(\gamma_{-} \circ L^{*}) | \hat{R}$ is the unique endomorphism of $\hat{R}'$ adjoint to $L | \mathcal{H}^{\dagger}(A)$ under the induced pairing of $\hat{R}$ with $\mathcal{H}^{\dagger}(A)$.

In particular consider the endomorphism $D_F = F^{-1} \circ x \frac{d}{dx} \circ F$ as componentwise mapping of $\hat{R}'$ into itself, its dual endomorphism in $\hat{R}'$ is $-F \circ \frac{d}{dx} \circ x F^{-1} = -\frac{F}{x} \circ x \frac{d}{dx} \circ x F^{-1} = -D_{xF^{-1}}$ and therefore the adjoint to $D_F | \mathcal{H}^{\dagger}(A)$ in $\hat{R}$ is $D_F^{*} = -\gamma_{-} D_{xF^{-1}}$. We shall denote by $\underline{K}_F$ the kernel of D_F^{*} in R. We claim that $\dim \underline{K}_F = \dim \underline{W}_F$ and therefore $\underline{K}_F$ can be identified with the dual space of $\underline{W}_F$ under the pairing $\langle , \rangle$.

Let $\underline{V}$ be the space spanned by the basis B (8.1.1), imbedded diagonally in R'. As observed already $\dim \underline{V} = \dim \underline{W}_F$.

Further for $\xi \in \hat{R}$, $D_F^{*}\xi = 0$ is equivalent to $D_{xF^{-1}} \xi \in \underline{V}$. So all we have to prove is that $D_{xF^{-1}}$ defines a bijection of $\underline{K}_F$ onto $\underline{V}$, i.e. that for each $\eta \in \underline{V}$, there exists $\xi_c \in \hat{R}_c$, such that $D_{xF^{-1}} \xi_c = \eta$ for all $c \in S$. Indeed we are exactly in the situation discussed in § 5.4.2 b) : in the generic disk of the circumference $|T_c| = r$, $r<1$ the solution $F(x)/x$ has radius of convergence r^{d_c+1} if $c \neq \infty$ (and $r^{d_\infty - 1}$ if $c = \infty$). Therefore the differential operator $T^{d_c+1} D_{xF^{-1}}$, if $c \neq 0, \infty$ ($T^{d_0} D_{xF^{-1}}$ if $c = 0$, $T^{d_\infty - 1} D_{xF^{-1}}$, if $c = \infty$) has index 0 as endomorphism of the space of analytic elements on the ball $|T_c| \leqslant r$, and therefore defines a surjective map from $\hat{R}_c$ onto itself, and thus $D_{xf^{-1}}$ defines a surjective map from $\hat{R}_c$ onto $T_c^{-(d_c+1)}\hat{R}_c$ if $c \neq 0, \infty$ (onto $T_0^{-d_0}\hat{R}_0$ if $c = 0$, onto $T_\infty^{-(d_\infty-1)}\hat{R}_\infty$, if $c = \infty$) which is all we wanted to prove.

8.3. The dual of the Frobenius map.

The mapping ψ of $\mathcal{H}^\dagger(A)$ into $\mathcal{H}^\dagger(A^\sigma)$ may be extended to a mapping of $\hat{R}'(S)$ into $\hat{R}'(S^\sigma)$. The mapping ϕ of $\mathcal{H}^\dagger(A^\sigma)$ into $\mathcal{H}^\dagger(A)$ extends to a mapping of $\hat{R}'(S^\sigma)$ into $\hat{R}'(S)$ by

$$\zeta = (\zeta_{c^\sigma})_{c \in S} \longmapsto (\zeta_{c^\sigma}(x^p))_{c \in S} = \phi\zeta .$$

Under the pairing $<,>$ of $\hat{R}'$ with itself, ϕ is the dual of $\psi \circ \frac{1}{x^{p-1}}$ ([DW 1], theorem 4.3).

Under the pairing $<,>$, the dual to the map $\psi \circ G : \mathcal{H}^\dagger(A) \longrightarrow \mathcal{H}^\dagger(A^\sigma)$ is the map $\omega := \gamma_-\, x^{p-1}\, G \circ \phi$ of $\hat{R}(S^\sigma)$ into $\hat{R}(S)$. Further ω maps $\underline{K}_{F^\sigma}$ into $\underline{K}_F$ and putting $\alpha_F^* := \omega|\underline{K}_{F^\sigma}$, we conclude that α_F^* is the dual of α_F. ([DW 1] lemma 4.4.3).

8.4. The symplectic structure.

We have seen in 8.2 that $D_{xF^{-1}}$ defines a bijection from $\underline{K}_F$ onto V, and in 8.1 that the natural map of $\mathcal{H}^\dagger(A)$ onto $\underline{W}_{xF^{-1}}$ defines a bijection of $\underline{V}$ onto $\underline{W}_{xF^{-1}}$. Taking the composition of these two maps we obtain a bijection $\hat{D}_F$ from $\underline{K}_F$ onto $\underline{W}_{xF^{-1}}$

$$\hat{D}_F : \underline{K}_F \xrightarrow{D_{xF^{-1}}} \underline{V} \xrightarrow{\text{proj.}} \underline{W}_{xF^{-1}} .$$

We claim that the following diagram commutes with a factor p

$$\begin{tikzcd} \underline{K}_F \arrow[r, "\hat{D}_F"] & W_{xF^{-1}} \\ \underline{K}^{\sigma}_F \arrow[u, "\alpha^{*}_F"] \arrow[r, "\hat{D}_{F^{\sigma}}"'] & \underline{W}_{x(F^{\sigma})^{-1}} \arrow[u, "\alpha^{-1}_{xF^{-1}}"'] \end{tikzcd} .$$

Explicitly

(8.4.1) $$\hat{D}_F \circ \alpha^{*}_F = p\, \alpha^{-1}_{xF^{-1}} \circ \hat{D}_{F^{\sigma}}$$

as map of $\underline{K}_{F^{\sigma}}$ into $\underline{W}_{xF^{-1}}$.

But $\alpha_{xF^{-1}} = \overline{\psi \circ x^{1-p} G^{-1}}$, $\alpha^{-1}_{xF^{-1}} = \overline{x^{p-1} G \circ \phi}$ (the $\overline{}$ indicating that we reduce to the quotient space). So we have to verify

$$(F/x) \circ x \frac{d}{dx} \circ xF^{-1} \circ \gamma_{-} x^{p-1} G \circ \phi = p x^{p-1} G \circ \phi \circ (F^{\sigma}/x) \circ x \frac{d}{dx} \circ x (F^{\sigma})^{-1} \bmod D_{xF^{-1}} \mathcal{H}^{\dagger}(A).$$

As for all $\zeta \in \hat{R}'$ $\gamma_{+}\, \zeta \in \mathcal{H}^{\dagger}(A)$, $D_{xF^{-1}}\, \gamma_{+}\, \zeta \in D_{xF^{-1}}\, \mathcal{H}^{\dagger}(A)$ and so we are reduced to the verification

$$D_{xF^{-1}} \circ x^{p-1} G \circ \phi = p x^{p-1} G \circ \phi \circ D_{x(F^{\sigma})^{-1}}$$

and this reduces to the well-known property

$$p \phi \circ x \frac{d}{dx} = x \frac{d}{dx} \circ \phi .$$

8.5. Estimate of the magnitude of the determinant of the Frobenius matrix.

In this paragraph we assume $h = 1$.

We now choose bases in our different spaces. In $\underline{W}_F$, $\underline{W}_{xF^{-1}}$, (resp. $\underline{W}_{F^{\sigma}}$, $\underline{W}_{x(F^{\sigma})^{-1}}$) we choose the basis image of the basis B (8.1.1) (resp. B^{σ}). We shall denote Γ_F the matrix of α_F related to these bases, and likewise we define $\Gamma_{xF^{-1}}$.

In $\underline{K}_F$ (resp. $\underline{K}_{F^\sigma}$) viewed as the dual space of ($\underline{W}_F$) (resp. $\underline{W}_{F^\sigma}$) we choose the dual basis. Then the matrix of α_F^* will be the transpose Γ_F^t of Γ_F.

Denote Δ_F and Δ_{F^σ} the matrices of $\hat{D}_F$ and $\hat{D}_{F^\sigma}$ associated to these bases. We deduce from (8.4.1) the relation

(8.5.1) $$\det \Delta_F \det \Gamma_F^t = p^\delta \det \Gamma^{-1}_{xF^{-1}} \det \Delta_{F^\sigma}$$

where $\delta = \sum_{c \in S} (d_c+1) - 2$ is the dimension of all these vector spaces. Now observe that the coefficients of Δ_F depend rationally over $\mathbb{Q}_p(\pi)$ upon the coefficients of f, and thus if we replace f by f^σ we see that $\det \Delta_{F^\sigma} = (\det \Delta_F)^\sigma$, so the quotient of these two determinants is a unit. Also it is known that $\det \Gamma_F^t = \det \Gamma_F$.

Observe that x is an invertible element of $\mathcal{H}^\dagger(A)$ and we have the commutative diagram

$$\begin{array}{ccc} \mathcal{H}^\dagger(A) & \xrightarrow{\text{mult by } x^{-1}} & \mathcal{H}^\dagger(A) \\ {\scriptstyle D_{F^{-1}}}\downarrow & & \downarrow{\scriptstyle D_{xF^{-1}}} \\ \mathcal{H}^\dagger(A) & \xrightarrow{\text{mult by } x^{-1}} & \mathcal{H}^\dagger(A) \end{array}$$

so we have, going to the quotient space, a natural isomorphism

$$\underline{W}_{F^{-1}} \xrightarrow{\overline{x^{-1}}} \underline{W}_{xF^{-1}} .$$

Likewise we have the commutative diagram

$$\begin{array}{ccc} \mathcal{H}^\dagger(A) & \xrightarrow{x^{-1}} & \mathcal{H}^\dagger(A) \\ {\scriptstyle \psi\circ G^{-1}}\downarrow & & \downarrow{\scriptstyle \psi\circ x^{1-p}G^{-1}} \\ \mathcal{H}^\dagger(A\ ) & \xrightarrow{x^{-1}} & \mathcal{H}^\dagger(A\ ) \end{array}$$

from which we deduce the commutative diagram

$$\begin{array}{ccc} \underline{W}_{F^{-1}} & \xrightarrow{\overline{x^{-1}}} & \underline{W}_{xF^{-1}} \\ {\scriptstyle \alpha_{F^{-1}}}\downarrow & & \downarrow{\scriptstyle \alpha_{xF^{-1}}} \\ \underline{W}_{(F^{\sigma})^{-1}} & \xrightarrow{\overline{x^{-1}}^{\sigma}} & \underline{W}_{x(F^{\sigma})^{-1}} \end{array} .$$

Taking bases, we see again that $\mathrm{Matrix}(\overline{x^{-1}}^{\sigma}) = \mathrm{Matrix}(\overline{x^{-1}})^{\sigma}$ and therefore $|\det \Gamma_{F^{-1}}| = \det|\Gamma_{xF^{-1}}|$.

Now $F(x) = \exp\pi f(x)$ and $F^{-1}(x) = \exp-\pi f(x)$. So we go from F to F^{-1} changing π into $-\pi$. The equation $x^{p-1}+p$ is irreducible over $\underline{T}$ and hence $\pi \longrightarrow -\pi$ defines an isomorphism of $\underline{T}(\pi)$ over $\underline{T}$. Since $\underline{T}$ is complete this isomorphism is an isometry

$$|\det \Gamma_{F^{-1}}| = \det|\Gamma_F| .$$

Eventually (8.5.1) gives

$$|\det \Gamma_F|^2 = |p^{\delta}|$$

or

$$|\det \Gamma_F| = p^{-\delta/2} .$$

8.6. Functional equation for the L-function.

We consider again the general case.

Let $m \in \mathbb{N}^*$ and let $q = p^m$. Consider the L-function introduced in § 6 $L(g;f,h;t)$. We shall denote by f (resp. h) a lifting of f (resp. h) in characteristic 0 such that $f^{\sigma^m} = f$ (resp. $h^{\sigma^m} = h$). Let A be the union of residue classes where $g \neq 0$. Then $A^{\sigma^m} = A$.

As the additive (resp. the multiplicative) character takes its values in μ_p (resp. in μ_{q-1}), the complex conjugate of $S_r(g;f,h)$ is $S_r(g;-f,h^{-1})$ and thus the complex conjugate of $L(g;f,h;t)$ is $L(g;-f,h^{-1};t)$.

In § 6.3 it has been proven that

$$L(g;f,h;t) = \det(1-t\overline{\alpha}) \tag{8.6.1}$$

with $\overline{\alpha} = \alpha_{F^{\sigma^{m-1}}} \circ \ldots \circ \alpha_{F^{\sigma}} \circ \alpha_F$ (where F and α_F are defined in § 8.1).

We observe that $F^{\sigma^m} = F$. It is clear that to replace (f,h) by $(-f,h^{-1})$ is equivalent to replacing F by F^{-1}. Therefore

(8.6.2) $$L(g;-f,h^{-1};t) = \det(1-t\overline{\beta})$$

with $\overline{\beta} = \alpha_{(F^{\sigma^{m-1}})^{-1}} \circ \ldots \circ \alpha_{F^{-1}}$.

Write (8.4.1) in the from

(8.6.3) $$\hat{D}_F \circ \alpha_F^* \circ \hat{D}_{F^{\sigma}}^{-1} = p\, \alpha_{xF^{-1}}^{-1} .$$

Then one deduces

(8.6.4) $$\hat{D}_F \circ \alpha_F^* \circ \ldots \circ \alpha_{F^{\sigma^{m-1}}}^* \circ \hat{D}_F^{-1} = q\, \alpha_{xF^{-1}}^{-1} \circ \ldots \circ \alpha_{x(F^{\sigma^{m-1}})^{-1}}^{-1}$$

$$= q\, \overline{x} \circ \alpha_{F^{-1}}^{-1} \circ \ldots \circ \alpha_{(F^{\sigma^{m-1}})^{-1}}^{-1} \circ \overline{x^{-1}}$$

with $\overline{x}$ and $\overline{x^{-1}}$ as defined in 8.5. This can also be written

(8.6.5) $$\hat{D}_F \circ (\overline{\alpha}) \circ \hat{D}_F^{-1} = q\, \overline{x} \circ (\overline{\beta})^{-1} \circ \overline{x}^{-1}$$

and therefore by (8.6.1) and (8.6.2)

$$L(g;f,h;t) = L(g;-f,h^{-1};1/qt).M$$

where M is an monomial in t chosen so as to make the right side a polynomial in t with constant term 1. Precisely $M = q^{\delta}t^{\delta}/\det\overline{\alpha}$ (δ as in § 8.5).

In particular

$$\det\overline{\alpha}\ \det\overline{\beta} = q^{\delta}$$

and therefore

$$|\det\overline{\alpha}|_{\mathbb{C}} = q^{\delta/2} .$$

In the case h = 1 one sees as in § 8.5 that $|\det\overline{\alpha}| = |\det\overline{\beta}|$ (p-adic valuation) and so $|\det\overline{\alpha}| = q^{\delta/2}$. We thus recover the result of § 8.5. We gave the proof of § 8.5 because it is valid even if f^{σ^m} is different of f for all m.

§ 9. THE GENERALIZED ADOLPHSON INDEX THEOREM.

In this section we abstract the technique of Aldolphson [Ad] so as to make it applicable to systems with irregular singular points. We use the notation of Adolphson but avoid characterization of singular points until the very end.

Let A be a $k \times k$ matrix with coefficients in $\Omega(X)$. Let

$\Omega^n[x]$ = vector space of polynomials of degree $< n$
$\Omega^n[(x-a)^{-1}]$ = vector space of polynomials of degree $< n$ in $\frac{1}{x-a}$.

For $a \in \Omega$, $r \in \mathbb{R}$, $r > 0$, let $F(a,r)$ be the set of all functions holomorphic on $D(a,r^+)^c$ and vanishing at infinity i.e.

$F(a,r) = \{\xi \in (x-a)^{-1}\ \Omega[[(x-a)^{-1}]] \mid \xi$ converges for $|x-a| > r\}$.

$F_n(a,r) = \{\xi_1 + \xi_2 \mid \xi_1 \in F(a,r),\ \xi_2 \in \Omega^n[x]\}$.

We define

$F(\infty,r) = \{\xi \in \Omega[[x]] \mid \xi$ converges for $|x| < r\}$.

We choose $P \in \Omega[x]$ such that the matrix P.A has polynomial entries. A polynomial having this property will be said to be suitable for A. Choose $N > 0$, such that $(D-A)$ maps $F(a,r)^k$ into $F_N(a,r)^k . \frac{1}{P}$. This N may be chosen independent of $a \in \Omega$. For example it is enough if

$$N \geqslant \deg(P.a_{ij}) - 1$$
$$N \geqslant \deg P - 2$$
$$N \geqslant 0 .$$

Such an N will be said to be suitable for A. We define (for $a \in \Omega$)

$$G_a = \left(\frac{1}{x-a}\ \Omega\left[\frac{1}{x-a}\right]\right)^k ,$$

$$G_a^n = \frac{1}{P(x)}\ \left(\frac{1}{x-a}\ \Omega\left[\frac{1}{x-a}\right] + \Omega^n[x]\right)^k$$

also

$$G_\infty = (\Omega[x])^k$$

$$G_\infty^n = \left(\frac{1}{P(x)}\ \Omega[x]\right)^k$$ (so n as element of $\mathbb{N}$ plays no role here).

9.1. Global Theory.

Let $\{a_1,\ldots,a_n\}$ be a set of points in Ω and $\{r_1,r_2,\ldots,r_n,r_\infty\}$ a set of positive real numbers. Let $F = F(r_\infty,r_1,\ldots,r_n)$ be the space of functions holomorphic in the region

$$S = D(O,r_\infty^-) - \bigcup_{c=1}^{n} D(a_i,r_i^+) .$$

We assume that each $D(a_i,r_i^+)$ lies in $D(O,r_\infty^-)$ and that the disks $D(a_i,r_i^+)$ are pairwise disjoint. By the p-adic Mittag-Leffler theorem

$$F = F(\infty,r_\infty) \oplus \bigoplus_{i=1}^{n} F(a_i,r_i) .$$

Let

$$E = \Omega[x , \frac{1}{x-a_1},\ldots,\frac{1}{x-a_n}] .$$

Finally we assume that each disk $D(a_i,r_i^-)$ contains at most one zero of P. If it does contain a zero of P then we assume the zero to be a_i itself. We assume that no zero of P lies in the disk $|x| > r$ at infinity.

THEOREM. We assume

$$H_1 \qquad \mathrm{Ker}(D-A,F(a_i,r_i)) \subset G_{a_i} \qquad i = 1,\ldots,r$$
$$\mathrm{Ker}(D-A,F(\infty , r_\infty)) \subset G_\infty$$

$$H_2 \qquad G_{a_i}^N/(D-A)G_{a_i} \simeq \frac{1}{P} F_N(a_i,r_i)^k/(D-A)F(a_i,r_i)^k \quad i=1,\ldots,r$$
$$G_\infty^N/(D-A)G_\infty \simeq \frac{1}{P} F(\infty,r_\infty)^k / (D-A)F(\infty , r_\infty)^k .$$

We conclude that

$$\mathrm{Ker}(D-A,F^k) \subset E^k \tag{9.1.1}$$

$$\frac{1}{P} E^K / (D-A)R^k \simeq \frac{1}{P} F^k / (D-A)F^k . \tag{9.1.2}$$

Proof :

Let $\xi \in \mathrm{Ker}(D-A),F^k)$,

$$\xi = \xi_1 + \xi_2 +\ldots+ \xi_n + \xi_\infty$$

where $\xi_i \in F(a_i,r_i)^k$ (resp. : $\xi_\infty \in F(\infty,r_\infty)^k$). Hence

$$-P(D-A)\xi_1 = P(D-A)\xi_2 + \ldots + P(D-A)\xi_\infty .$$

The right side is analytic on $S \cup D(a_1, r_1^+)$ while the left side is, aside from a pole at infinity, analytic on $D(a_i, r_i^+)^c$. Hence $P(D-A)\xi_1$ is a polynomial and so by hypothesis H_2, there exists $\eta_1 \in G_{a_1}$ such that

$$(D-A)\xi_1 = (D-A)\eta_1 .$$

But $\xi_1 - \eta_1 \in F(a_1, r_1)^k$ and so by H_1 lies in G_{a_1}. Thus $\xi_1 \in G_{a_1}$ and likewise for the other components $\xi_2, \ldots, \xi_n, \xi_\infty$. This completes the proof of (9.1.1).

For the proof of (9.1.2) we observe that there is a natural mapping of the algebraic factor space on the left into the analytic factor space of the right. We assert that the mapping is injective. Let the $\eta \in P^{-1}E^k$, $\xi \in F^k$ and suppose

$$\eta = (D-A)\xi .$$

Thus

$$P\eta = P\eta_1 + \ldots + P\eta_n + P\eta_\infty$$

where

$$P\eta_i \in G_{a_i}$$

$$P\eta_\infty \in G_\infty .$$

Also

$$\xi = \xi_1 + \ldots + \xi_n + \xi_\infty$$

and

$$P(D-A)(\xi_1 + \ldots + \xi_n + \xi_\infty) = P\eta_1 + \ldots + P\eta_\infty .$$

Thus by a previous argument $P(D-A)\xi_1 - P\eta_1$ is, aside from a pole at infinity, analytic on $S \cup D(a_1, r_1^+)$ and on $D(a_1, r_1^+)^c$. Thus

$$P(D-A)\xi_1 \equiv P\eta_1 \mod \Omega[x]^k .$$

Hence by H_2

$$(D-A)\xi_1 = (D-A)\tilde{\xi}_1 , \quad \tilde{\xi}_1 \in G_{a_1} ,$$

and so by H_1, $\xi_1 - \tilde{\xi}_1 \in G_{a_1}$, i.e. $\xi_1 \in G_{a_1}$. Likewise for $\xi_2, \ldots, \xi_n, \xi_\infty$. Thus $\xi \in E^k$. This complete the proof of injectivity.

We now demonstrate surjectivity. Let $\eta \in \frac{1}{P} F^k$, then

$$P\eta = P\eta_1 + \ldots + P\eta_n + P\eta_\infty$$

where $P\eta_i$ (resp. $P\eta_\infty$) $\in F(a_i,r_i)^k$ (resp. : $F(\infty,r_\infty)^k$). By H_2, for $i = 1,\ldots,n$

$$P\eta_i = P\xi_i + Pq_i + P(D-A)\mu_i$$

where

$$P\xi_i \in G_{a_i},\ Pq_i \in \Omega[x^k],\ \mu_i \in F(a_i,r_i)$$

and likewise

$$P\eta_\infty = P\xi_\infty + P(D-A)\mu_\infty$$

where $\mu_\infty \in F(\infty,r_\infty)^k$, $P\xi_\infty \in \Omega[x]^k$. Thus

$$P\eta = P(\xi_1+\ldots+\xi_\infty) + P(q_i+\ldots+q_n) + P(D-A)(\mu_i+\ldots+\mu_n+\mu_\infty)\ .$$

However

$$P(\xi_1+\ldots+\xi_\infty) + P(q_1+\ldots+q_n) \in E^K$$

and so

$$\eta \in \frac{1}{P} E^K + (D-A)F^k\ .$$

This completes the proof of the theorem.

9.2. Local Theory.

The object of this section is to give criteria which permit us to conclude that the local hypotheses H_1, H_2 are satisfied.

These local hypotheses may be verified point by point. For this reason we may restrict our attention to one point, say $a_1 = 0$, $r_1 = r$.

Local Comparison Theorem.

Let A and B be $k\times k$ matrices with coefficients in $\Omega(x)$. Let P, N be polynomial and integer suitable for both A and B. We assume D-A and D-B are locally equivalent on $D(0,R)$ with $R > r$, i.e. $\exists V$, a $k\times k$ matrix with coefficients holomorphic on $D(0,R)$ such that det V has no zero in this disk except possibly at $x = 0$, and such that

$$D-A = V\circ(D-B)\circ V^{-1}$$

$H_1(B)$ $\quad \mathrm{Ker}(D-B,F(0,r) \subset G_0$

$H_2(B)$ $\quad G_0^N / (D-B)G_0 \simeq \frac{1}{P} F_N(0,r)^k / (D-B)F(0,r)^k\ .$

We conclude that $H_1(A)$ and $H_2(A)$ are valid.

Note : H_1 and H_2 as used here refer to just one point.

Proof : For z analytic in $D(0,R^-) - D(0,r^+)$ we have the Mittag-Leffler decomposition which we write

$$z = \gamma_- r + \gamma_+ z$$

where $\gamma_+ z$ is analytic in $D(0,R^-)$ and $\gamma_- z \in F(0,r)$. To verify $H_1(A)$ let $\xi \in F(0,r)$, $(D-A)\xi = 0$. Then

$$P(D-B)V^{-1}\xi = 0$$

and so

$$\gamma_- \; P(D-B)V^{-1}\xi = 0$$

i.e.

$$\gamma_- \; P(D-B)(\gamma_- V^{-1}\xi + \gamma_+ V^{-1}\xi) = 0 \; .$$

Now P(D-B) is stable on k-tuples whose components are analytic on $D(0,R^-)$ and so

$$\gamma_- P(D-B)\gamma_+ V^{-1}\xi = 0 \; .$$

Thus

$$\gamma_- P(D-B)\gamma_- V^{-1}\xi = 0 \; .$$

Since P(D-B) maps $F(0,r)^k$ into $F_N(0,r)$, it now follows that

$$P(D-B)\gamma_- V^{-1}\xi \in (\Omega^N[x])^k$$

and so by $H_2(B)$,

$$(D-B)\gamma_- V^{-1}\xi = (D-B)\xi_0, \; \xi_0 \in G_0$$

and then by $H_1(B)$,

$$\gamma_- V^{-1}\xi = \xi_1 \in G_0 \; .$$

We apply $\gamma_- V$ and obtain

$$\gamma_- V\gamma_- V^{-1}\xi = \gamma_- V\xi_1 \in G_0 \; .$$

The left side is

$$\begin{aligned}\gamma_- V(V^{-1}\xi - \gamma_+ V^{-1}\xi) &= \gamma_-\xi - \gamma_- V\gamma_+ V^{-1}\xi \\ &= \gamma_-\xi \; = \xi \; .\end{aligned}$$

We conclude $\xi \in G_0$ as asserted.

To verify $H_2(A)$ we first show that the natural mapping

$$G_0^N/(D-A)G_0 \longrightarrow \frac{1}{P} F_N(0,r)^k \,/\, (D-A)F(0,r)^k$$

is injective.

Let then $\eta \in G_0^N$, $\xi \in F(0,r)^k$

$$\eta = (D-A)\xi$$

i.e.

$$V^{-1}\eta = (D-B)V^{-1}\xi$$

and so

$$V^{-1}P\eta = P(D-B)V^{-1}\xi .$$

Applying γ_- we obtain by a previous argument

$$\gamma_- V^{-1}P\eta = \gamma_- P(D-B)\gamma_- V^{-1}\xi .$$

But $P\eta \in G_O$ and so $\gamma_- V^{-1}P\eta \in G_O$. The key point is that $\gamma_- V^{-1}\xi \in F(O,r)^k$ and so

$$P(D-B)\gamma_- V^{-1}\xi \in F(O,r)^k + (\Omega^N[x])^k .$$

Thus

$$P(D-B)\gamma_- V^{-1}\xi \in G_O + (\Omega^N[x])^k$$

i.e.

$$(D-B)\gamma_- V^{-1}\xi \in G_O^N .$$

We now use $H_2(B)$ to conclude that $\exists\, \xi_O \in G_O$ such that

$$(D-B)\xi_O = (D-B)(\gamma_-(V^{-1}\xi)) .$$

Using $H_1(B)$ we conclude that

$$\gamma_-(V^{-1}\xi) = \xi_1 \in G_O .$$

We now apply $\gamma_- V$ and compute

$$\gamma_- V\gamma_- V^{-1}\xi = \gamma_- V\xi_1 \in G_O .$$

The left side is

$$\gamma_- V V^{-1}\xi = \gamma_-\xi = \xi .$$

This shows that $\xi \in G_O$ which completes the proof of injectivity.

To complete the proof of $H_2(A)$ we must prove surjectivity. Let then $\eta \in F_N(O,r)^k$, so by $H_2(B)$ there exists $z \in P.\ G_O^N$, $\xi \in F(O,r)^k$ such that

$$z + P(D-B)\xi = \gamma_- V^{-1}\eta$$

i.e. applying V on the left,

$$Vz + P(D-A)V\xi = V\gamma_- V^{-1}\eta$$

and so

$$\gamma_- Vz + \gamma_- P(D-A)\gamma_- V\xi = \gamma_- V\gamma_- V^{-1}\eta = \eta .$$

Since $\gamma_- Vz \in G_O$

$$\gamma_+ P(D-A)\gamma_- V\xi \in \Omega^N[x]^k$$
$$\gamma_- V\xi \in F(0,r)^k$$

it is clear that

$$\eta \in G_0^N + (D-A)F(0,r) .$$

This completes the proof of $H_2(A)$.

A similar local comparison theorem holds at infinity. We need not pause to give the proof.

9.3. Local reduction theorem.

We consider a system of linear differential equations, L, deduced from a first order scalar equation

$$\ell = D - \theta$$

($\theta \in \Omega(x)$) by writting

$$L \begin{pmatrix} \xi_1 \\ \vdots \\ \xi_k \end{pmatrix} = \begin{pmatrix} \ell\xi_1 \\ \ell\xi_2 \\ \vdots \\ \ell\xi_k \end{pmatrix} - \frac{1}{x} \begin{pmatrix} \xi_2 \\ \xi_3 \\ \vdots \\ \xi_{k-1} \\ 0 \end{pmatrix} .$$

We choose P, N suitable for ℓ and L .

THEOREM. We assume that $H_1(\ell)$ and $H_2(\ell)$ are satisfied (with $k = 1$). We assert that $H_1(L)$ and $H_2(L)$ are satisfied.

Proof : If

$$L\xi = 0, \ \xi \in F(0,r)^k ,$$

then ξ_k the last component of ξ lies in $\mathrm{Ker}(\ell, F(0,r))$ and hence by $H_1(\ell)$ lies in G_0 . We now use induction and assume $\xi_{i+1} \in G_0$ and so

$$P\ell\xi_i = P\frac{1}{x}\xi_{i+1} \in \frac{1}{x}\Omega[\frac{1}{x}] + \Omega^N[x] .$$

By $H_2(\ell)$, there exists $\eta \in \frac{1}{x}\Omega[\frac{1}{x}]$ such that

$$\ell(\xi_i - \eta) = 0$$

and hence again by $H_1(\ell)$, $\xi_i - \eta \in \frac{1}{x}\,\Omega[\frac{1}{x}]$. This shows that $\xi_i \in \frac{1}{x}\,\Omega[\frac{1}{x}]$ and so $\xi \in G_O$ as asserted. This completes the proof of $H_1(L)$. To examine $H_2(L)$ we first verify that the natural map

$$G_O^N/LG_O \longrightarrow F_N(O,r)^k/LF(O,r)^k$$

is injective. Thus let $\xi \in F(O,r)^k$, $\eta \in G_O^N$

$$\eta = L\xi \ .$$

Hence

$$\ell\xi_k = \eta_k \in \frac{1}{P}(\frac{1}{x}\,\Omega[\frac{1}{x}] + \Omega^N[x])$$

which shows by $H_2(\ell)$ that there exists $\tilde{\xi}_k \in \frac{1}{x}\,\Omega[\frac{1}{x}]$ such that

$$\ell(\xi_k - \tilde{\xi}_k) = 0$$

and hence by $H_1(\ell)$ that $\xi_k \in \frac{1}{x}\,\Omega[\frac{1}{x}]$. Again we use induction, assume $\xi_{i+1} \in \frac{1}{x}\,\Omega[\frac{1}{x}]$ and use

$$\ell\xi_i = \eta_i + \frac{1}{x}\,\xi_{i+1} \in \frac{1}{P}\,[\frac{1}{x}\,\Omega[\frac{1}{x}] + \Omega^N[x])$$

and again conclude that $\xi_i \in \frac{1}{x}\,\Omega[\frac{1}{x}]$. This completes the proof of injectivity.

For surjectivity let $\eta \in \frac{1}{P}\,F_N(O,r)^k$. We will find $u \in G_O^N$, $\xi \in F(O,r)^k$ such that

$$u + L\xi = \eta \ .$$

For the last component the condition is

$$u_k + \ell\xi_k = \eta_k$$

which by $H_2(\ell)$ may be satisfied with $\xi_k \in F(O;r)$,

$$Pu_k \in \frac{1}{x}\,\Omega[\frac{1}{x}] + \Omega^N[\frac{1}{x}] \ .$$

We now use induction, for i^{th} component we need

$$u_i + \ell\xi_i = \eta_i + \frac{1}{x}\,\xi_{i+1} \ .$$

Since $\eta_i \in \frac{1}{P}\,F_N(O,r)$, $\xi_{i+1} \in F(O,r)$, we may by $H_2(\ell)$ choose $u_i \in \frac{1}{P}(\frac{1}{x}\,\Omega[\frac{1}{x}] + \Omega^N[x])$ and $\xi_i \in F(O,r)$ such that the indicated equality holds. This completes the proof of the theorem.

9.4. The Turrittin decomposition theorem asserts that a given linear system D-A is formally equivalent at $x=0$ (i.e. the matrix V in § 9.2 may have zero radius of convergence) to a direct sum of systems L (as in § 9.3) deduced from a scalar equation whose solution is $x^{\alpha} \exp \Delta(x^{-1/k!})$ where Δ is a polynomial. If the differences of the exponents α are all p-adically non-Liouville then the equivalence is not jus formal (Baldassarri [Ba]). The radii of convergence of the transformation matrices, V, are not well understood. When the singularity is regular and $\Delta = 0$ a recent result of Christol [Ch] gives informations on the radius of convergence of the transformation matrix V.

REFERENCES

[Ad] A. ADOLPHSON, An index theorem for p-adic differential operators. Trans. AMS 216 (1976), 279-293.

[A-S] A. ADOLPHSON - S. SPERBER, Twisted Kloosterman sums and p-adic Bessel functions, Amer. J. of Math., to appear.

[Am] Y. AMICE, Les nombres p-adiques. P.U.F. (1975).

[Ba] F. BALDASSARRI, Differential modules and singular points of p-adic differential equations. Adv. in Math. 44 (1982), 155-179.

[Bo] M. BOYARSKY, p-adic Gamma functions and Dwork cohomology. Trans. AMS 257 (1980), 356-369

[Cl] D. CLARK, A note on the p-adic convergence of solution of linear differential equations. Proc. AMS 17 (1966), 262-269.

[Ch] G. CHRISTOL, Un théorème de transfert pour les disques singuliers réguliers. Astérisque,

[Dw 1] B. DWORK, Lectures on p-adic differential equations. GL 253, Springer-Verlag, 1983.

[Dw 2] B. DWORK, On the Boyarsky principle, Amer. J. Math. Amer. J. of Math. 105 (1983)? 115-156.

[Dw 3] B. DWORK, On p-adic differential equations II Ann. of Math. 98 (1973), 366, 376.

[Dw 4] B. DWORK, Bessel functions as p-adic functions of the argument Duke Math. J. 41 (1974) 771-738.

[Dw 5] B. DWORK, On the zeta function of a hypersurface I.H.E.S. n° 12 (1962) 5-68.

[Ka 1] N. KATZ, Nilpotent connections and the monodromy theorem. Pub. Math. I.H.E.S. 39 (1970), 175-232.

[Ka 2] N. KATZ, Slope Filtration of F-crystals. Astérisque 63 (1979), 113-166.

[La] S. LANG, Cyclotomic fields II. Springer-Verlag (1980).

[Lu] E. LUTZ, Sur l'équation $y^2 = x^3 - Ax - B$ dans les corps p-adiques. T. Reine Angew. Math 177 (1937), 238-247.

[Ma] Y.I. MANIN, The theory of Commutative formal groups over fields of finite characteristic, Russian Math. Surveys 18 (1963), 1-83.

[Mo] P. MONSKY, Formal cohomology III. Ann. of Math. 93 (1971), 315-343.

[Re] D. REICH, A p-adic fixed point formula. Am. J. of Math. 91 (1969), 835-850.

[Ro 1] P. ROBBA, Fonctions analytiques sur les corps values ultra-metriques complets. Astérisque 10 (1973), 109-220.

[Ro 2] P. ROBBA, On the index of p-adic differential operator, Ann. of Math. 101 (1975), 280-316.

[Ro 3] P. ROBBA, On the index of p-adic differential operators II Duke Math. J. 43 (1976) 19-31.

[Se] J.-P. SERRE, Majoration de sommes exponentielles, Astérisque 41-42 (1977), 111-126.

[Sp] S. SPERBER, Congruences properties of the hyperkloosterman sum. Compositio mathematica 40 (1980), 3-33.

[We] A. WEIL, Basic number 3rd edit. Springer-Verlag (1974).

Université Paris IX
Département de Mathématiques
91406 ORSAY, FRANCE