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NON LINEAR PERTURBATIONS OF ISOMETRIES

by

Joram LINDENSTRAUSS and Andrzej SZANKOWSKI

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Section 1. Introduction

A classical result in Banach space theory is a theorem of Mazur and Ulam [4] which states that every isometry I from one real Banach space X onto another such space Y which maps 0 into 0 is linear. In 1945 Hyers and Ulam [3] asked whether any map T from X onto Y as above which satisfies

$$(1) \qquad \big|\, \|Tu-Tv\| - \|u-v\| \,\big| < \varepsilon \qquad u,v\in X$$

for some fixed $\varepsilon>0$ is close to an isometry. In view of the Mazur Ulam theorem this question is equivalent to the problem of how far a surjective map satisfying (1) can be from an affine map (i.e. a translation of a linear map). The question of Hyers and Ulam was solved recently by Gevirtz [1] who proved that if a surjective map T satisfies (1) then there is a (necessarily affine) isometry I from X onto Y so that

$$(2) \qquad \|Tu-Iu\| \le 5\varepsilon \text{ for all } u\in X.$$

The proof of Gevirtz makes an essential use of some ideas of Vogt [5] who proved a generalization of the Mazur Ulam theorem and of a paper of Gruber [2] where partial results on the Hyers Ulam problem were obtained (paper [2] contains also a short survey of earlier partial results on the Hyers Ulam problem).

The question which arises naturally in view of the result of Gevirtz is whether weaker assumptions than (1) imply that T is close to an isometry. Consider again a map T from a real Banach space X onto a real Banach space Y and put for $t\ge 0$

$$(3) \qquad \varphi_T(t) = \sup\{\, \big|\, \|Tu-Tv\|-\|u-v\| \,\big| \,:\, \|u-v\|\le t \text{ or } \|Tu-Tv\|\le t\}.$$

As pointed out by Gevirtz (private communication) the methods used in [1] can be used to show that if $\varphi_T(t)$ is unbounded but tends to ∞ with t sufficiently slowly then T must be a small perturbation of an isometry. On the other hand it is easy to see that unless $\varphi_T(t)$ is $o(t)$ as $t\to\infty$, no reasonable perturbation result generalizing (2) can be obtained.

In this paper we investigate the question what is the order of growth of $\varphi_T(t)$ as a function of t for which the perturbation theorem of Gevirtz can be generalized. It turns out quite surprisingly that it is possible to obtain a sharp result on the allowable order of growth and that this result does not depend on the special properties of X and Y. The critical order of growth of $\varphi_T(t)$ turns out to be like $t/\ell n t$. Our main positive result is :

<u>*Theorem 1.*</u> *Let* T *be a map from a real Banach space* X *onto a real Banach space* Y. *Let* $\varphi_T(t)$ *be defined by* (3). *If*

$$(4) \qquad \int_1^\infty \frac{\varphi_T(t)}{t^2}\, dt < \infty$$

then there is an isometry I *from* X *onto* Y *so that*

$$(5) \qquad \|Tu-Iu\| = o(\|u\|) \quad \textit{as} \quad \|u\|\to\infty \quad .$$

More precisely we have

$$(6) \qquad \|Tu-Iu\| \leq K(\|u\| \int_{\|u\|}^\infty \frac{\varphi(t)}{t^2}\, dt + \int_1^{\|u\|} \frac{\varphi(t)}{t}\, dt)$$

where K *is a universal constant and* $\varphi(t)=\max\{1,\varphi_T(t)\}$.

(To see that $\int_1^{\|u\|} \frac{\varphi(t)}{t}\, dt=o(\|u\|)$ let us observe that, by (4), $\varphi(t)=o(t)$).

Note that we assume besides (4) only that T is surjective. T need not be continuous or injective. That the surjectivity assumption is important here (as well as in the Hyers Ulam problem) is easy to see and is discussed in detail in [3] and [2].

Our second theorem shows that the condition (4) cannot be relaxed. Before we formulate it, let us mention that, for every T, the function φ_T is monotone increasing (obvious) and has the following property (cf. Lemma 1 below):

$$(7) \qquad \frac{\varphi_T(t)}{t} \leq 2\frac{\varphi_T(s)}{s} \text{ if } t\geq s>0.$$

<u>*Theorem 2.*</u> *Let* $\psi(t)$ *be a monotone increasing function on* $(0,\infty)$ *so that*

$$(8) \qquad \frac{\psi(t)}{t} \leq 2\,\frac{\psi(s)}{s} \quad \textit{if } t\geq s>0$$

and so that

$$(9) \qquad \int_1^\infty \frac{\psi(t)}{t^2}\, dt = \infty \quad .$$

Then there exist two non isometric real Banach spaces X *and* Y *and a homeomorphism* T *from* X *onto* Y *so that* $\varphi_T(t) \leq \psi(t)$ *for all* t.

The construction which verifies Theorem 2 takes place essentially in a Hilbert space. We construct a map T from ℓ_2 onto itself which satisfies $\varphi_T(t) \leq \psi(t)$ so that for every linear operator L on ℓ_2 and every sufficiently large t there is a $u \in \ell_2$ so that $\|u\|=t$ and $\|Tu-Lu\| \geq \|u\|$. (This justifies the assertion hinted to above that, even in the "best" space, condition (4) cannot be relaxed). The non isometric spaces X and Y appearing in the statement of Theorem 2 are obtained from ℓ_2 by using small renormings. Though X and Y are non isometric they are very close in the sense that their Banach Mazur distance d(X,Y) is 1 (i.e. for every $\varepsilon > 0$ there is an isomorphism U_ε from X onto Y with $\|U_\varepsilon\| \, \|U_\varepsilon^{-1}\| \leq 1+\varepsilon$).
It is possible that whenever X and Y are Banach spaces for which there is a mapping T from X onto Y satisfying $\big| \|Tu-Tv\| - \|u-v\| \big| = o(\|u-v\|)$ then $d(X,Y)=1$. This seems to be most likely the case if X and Y have the Radon Nikodym property.

Before we pass to the proofs we want to make one further comment on Theorems 1 and 2 and this concerns the estimate given in (6).
Put

$$(10) \qquad g(s) = s \int_s^\infty \frac{\varphi_T(t)}{t^2}\,dt, \quad h(s) = \int_1^s \frac{\varphi_T(t)}{t}\,dt$$

The construction used in the proof of Theorem 2 if applied to a function ψ with $\int_1^\infty \frac{\psi(t)}{t^2}\,dt < \infty$ produces a map T from ℓ_2 onto itself with $\varphi_T(t) \leq \psi(t)$ which is close to an isometry I from ℓ_2 into itself (as it has to be, by Theorem 1) so that for all sufficiently large s and an absolute constant $\gamma > 0$ there exists u with $\|u\|=s$ and such that

$$\|Tu-Iu\| \geq \gamma g(s)$$

Thus the first summand in the right hand side of (6) has to appear. We do not know whether the summand $h(\|u\|)$ can be dropped. The function h(s) is essentially larger than g(s) for large s only if $\varphi_T(t)$ grows very slowly (e.g. if $\lim_{t\to\infty} \varphi_T(t)/t^\alpha = 0$ for $\alpha > 0$). The main point in Grubers paper [2] is a proof that for very small φ_T (i.e. for bounded φ_T) the summand $h(\|u\|)$ is not needed in (6). For $\varphi_T(t)$ close to t (e.g. for $\varphi_T(t) \approx t/(\log t)^\alpha$ with $\alpha > 1$) we have that $\varphi_T(\|u\|) = o(g(\|u\|))$ as $\|u\| \to \infty$. Consequently it follows by the observation made above that in contrast to the solution to the Hyers Ulam problem given by (2) we cannot replace (6) in Theorem 1 by an estimate of the form $\|Tu-Iu\| \leq C\,\varphi_T(\|u\|)$ even if C is allowed to depend on T.

Section 2 below contains just some simple preliminary observations. Section 3 is devoted to the proof of Theorem 1. The main step in the proof is a proposition concerning almost isometries in general (i.e. not necessarily linear) metric spaces. Theorem 2 is proved in Section 4.

Section 2. Preliminaries.

In this section we prove three simple remarks which will be used in the subsequent sections.

Lemma 1. Let T be a map from a Banach space X onto a Banach space Y and define $\varphi_T(t)$ by (3). Then

$$\text{(11)} \qquad \varphi_T(r+t) \le \varphi_T(r) + \varphi_T(t) \qquad r,t \ge 0$$

and consequently also (7) holds.

Proof. Assume that $\|u-v\| \le r+t$ and that $\|Tu-Tv\| \ge \|u-v\|$. Let w be a point on the segment $[u,v]$ so that $\|u-w\| \le r$ and $\|w-v\| \le t$. Then

$$\big|\|Tu-Tv\| - \|u-v\|\big| = \|Tu-Tv\| - \|u-v\|$$

$$\le \|Tu-Tw\|-\|u-w\| + \|Tw-Tv\|-\|w-v\| \le \varphi_T(r) + \varphi_T(t).$$

Assume now that $\|Tu-Tv\| \le \|u-v\|$ and that $\|Tu-Tv\| \le r+t$. By the surjectivity of T there is a $w \in X$ so that Tw belongs to the segment $[Tu,Tv]$ and so that $\|Tu-Tw\| \le r$, $\|Tw-Tv\| \le t$. Then

$$\big|\|Tu-Tv\|-\|u-v\|\big| = \|u-v\|-\|Tu-Tv\| \le$$

$$\|u-w\|-\|Tu-Tw\| + \|w-v\|-\|Tw-Tv\| \le \varphi_T(r) + \varphi_T(t).$$

This verifies (11). From (11) and the monotonicity of φ_T it follows that for $t \ge s$

$$\frac{\varphi_T(t)}{t} \le \left(\left[\frac{t}{s}\right]+1\right) \frac{s}{t} \frac{\varphi_T(s)}{s} \le 2\frac{\varphi_T(s)}{s}$$

Lemma 2. Assume that $\varphi(t)$ is a non negative monotone increasing function. Then $\sum_{k=1}^{\infty} \varphi(2^k)2^{-k}$ converges if and only if $\int_1^{\infty} \varphi(t)\, t^{-2}dt<\infty$.

Proof. Obvious.

Lemma 3. Let X and Y be Banach spaces and let $T:X\to Y$ be a surjective map such that $\varphi_T(t) = o(t)$ as $t\to\infty$. Then for every $\varepsilon>0$ there is a bijective (= injective and surjective) map $S:X\to Y$ so that $\|Su-Tu\| \le \varepsilon/2$ for every $u \in X$.

Consequently, we have $\varphi_S(t) \le \varphi_T(t+\varepsilon) + \varepsilon$ for every t.

Proof. We shall first prove that card X = card Y (i.e. that X and Y are equinumerous). Clearly card X $\geq$ card Y. To show the converse inequality it suffices to prove that the density character of X is not larger than that of Y. Let τ be such that $\varphi_T(t) \leq t/2$ for $t \geq \tau$. Then $\|Tu-Tv\| \geq \tau/2$ whenever $\|u-v\| \geq \tau$. Let $A \subset X$ be a maximal subset with the property that $\|u-v\| \geq \tau$ for every $u \neq v$ in A. The density character of X is equal to the cardinality of A and, since $\|Tu-Tv\| \geq \tau/2$ for $u \neq v$ in A, this cardinal is not larger than the density character of Y.

Let $\varepsilon>0$ and let $Y = \cup Y_\alpha$ where Y_α are pairwise disjoint sets such that for every Y_α, diam $Y_\alpha \leq \varepsilon/2$ and card Y_α = card Y(= card X). Put $X_\alpha = T^{-1}Y_\alpha$. Since card X_α = card Y_α, we can find for every α a bijective map S_α from X_α onto Y_α. We define now $Su = S_\alpha u$ for $u \in X_\alpha$. Clearly $\|Su-Tu\| \leq \varepsilon/2$ for every $u \in X$.

The last statement of the Lemma is quite obvious: suppose that $\|u-v\| \leq t$ or $\|Su-Sv\| \leq t$. Then $\min\{\|u-v\|, \|Tu-Tv\|\} \leq t + \varepsilon$ and we have

$$(12)\qquad \big|\|Su-Sv\| - \|u-v\|\big| \leq \big|\|Tu-Tv\| - \|u-v\|\big| + \varepsilon \leq \varphi_T(t+\varepsilon) + \varepsilon.$$

Section 3. The proof of Theorem 1.

As mentioned in the introduction, the main part of the proof presented here will be carried out in the setting of general metric spaces. For a mapping T between metric spaces X and Y the function φ_T is defined by

$$(13)\qquad \varphi_T(t) = \sup\{|d(Tu,Tv) - d(u,v)| : d(u,v) \leq t \text{ or } d(Tu,Tv) \leq t\}$$

Let X be a metric space and let $u, v \in X$. The point $x \in X$ will be called a *metric center* for u,v if there exists an isometry $a:X\to X$ such that $au = v$ and for every $w \in X$

$$(14)\qquad a(aw) = w,\ d(w,aw) = 2d(w,x).$$

Clearly (14) implies in particular that x is a fixed point of a. Obviously if X is a normed space, then the point $x = (u+v)/2$ is a metric center of the pair u,v in X.

Proposition. Let T *be a bijective map from a metric space* X *onto a metric space* Y. *Let* φ *be any function such that* $\varphi(t) \geq \varphi_T(t)$ *and* $\varphi(2t) \leq 2\varphi(t)$ *for every* t. *Let* $u, v \in X$ *and assume that this pair has a metric center* x *and that* Tu,Tv *has a metric center* y. *Put* $d = d(u,v)$ *and let* n *be an integer so that* $2^n \geq d/\varphi(d)$. *Then*

$$(15)\qquad d(y,Tx) \leq 19\varphi(d) + \varphi(\tfrac{1}{2}d) + \varphi(\tfrac{1}{4}d) + \ldots + \varphi(2^{-n}d).$$

Before proving the Proposition we shall show how to deduce Theorem 1 from it.

Proof of Theorem 1. Let T be a surjective map from the Banach space X onto the Banach space Y so that (4) holds. By Lemma 3 there is no loss of generality to assume that T is bijective. Also we assume, as we may without loss of generality, that $T0 = 0$. The main point in the proof is to show that

$$(16)\qquad Iw = \lim_{n\to\infty} 2^{-n}\, T(2^n w)$$

exists for every $w \in X$.

Let $\varphi(t) = \max\{1, \varphi_T(t)\}$. By (11), we have $\varphi(2t) \le 2\varphi(t)$. By applying the proposition with $u = 0$, $v = 2^n w$, $x = (u+v)/2 = 2^{n-1}w$ and $y = (Tu + Tv)/2 = \frac{1}{2}T(2^n w)$ we deduce that if $\|w\| \le 2^m$ with $m \ge 0$ then

$$(17)\qquad \|T_n w\| \le 2^{-n+1}\,(19\,\varphi(2^{n+m}) + \varphi(2^{n+m-1}) + \varphi(2^{n+m-2}) + \ldots + \varphi(1))$$

where

$$(18)\qquad T_n w = 2^{-n}T(2^n w) - 2^{-n+1}T(2^{n-1}w)\ ,\ n = 1,2,\ldots$$

By (17),

$$(19)\qquad \sum_{n=1}^{\infty} \|T_n w\| \le 19 \sum_{n=1}^{\infty} 2^{-n+1}\,\varphi(2^{n+m}) + \sum_{j=m+1}^{\infty} \varphi\,(2^j) \sum_{n=j-m}^{\infty} 2^{-n} + \sum_{j=0}^{m} \varphi(2^j) \sum_{n=0}^{\infty} 2^{-n} \le$$

$$\le 20\cdot 2^m \sum_{n=m+1}^{\infty} 2^{-n}\,\varphi(2^n) + 2\sum_{j=0}^{m} \varphi(2^j).$$

By (4) and Lemma 2, the left hand side of (19) is finite. Hence

$$Iw = Tw + \sum_{n=1}^{\infty} T_n w$$

exists. Moreover, since for w of norm ≥ 1 we can clearly take m so that $\|w\| \ge 2^{m-1}$, it follows from (19) and Lemma 2 that (6) holds.

It remains to verify that the map I defined by (16) is a surjective isometry from X onto Y. Let $u,v \in X$ and let $n \ge 1$ be an integer. By applying the proposition to $2^n u, 2^n v$ and using the fact that $\varphi_T(t) = o(t)$ as $t\to\infty$ it follows that

$$\|2^{-n}T(2^n \tfrac{1}{2}(u+v)) - \tfrac{1}{2}(2^{-n}T(2^n u) + 2^{-n}T(2^n v))\| = o(1)$$

as $n\to\infty$. Consequently

$$(20) \qquad I(\tfrac{1}{2}(u+v)) = \tfrac{1}{2}(Iu+Iv).$$

From the definition of φ_T and I and the fact that $\varphi_T(t)=o(t)$ as $t\to\infty$ it follows that I is an isometry and thus, by (20), it is a linear isometry. Finally we show that I is surjective. Assume it is not. Then there is a $y\in Y$ with $\|y\|=1$ so that $d(y,IX)\geq\frac{1}{2}$. Let $x_n\in X$ satisfy $Tx_n=ny$. Then $\|x_n\|=O(n)$ and $\|Tx_n-Ix_n\|\geq n/2$. On the other hand, by (5), $\|Tx_n-Ix_n\|=o(n)$, a contradiction.

We pass now to the proof of the Proposition. Let u,v,x,y,X,Y and T be as in the statement of the Proposition. We define inductively two sequences $\{x_j\}_{j=-\infty}^{\infty}\subset X$ and $\{y_j\}_{j=-\infty}^{\infty}\subset Y$ by putting

$$x_0 = x \quad , \quad y_0 = y$$

$$(21) \qquad x_{j+1} = T^{-1}y_{-j} \quad , \quad y_{j+1} = Tx_{-j} \qquad j=0,1,2,\ldots$$

$$x_{-j}= ax_j \ , \ y_{-j}= bx_j$$

where a and b are the isometries of X, respectively Y, associated with the definition of x as a metric center of u,v, respectively y as a metric center of Tu,Tv. Note that

$$(22) \qquad Tx_j=y_{-j+1} \ , \quad T^{-1}y_j=x_{-j+1} \qquad j=0,\pm1,\pm2,\ldots$$

We define further the following numbers for j=0,1,2,...

$$(23) \qquad \delta_{2j} = \max\{d(x_{2k},x_{2\ell}) \ ; \ |k-\ell|=j, \ -j\leq k,\ell\leq j\}$$

$$(24) \qquad \Delta_{2j}= \max_{0\leq i\leq j} \ \max\{d(x_{2i},u) \ , \ d(x_{2i},v)\}$$

We need a couple of lemmas that describe the behaviour of the quantities defined in (23) and (24).

Lemma 4. With the notation as above

$$(25) \qquad \delta_{2^{m+1}} \geq 2\delta_{2^m} - 2^m \varphi_T(\delta_{2^m}) \ , \ m=1,2,\ldots$$

Proof. Assume that $|k-\ell|=j$, $-j+1 \leq k,\ell \leq j$. We have

$$(26) \qquad |d(x_{2k-2}, x_{2\ell-2}) -d(x_{2k},x_{2\ell})| \leq 2\varphi_T(\delta_{2j})$$

Indeed,

$$|d(x_{2k-2}, x_{2\ell-2}) - d(x_{2k}, x_{2\ell})| = |d(ax_{-2k+2}, ax_{-2\ell+2}) - d(x_{2k}, x_{2\ell})| =$$

$$= |d(x_{-2k+2}, x_{-2\ell+2}) - d(x_{2k}, x_{2\ell})| \le |d(x_{-2k+2}, x_{-2\ell+2}) - d(Tx_{-2k+2}, Tx_{-2\ell+2})| +$$

$$+ |d(Tx_{-2k+2}, Tx_{-2\ell+2}) - d(Tx_{2k}, Tx_{2\ell})| + |d(Tx_{2k}, Tx_{2\ell}) - d(x_{2k}, x_{2\ell})| \le$$

$$\le 2\varphi_T(\delta_{2j})$$

because

$$d(Tx_{-2k+2}, Tx_{-2\ell+2}) = d(y_{2k-1}, y_{2\ell-1}) = d(by_{2k-1}, by_{2\ell-1})$$

$$= d(y_{-2k+1}, y_{-2\ell+1}) = d(Tx_{2k}, Tx_{2\ell}).$$

Let now k, ℓ be such that $|k-\ell| = 2^{m-1}$, $-2^{m-1} \le k, \ell \le 2^{m-1}$ and

$$\delta_{2^m} = d(x_{2k}, x_{2\ell}). \tag{27}$$

Since $d(x_{2k}, x_{2\ell}) = d(x_{-2k}, x_{-2\ell})$ there is no loss of generality to assume that $k \ge 2^{m-2}$. By applying (26) $2^{m-1}-k$ times we obtain that

$$|d(x_{2k}, x_{2\ell}) - d(x_{2^m}, x_0)| \le 2(2^{m-1}-k)\varphi_T(\delta_{2^m}) \le 2^{m-1}\varphi_T(\delta_{2^m}).$$

This inequality together with (27) and the fact that

$$\delta_{2^{m+1}} \ge d(x_{-2^m}, x_{2^m}) = 2\, d(x_{2^m}, x_0)$$

imply (25).

<u>Lemma 5</u>. Let Δ_{2j} be defined by (24). If $j \le \frac{d}{8\varphi_T(d)}$, then $\Delta_{2j} \le d$.

<u>Proof</u>. Without loss of generality we can assume that

$$\varphi_T(d) \le \frac{1}{4}d \tag{28}$$

(otherwise there is nothing to prove). Let

$$N = \max\{j : \Delta_{2j} \le \frac{3}{4} d\}.$$

For every $j \le N$ we have the following estimate

$$|d(x_{2j+2}, u) - d(x_{2j}, u)| \le 2\varphi_T(d). \tag{29}$$

Indeed,

$$|d(x_{2j+2},u) - d(x_{2j},u)| = |d(x_{2j+2},u) - d(ax_{2j},au)| =$$

$$= |d(x_{2j+2},u) - d(x_{-2j},v)| \leq |d(x_{2j+2},u) - d(Tx_{2j+2},Tu)| +$$

$$+ |d(Tx_{2j+2}, Tu) - d(Tx_{-2j}, Tv)| + |d(Tx_{-2j}, Tv) - d(x_{-2j}, v)|.$$

We have

$$(30) \quad d(x_{-2j},v) = d(ax_{-2j}, av) = d(x_{2j}, u) \leq \Delta_{2j} \leq \frac{3}{4}d,$$

therefore

$$(31) \quad |d(Tx_{-2j},Tv) - d(x_{-2j},v)| \leq \varphi_T(\tfrac{3}{4}d) \leq \varphi_T(d).$$

Hence, by (28)

$$(32) \quad d(Tx_{-2j},Tv) \leq d(x_{-2j},v) + \varphi_T(d) \leq \frac{3}{4}d + \frac{1}{4}d = d.$$

Secondly,

$$(33) \quad d(Tx_{2j+2},Tu) = d(y_{-2j-1},Tu) = d(by_{-2j-1},bTu) = d(y_{2j+1},Tv) =$$

$$= d(Tx_{-2j},Tv)$$

Therefore, by (32),

$$(34) \quad |d(x_{2j+2},u) - d(Tx_{2j+2},Tu)| \leq \varphi_T(d(Tx_{2j+2},Tu)) =$$

$$= \varphi_T(d(Tx_{-2j},Tv)) \leq \varphi_T(d).$$

Now, (31), (33) and (34) yield (29). Of course the same inequality holds with u replaced by v. Consequently,

$$\Delta_{2j+2} \leq \Delta_{2j} + 2\varphi_T(d) \text{ if } j \leq N.$$

Since $\Delta_0 = \frac{1}{2}d$, by induction we obtain

$$\Delta_{2j+2} \leq \frac{1}{2}d + (2j+2)\, \varphi_T(d) \text{ if } j \leq N.$$

From this follows, again by induction, that if $j \leq \dfrac{d}{8\varphi_T(d)}$,

then

$$\Delta_{2j} \leq \frac{1}{2}d + \frac{1}{4}d \leq \frac{3}{4}d,$$

thus proving the lemma.

Proof of the Proposition. Let us recall that φ is any function such that $\varphi \geq \varphi_T$ and

(35) $\quad \varphi(2t) \leq 2\varphi(t)$ for every t.

Evidently, Lemmas 4 and 5 hold with φ_T replaced by φ.

Let us notice that, by the definition of δ_{2j} and Δ_{2j},

(36) $\quad \delta_{2j} \leq 2\Delta_{2j}$ for every j.

Let n be the largest integer such that $2^n \leq \frac{d}{8\varphi(d)}$.

By Lemma 5, $\Delta_{2^{n+1}} \leq d$. Consequently, by (36),

(37) $\quad \delta_{2^{n-k}} \leq 2\,\Delta_{2^{n+1}} \leq 2d$ for $k = -1,0,1,\ldots,n$.

and, by (35),

(38) $\quad \varphi(\delta_{2^{n-k}}) \leq \varphi(2d) \leq 2\varphi(d)$ for $k = -1,0,1,\ldots,n$.

Now we shall prove that

(39) $\quad \delta_{2^{n-k}} \leq \left(\frac{1}{\sqrt{2}}\right)^{k-1} d$ for $k = -1,0,1,\ldots,n-1$.

By (37), this is true for $k = -1$. By Lemma 4 and by (38), we have, for $k = 0,1,2,\ldots,n-1$

$$\delta_{2^{n-k}} \leq \frac{1}{2}\,\delta_{2^{n-k+1}} + 2^{n-k-1}\,\varphi(\delta_{2^{n-k}}) \leq$$
$$\leq \frac{1}{2}\,\delta_{2^{n-k+1}} + \frac{d}{2\varphi(d)} \cdot 2^{-k-3}\varphi(\delta_{2^{n-k}}) \leq$$
$$\leq \frac{1}{2}\,\delta_{2^{n-k+1}} + 2^{-k-3}d.$$

An induction on k and a trite calculation show that (39) holds.

Let us next observe that, by repetetive use of Lemma 4, the following inequality holds for $m = 1,2,\ldots$

(40) $\quad \delta_{2^{m+1}} \geq 2^m\,\delta_2 - 2^m\,[\varphi(\delta_{2^m}) + \varphi(\delta_{2^{m-1}}) + \ldots + \varphi(\delta_2)]$.

Since $\frac{d}{8\varphi(d)} \leq 2^{n+1}$, by (39) and by (40) we have

$$\delta_2 + \varphi(\delta_2) \leq 2^{-n}\delta_{2^{n+1}} + \varphi(\delta_{2^n}) + \varphi(\delta_{2^{n-1}}) + \ldots + \varphi(\delta_2) + \varphi(\delta_2) \leq$$

$$\leq \frac{16\varphi(d)}{d}\,2d + \varphi(\sqrt{2}d) + \varphi(d) + \ldots + \varphi(2^{-\frac{n}{2}+1}d) + \varphi(2^{-\frac{n}{2}+1}d) \leq$$

$$\leq 32\varphi(d) + 2\varphi(2d) + 2\varphi(d) + 2\varphi(\tfrac{1}{2}d) + \ldots + 2\varphi(2^{-n}d).$$

Since $\varphi(2t) \leq 2\varphi(t)$, we have

$$\delta_2 + \varphi(\delta_2) \leq 38\varphi(d) + 2(\varphi(\tfrac{1}{2}d) + \ldots + \varphi(2^{-n}d)).$$

To prove the proposition it is enough to notice that

$$d(y,Tx) = d(y_0,y_1) = \tfrac{1}{2}d(y_{-1},y_1) = \tfrac{1}{2}d(Tx_2,Tx_0) \leq$$

$$\leq \tfrac{1}{2}(\delta_2 + \varphi(\delta_2)).$$

Section 4. The proof of Theorem 2.

Let us define an auxiliary function $\varphi(t)$ by

$$\varphi(t) = \min\{\tfrac{1}{2}t,\ \psi(\tfrac{1}{2}t)\}$$

It is easy to see that φ is an increasing function which has the properties (8) and (9), with ψ replaced by φ, of course.

Let $\ell_2 = \ell_2(\{0,1,2,\ldots\})$ with the usual norm

$$|||(x_i)_{i=0}^{\infty}||| = (\sum_{i=0}^{\infty}|x_i|^2)^{\frac{1}{2}}.$$

The spaces X and Y are both equal to ℓ_2 with equivalent norms $\|\ \|_X$ and $\|\ \|_Y$, respectively. The homeomorphism $T:X\to Y$ is actually an ℓ_2-isometry of every sphere of ℓ_2 onto itself, which, globally, "slowly rotates" the spheres of ℓ_2.

In order to define $\|\ \|_X$, $\|\ \|_Y$ and T we shall need a function $f:[0,\infty)\to[0,\infty)$, which depends on φ and a sequence of positive numbers $(\lambda_n)_{n=1}^{\infty}$, which depends on f.

The function $f:[0,\infty)\to[0,\infty)$ is defined by

$$f(t)=2 \text{ for } 0\leq t\leq 1\ ;$$

$$f(t)=2 + \frac{1}{10}\int_1^t \frac{\varphi(x)}{x^2}\,dx \text{ for } 1\leq t<\infty.$$

First let us observe that

(41) $\qquad t(f(t+s) - f(t)) \leq \frac{1}{5}\varphi(s) \quad \text{if } t\geq 0,\ s>0.$

Indeed, if $t \geq 1$, then

$$f(t+s)-f(t) = \frac{1}{10}\int_t^{t+s} \frac{\varphi(x)}{x^2}\,dx \leq \frac{1}{10}\varphi(t+s)\int_t^{t+s}\frac{dx}{x^2} = \frac{1}{10}\varphi(t+s)\cdot\frac{s}{t(t+s)}$$

and (41) follows because $\frac{\varphi(t+s)}{t+s} \leq 2\frac{\varphi(s)}{s}$. The case $0<t<1$ follows by observing that $f(t) = f(1)$ for $0<t<1$.

Secondly let us notice that, by (9), $\lim\limits_{t\to\infty} f(t)=\infty$. Moreover, f is a continuous function. It is therefore possible to find numbers t_n so that

$$(42) \qquad f(\tfrac{1}{3}t_n) = n+2 \quad \text{for } n=1,2,\ldots$$

Let us now pick a decreasing sequence of positive numbers $(\lambda_n)_{n=1}^{\infty}$ so that the following two conditions are satisfied

$$(43) \qquad \Big(\sum_{n=1}^{\infty}\lambda_n^2\Big)^{\frac{1}{2}} \leq \frac{1}{4}$$

$$(44) \qquad 0 < \lambda_n \leq \frac{\varphi(t_n)}{8t_n} \quad \text{for } n=1,2,\ldots$$

Finally we shall define some unitary operators $T_t:\ell_2\to\ell_2$ for $t\in[1,\infty)$. Let $\{e_i\}_{i=0}^{\infty}$ denote the unit vectors in ℓ_2. For $n=1,2,\ldots$ and $0\leq\Theta\leq 1$ we define

$$T_{n+\Theta}(e_i) = \begin{cases} e_i & \text{for } i\geq n+1,\\ -e_{i-1} & \text{for } 0<i\leq n-1,\\ \cos\frac{\Theta\pi}{2}\cdot e_{n-1}+\sin\frac{\Theta\pi}{2}\cdot e_n & \text{for } i=0,\\ -\sin\frac{\Theta\pi}{2}\cdot e_{n-1}+\cos\frac{\Theta\pi}{2}\cdot e_n & \text{for } i=n,\end{cases}$$

It is easy to verify that for $t\geq 1$, $s>0$,

$$(45) \qquad |||T_{t+s}-T_t||| \leq K\cdot s \text{ where } K=\frac{\pi}{2}<2.$$

We define now the norms $\|\ \|_X$, $\|\ \|_Y$ and the homeomorphism $T:X\to Y$ by the following formulas (as we said above, we define $X=(\ell_2,\|\ \|_X)$, $Y=(\ell_2,\|\ \|_Y)$):

$$\|x\|_X = |||x||| + \sum_{i=1}^{\infty} \lambda_i \; |x_{i+1}|$$

$$\|x\|_Y = |||x||| + \sum_{i=1}^{\infty} \lambda_i \; |x_i|$$

(here $x=(x_i)_{i=0}^{\infty} \in \ell_2 = X = Y$) and

$$T\,x = T_{f(|||x|||)}(x) \quad \text{for } x \in \ell_2.$$

The spaces X and Y are not isometric since span $\{e_0,e_1\}$ in X is isometric to ℓ_2^2 while Y fails to have a two dimensional subspace with differentiable norm.

It is also evident that T is a 1-1 surjective map. It remains thus to show that $\varphi_T \le \psi$.

Note that, by (43), we have for every $x \in \ell_2$

(46) $\quad |||x||| \le \|x\|_X \le \frac{5}{4}\, |||x|||, \quad |||x||| \le \|x\|_Y \le \frac{5}{4}\, |||x|||.$

Let $n+1 \le r < n+2$. By writing down the formulas for $\|T_r x\|_Y$ and $\|x\|_X$ we see easily that there exist α,β with $\alpha^2+\beta^2 = 1$ such that

$$\|T_r\, x\|_Y - \|x\|_X =$$

$$= \lambda_n |\alpha x_0 - \beta x_{n+1}| + \lambda_{n+1} |\beta x_0 + \alpha x_{n+1}| - \lambda_n |x_{n+1}| - \sum_{i=n+2}^{\infty} |x_i| (\lambda_{i-1} - \lambda_i)$$

Hence,

(47) $\quad \big|\|T_r x\|_Y - \|x\|_X\big| \le 2\lambda_n |||x||| \le 2\lambda_n \|x\|_X$ if $n+1 \le r < n+2$.

Let now $u,v \in \ell_2$ with $|||u||| \ge |||v|||$. Let us denote $t=|||v|||$, $s=|||u|||-|||v|||$. We have

(48) $\quad \|u-v\|_X \ge |||u-v||| \ge |||u|||-|||v||| = s$

By (46), (45) and (41) we have

(49) $\quad \|T_{f(t+s)}(v) - T_{f(t)}(v)\|_Y \le 1.25\; |||T_{f(t+s)}(v)-T_{f(t)}(v)||| \le$

$$\le 2\cdot 1.25(f(t+s)-f(t))\, |||v||| = 2.5t(f(t+s)-f(t)) \le$$
$$\le \frac{1}{2}\, \varphi(s) \le \frac{1}{4} s \le \frac{1}{4}\, \|u-v\|_X$$

We have

$$\|Tu-Tv\|_Y \geq |||Tu-Tv||| = |||T_{f(t+s)}(u)-T_{f(t)}(v)||| \geq$$

$$\geq |||T_{f(t+s)}(u) - T_{f(t+s)}(v)||| - |||T_{f(t+s)}(v) - T_{f(t)}(v)|||$$

The first term is equal to $|||u-v|||$, because T_α is a unitary operator on ℓ_2, the second term is estimated by (49). We thus obtain, by (46) and (48)

$$(50)\quad \|Tu-Tv\|_Y \geq |||u-v|||-\tfrac{1}{2}\varphi(s) \geq |||u-v|||-\tfrac{1}{2}\varphi(|||u-v|||)$$

$$\geq \tfrac{3}{4}\,|||u-v||| \geq \tfrac{3}{5}\,\|u-v\|_X$$

Now we proceed to estimate the difference

$$(51)\quad \big|\|Tu-Tv\|_Y-\|u-v\|_X\big| \leq \|T_{f(t+s)}(v)-T_{f(t)}(v)\|_Y +$$

$$+ \big|\|T_{f(t+s)}(u)-T_{f(t+s)}(v)\|_Y-\|u-v\|_X\big|.$$

The first term is estimated by (49). In order to estimate the second one, let us denote $r=f(t+s)$ and let n be the integer such that $n+1\leq r<n+2$.

Since f is an increasing function, by (42) and (46), we have

$$\tfrac{1}{3}t_n \geq t+s = |||u||| \geq \tfrac{1}{2}\,|||u-v||| \geq \tfrac{1}{3}\,\|u-v\|_X,$$

thus

$$\|u-v\|_X \leq t_n$$

Hence, by (44), (47) and (8),

$$\big|\|T_{f(t+s)}(u) - T_{f(t+s)}(v)\|_Y-\|u-v\|_X\big| \leq 2\lambda_n\|u-v\| \leq$$

$$\frac{\varphi(t_n)\|u-v\|_X}{4t_n} \leq \tfrac{1}{2}\varphi(\|u-v\|_X).$$

This, together with (49) and (51) gives the estimate

$$\big|\|Tu-Tv\|_Y - \|u-v\|_X\big| \leq \varphi(\|u-v\|_X) .$$

Suppose now that $\min\{\|u-v\|_X, \|Tu-Tv\|_Y\} = \alpha$.

By (50),

$$\|u-v\|_X \leq \frac{5}{3}\alpha \leq 2\alpha.$$

Consequently,

$$\left|\|Tu-Tv\|_Y - \|u-v\|_X\right| \leq \varphi(2\alpha) \leq \psi(\alpha),$$

which proves the desired inequality $\varphi_T(\alpha)\leq\psi(\alpha)$.

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