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RECURRENCE OF BROWNIAN MOTIONS ON COMPACT MANIFOLDS

by J.R. Baxter and G.A. Brosamler

§1. INTRODUCTION AND SUMMARY

Let M be a compact C^∞ manifold of dimension d. A C^∞ metric $\mathcal{g}$ and a C^∞ vector field V on M determine a "Brownian motion" on M, i.e. a strong Markov process $\{\Omega, \mathcal{A}; P^x, x \in M; X_t: \Omega \to M, \mathcal{F}_t, \theta_t, t \geq 0\}$ with continuous sample paths and generator

$$(1.1) \qquad L = \frac{1}{2}\Delta + V \quad ,$$

where Δ is the Laplace-Beltrami operator associated with the metric $\mathcal{g}$. If m denotes the Riemann measure induced by $\mathcal{g}$, then $P^x\{X_t \in B\} = \int_B p(t,x,y)dm(y)$, $t > 0$, $x \in M$, B Borel $\subseteq$ M, and the transition density p is the fundamental solution of

$$L'_y p(t,x,y) = \frac{\partial}{\partial t} p(t,x,y) \quad ,$$

where L' is the dual of L with respect to m.

We shall discuss certain aspects of the recurrence pattern of such Brownian motions, which - appropriate for the occasion - are connected with Schwartz distributions. To be precise, we are interested in the asymptotic behaviour of functionals

$$(1.2) \qquad L_t(f) = L_t(f,\omega) = \int_0^t f(X_s)ds.$$

These functionals are well-defined for all $\omega \in \Omega$ if $f : M \to R$ is bounded Borel, and are well-defined P^m-a.e. if $f \in L^1(M)$. The exceptional Ω-set depends of course on f. The case $d = 1$ plays a special role. In this case there exists Ω_0 such that $P^x(\Omega_0) = 1$, $x \in M$, and $f \circ X_. \in L^1_{loc}(R^+)$ for $\omega \in \Omega_0$, $f \in L^1(M)$. Just take as Ω_0 the Ω-set where the local time $L(t,x)$ is a well-defined continuous function on $R^+ \times M$, and observe

$$(1.3) \qquad L_t(f) = \int_M f(x)L(t,x)dm(x).$$

The connection between the functionals (1.2) and the recurrence pattern of the Brownian motion X is rather obvious: We recall that X has a unique invariant probability measure λ, and that $d\lambda = \phi dm$, $\phi > 0$, $\phi \in C^\infty(M)$, $L'\phi = 0$ (see e.g. [1]). The ergodic theorem gives for all $f \in L^1(M)$

$$(1.4) \qquad P^\lambda\{\lim_{t\to\infty} t^{-1}L_t(f) = S(f)\} = 1,$$

with $S(f) = \int_M f d\lambda$. A trivial consequence is that for all $x \in M$

$$(1.5) \qquad P^x\{\lim_{t\to\infty} t^{-1}L_t(f) = S(f),\ f \in C(M)\} = 1,$$

which of course implies that P^x-a.e. the cluster set as $t\to\infty$, of $X_t(\omega)$ equals M.

More delicate investigations of the recurrence pattern of X involve naturally the fluctuations of $L_t(f)$. In [1] a central limit theorem and a $\log_2$-law were obtained for these fluctuations for f bounded Borel. The normalized asymptotic variance was obtained as the self-energy of f, i.e.

$$(1.6) \qquad \sigma_f^2 = 2(f,Gf)_{L^2(d\lambda)} = \int_M |\text{grad } Gf|^2 d\lambda,$$

where $G : L^1(M) \to L^1(M)$ is defined as the Green operator

$$(1.7) \qquad (Gf)(x) = \int g(x,y)f(y)dm(y)$$

with the Green kernel

$$(1.8) \qquad g(x,y) = \int_0^\infty \{p(s,x,y) - \phi(y)\}ds.$$

Recall that G is positive, i.e. $(Gf,f)_{L^2(d\lambda)} \geq 0$ and $(Gf,f)_{L^2(d\lambda)} = 0$ iff $f = c$ m-a.e.

Our restriction to f bounded Borel was of course unnecessary. The natural class of functions in this context is the class P(M) of "functions with bounded potentials", at least if one insists on P^x-a.e. results for <u>all</u> $x \in M$.

We define

$$P(M) = \begin{cases} \{f \in L^1(M);\ \sup_x \int [r(x,y)]^{-d+2} |f(y)| dm(y) < \infty\} & \text{if } d \geq 3 \\ \{f \in L^1(M);\ \sup_x \int \log^- r(x,y) |f(y)| dm(y) < \infty\} & \text{if } d = 2 \\ L^1(M) & \text{if } d = 1\ , \end{cases}$$

where r is the geodesic distance of x, y for a C^∞ metric g. Obviously, the space P(M) is an invariant of the differentiable manifold, i.e. does not depend on the specific metric chosen. Moreover, if $f \in P(M)$ then $\|Gf\|_\infty < \infty$ and for all $x \in M$, P^x-a.e. $f \circ X. \in L^1_{loc}(R^+)$. The following theorem strengthens results in [1].

(1.9) THEOREM: Let $f \in P(M)$.

(a) For any probability measure ν on M, the P^ν-law of $t^{-1/2}\{L_t(f)-tS(f)\}$ converges weakly to the normal distribution $N(0,\sigma_f^2)$.

(b) For all $x \in M$, P^x-a.e. $\underset{t \to \infty}{\text{cluster set}} \dfrac{L_t(f)-tS(f)}{\sqrt{2t \log \log t}} = [-\sigma_f, +\sigma_f]$.

In [1] we posed the problem to prove a "universal" $\log_2$-law for a natural class of functions on M with one single exceptional null-set, similar to the "universal" law of large numbers (1.5). The difficulty is obvious, as it is a priori not clear how to obtain the "universal" law even for bounded functions, by approximation from the law (1.9) (b) for countably many functions. Such an approximation argument leads trivially from (1.4) to (1.5). If the class of functions may contain unbounded functions, simultaneous local integrability of $f \circ X.$ has to be checked.

Our problem was solved in [2] for $M = T^d$, the d-dimensional torus, g the flat metric on M, V = 0, by choosing as function class the Sobolev space $H^\alpha(T^d)$, $\alpha > \frac{d}{2} - 1$. The treatment in [2] is based on the description of $H^\alpha(T^d)$ in terms of its Fourier transform and on very special properties of the trigonometric functions. It is not clear how to adapt this technique even to the case of the non-flat torus.

In [3] the results of [2] were partially extended to general (M, g), $V = 0$, by making use of the Riesz kernel. In the present paper we refine the arguments of [3] and thereby obtain the results of [2] for general M, g, V without the restriction on the index α that was necessary in [3].

Just as in [3] we shall use the notion of Sobolev spaces $\{H^\alpha(M), \alpha>0\}$ as defined (invariantly) in [8]. These function spaces are "Hilbertian" spaces. They are associated intrinsically with the C^∞ manifold M (and do not depend on g or V), just as are the function spaces $\{L^p(M),\ p\geq 1\}$ and $P(M)$. Via the regularity theorem for elliptic differential equations with C^∞ coefficients, the spaces $H^\alpha(M)$ can be made accessible to our Brownian motion by a Hilbertian space isomorphism $K^s_\alpha : L^2(M) \to H^\alpha(M)$, where the operator K^s_α is given by a kernel, defined in terms of the invariants g and V of our Brownian motion. For details see §3. Recall that $H^0(M) = L^2(M)$ and that $C^\infty(M) = H^\infty(M) = \varprojlim H^k(M)$. Moreover, $H^\alpha(M) \subseteq C(M)$ if $\alpha > \frac{d}{2}$ (Sobolev's Theorem). If $\alpha \leq \frac{d}{2}$, $f \in H^\alpha(M)$ will in general not be bounded. But if $d \geq 2$, $\alpha > \frac{d}{2} - 1$ or $d = 1$, $\alpha \geq 0$, then Gf, grad Gf are continuous for $f \in H^\alpha(M)$ (Lemma (8.1)). Also if $d \geq 3$, $\alpha > \frac{d}{2} - 2$ then $H^\alpha(M) \subseteq P(M)$, whereas for $d = 2$, $H^0(M) \subseteq P(M)$.

We shall prove the following theorems.

(1.10) THEOREM: Let $d \geq 2$.

There exists a measurable set $\Omega_0 \subseteq \Omega$ such that:

(a) $\theta_t\Omega_0 \subseteq \Omega_0$, all $t \geq 0$.

(b) $P^x(\Omega_0) = 1$, all $x \in M$.

(c) $f \circ X_\cdot(\omega) \in L^1_{loc}(R^+)$ for $\omega \in \Omega_0$, $f \in \bigcup_{\alpha > \frac{d}{2}-1} H^\alpha(M)$.

(d) For all $\alpha > \frac{d}{2} - 1$, $\omega \in \Omega_0$, $t \geq 0$, $f \mapsto L_t(f,\omega)$ is continuous on $H^\alpha(M)$.

By what was said above about the case $d = 1$, the statements of the theorem remain true for $d = 1$, if every $H^\alpha(M)$ is replaced by $L^1(M)$.

(1.11) THEOREM: For all $x \in M$, P^x-a.e.

$$\underset{t\to\infty}{\text{cluster set}} \frac{L_t(f) - tS(f)}{\sqrt{2t \log \log t}} = [-\sigma_f, +\sigma_f] \tag{1.12}$$

for all f $\in \bigcup_{\alpha > \frac{d}{2}-1} H^{\alpha}(M)$ if d $\geq$ 2, and for all f $\in L^2(M)$ if d = 1.

Notice that this "universal" $\log_2$-law implies a "universal" law of large numbers: For all $x \in M$, P^x-a.e.

(1.13) $$\lim_{t\to\infty} t^{-1} L_t(f) = S(f)$$

for all f $\in \bigcup_{\alpha > \frac{d}{2}-1} H^{\alpha}(M)$ if d $\geq$ 2, and for all f $\in L^2(M)$ if d = 1.

Notice that the functions f in (1.13) need not be bounded as in (1.5).

It is possible to give a function space (or rather distribution space) version of the $\log_2$-law and of the central limit theorem. To this end we bring in the spaces $\{H^{-\alpha}(M),\ \alpha>0\}$ which are the duals of the Hilbertian spaces $\{H^{\alpha}(M),\alpha>0\}$. We let $H^{-\infty}(M) = \varinjlim H^{-k}(M)$ = dual of $H^{\infty}(M)$. For more details see §3.

We define $\mathcal{L}: R^+ \times \Omega \to H^{-\infty}(M)$ by $\mathcal{L}_t(\omega)(f) = L_t(f,\omega)$, $f \in C^{\infty}(M)$. In view of (1.2), $\mathcal{L}_t(\omega)$ can also be thought of as the image measure on M of Lebesgue measure on $[0,t]$ under the random mapping $X.(\omega)\colon R^+ \to M$. By (1.5) this highly singular random measure on M converges if normalized, weakly to λ, P^x-a.e., $x \in M$.

The following theorem deals with the singularity of $\mathcal{L}_t$.

(1.14) THEOREM: Let d $\geq$ 2.

(a) On the set Ω_0 of theorem (1.10), we have for all $t \geq 0$, $\mathcal{L}_t(\omega) \in \bigcap_{\alpha > \frac{d}{2}-1} H^{-\alpha}(M)$ and $\mathcal{L}_t(\omega)(f) = L_t(f,\omega)$ for $f \in \bigcup_{\alpha > \frac{d}{2}-1} H^{\alpha}(M)$.

(b) There exists a measurable set Ω_0 such that in addition to (a)-(d) of Theorem (1.10) we have:

For every $\alpha > \frac{d}{2} - 1$, the process $\{\mathcal{L}_t,\ t\geq 0\}$ on Ω_0 is a strongly continuous $H^{-\alpha}(M)$-valued additive functional with respect to the Borel sets on $H^{-\alpha}(M)$.

Notice that for every $\omega \in \Omega_0$ of theorem (1.14)(b), $L.(f,\omega)$ is continuous for all $f \in \bigcup_{\alpha > \frac{d}{2}-1} H^{\alpha}(M)$. -Again, by what was said about the case $d = 1$, we have that for $d = 1$, $\{L_t(\omega),\ t \geq 0\}$ is a strongly continuous $(L^2(M))^*$-valued process on the set Ω_0, where the local time is defined.

As for the asymptotic behaviour of L_t in each $H^{-\alpha}(M)$ we have the following two theorems. In these theorems B denotes the unit ball in $L^2(d\lambda)$ and $G^*: L^2(d\lambda) \to L^2(d\lambda)$ is the adjoint of G. We identify $H^{-0}(M)$ and $H^0(M) = L^2(d\lambda)$.

(1.15) THEOREM (LOG_2-LAW): Let $d \geq 2$. For all $x \in M$, P^x-a.e. for all $\alpha > \frac{d}{2} - 1$:

(a) The random set $\left\{\frac{L_t(\omega) - tS}{\sqrt{2t \log \log t}},\ t > e^2\right\}$ in $H^{-\alpha}(M)$ is relatively strongly compact in $H^{-\alpha}(M)$.

(b) The strong $H^{-\alpha}(M)$-cluster set as $t \to \infty$ of $\frac{L_t(\omega) - tS}{\sqrt{2t \log \log t}}$ equals $(G+G^*)^{1/2}B$.

If $d = 1$, statements (a) and (b) hold if "all $\alpha > \frac{d}{2} - 1$" is replaced by "$\alpha = 0$".

Notice that the cluster set in (b) is a compact set in $L^2(M)$. As it is also compact in $H^{-\alpha}(M)$, (b) implies a strong law of large numbers for L_t. This law strengthens the weak convergence (1.5) of $\frac{1}{t} L_t$.

(1.16) THEOREM (CENTRAL LIMIT THEOREM): If $d \geq 2$, $\alpha > \frac{d}{2} - 1$, or $d = 1$, $\alpha = 0$, then there exists exactly one Gaussian measure μ_α on the Hilbertian space $H^{-\alpha}(M)$ with mean 0 such that

$$(1.17) \qquad \int_{H^{-\alpha}(M)} \ell(f_1)\ell(f_2)\,d\mu_\alpha(\ell) = ((G+G^*)f_1, f_2)_{L^2(d\lambda)}, \quad f_1, f_2 \in H^{\alpha}(M)$$

and for every probability measure ν on M, the P^ν-law of the $H^{-\alpha}(M)$-valued random variable $t^{-1/2}\{L_t - tS\}$ converges weakly to μ_α.

The covariance operator $S_\alpha : H^\alpha(M) \to H^\alpha(M)$, defined uniquely by

(1.18) $$\int_{H^{-\alpha}(M)} \ell(f_1)\ell(f_2)d\mu_\alpha(\ell) = (S_\alpha f_1, f_2)_{H^\alpha(M)}, \quad f_1, f_2 \in H^\alpha(M),$$

depends of course on the specific choice of the inner product in the Hilbertian space $H^\alpha(M)$.

In a forthcoming paper we shall discuss limit laws for general additive functionals.

§2. RIESZ KERNELS AND OPERATORS

We begin with a few remarks on the behaviour of the transition density p for small t:

There exist α, $C > 0$ such that

(2.1) $$p(t,x,y) \leq C\{t^{-\frac{d}{2}} e^{-\frac{\alpha[r(x,y)]^2}{t}} + 1\}, \quad x,y \in M; \quad t \leq 1,$$

and

(2.2) $$|\mathrm{grad}_x p(t,x,y)| \leq C\{t^{-\frac{d+1}{2}} e^{-\frac{\alpha[r(x,y)]^2}{t}} + 1\}, \quad x,y \in M; \quad t \leq 1.$$

Here r and grad denote geodesic distance and gradient defined by g.

Estimate (2.1) is given in [5] for p_0, the fundamental solution of $\frac{1}{2}\Delta p_0 = \frac{\partial}{\partial t} p_0$; estimate (2.2) for p_0 is obtained in essentially the same way. The two estimates for p itself follow from

(+) $$p(t,x,y) = p_0(t,x,y) + \int_0^t ds \int dm(z) p_0(t-s,z,y)\phi(s,x,z)$$

where $$\phi = \sum_1^\infty \psi^{*k},$$

$$\psi(t,x,y) = \psi^{*1}(t,x,y) = V(y)\cdot \mathrm{grad}_y p_0(t,x,y),$$

$$\psi^{*k+1}(t,x,y) = \int_0^t ds \int dm(z)\psi^{*k}(s,x,z)\psi(t-s,z,y).$$

For the integration in (+) use

$$\int_M \frac{e^{-\frac{[r(x,z)]^2}{2t_1}}}{t_1^{d/2}} \frac{e^{-\frac{[r(z,y)]^2}{2t_2}}}{t_2^{d/2}} dm(z) \leq c_1 \frac{e^{-\frac{c_2[r(x,y)]^2}{2(t_1+t_2)}}}{(t_1+t_2)^{d/2}}, \quad x,y \in M;\ t_1,t_2 > 0.$$

The fact that (+) gives the fundamental solution of $L_y' p(t,x,y) = \frac{\partial}{\partial t} p(t,x,y)$, is verified by (weak) differentiation.

Estimates (2.1) and (2.2) imply for $\alpha > 0$

$$(2.3) \qquad \int_0^1 t^{\alpha-1} p(t,x,y)dt \leq \begin{cases} C & \text{if } \alpha > \frac{d}{2} \\ C\{1 + \log^- r(x,y)\} & \text{if } \alpha = \frac{d}{2} \\ C[r(x,y)]^{-d+2\alpha} & \text{if } \alpha < \frac{d}{2} \end{cases}$$

and

$$(2.4) \qquad \left|\operatorname{grad}_x \int_0^1 p(t,x,y)dt\right| \leq C[r(x,y)]^{-d+1}.$$

For large t we shall use the estimate (see [1])

$$(2.5) \qquad |p(t,x,y)-\phi(y)| \leq Ae^{-\alpha t}, \quad x,y \in M, \quad t \geq 1,$$

which, together with $p(t+1,x,y) = \int p(1,x,z)p(t,z,y)dm(z)$, implies also

$$(2.6) \qquad \left|\operatorname{grad}_x \int_1^\infty \{p(t,x,y)-\phi(y)\}dt\right| \leq C, \quad x,y \in M.$$

As was done in [3], we shall make use of the "Riesz kernels"

$$(2.7) \qquad g_\alpha(x,y) = \frac{1}{\Gamma(\alpha)} \int_0^\infty t^{\alpha-1}\{p(t,x,y)-\phi(y)\}dt, \quad \alpha > 0,\ x,y \in M.$$

The $\{g_\alpha,\ \alpha>0\}$ form a semigroup with respect to dm and we have $g_1 = g$. This latter kernel was introduced in [1], where it was shown that

$$(2.8) \qquad L_x g(x,y) = -\delta_y(x) + \phi(y).$$

All g_α are bounded below. From (2.3) and (2.5) we obtain

$$(2.9) \qquad g_\alpha(x,y) \leq \begin{cases} C & \text{if } \alpha > \frac{d}{2} \\ C\{1 + \log^- r(x,y)\} & \text{if } \alpha = \frac{d}{2} \\ C[r(x,y)]^{-d+2\alpha} & \text{if } \alpha < \frac{d}{2} \end{cases} .$$

From (2.4) and (2.6) we have

$$(2.10)\qquad |\mathrm{grad}_x g(x,y)| \leqslant C[r(x,y)]^{-d+1} .$$

Notice that $\sup_x \int |g_\alpha(x,y)|dm(y) < \infty$, $\sup_x \int |\mathrm{grad}_x g(x,y)|dm(y) < \infty$. We shall also use the following estimates which enable us to deal with convolutions of the expressions on the right side in (2.3) and (2.9): For $0 \leqslant \sigma < d$, $0 \leqslant \tau < d$

$$(2.11)\qquad \int_M [r(x,z)]^{-\sigma}[r(z,y)]^{-\tau}dm(z) \leqslant \begin{cases} C & \text{if } \sigma+\tau<d \\ C\{1+\log^- r(x,y)\} & \text{if } \sigma+\tau=d \\ C[r(x,y)]^{d-\sigma-\tau} & \text{if } \sigma+\tau>d \end{cases}$$

(see also [3]).

For convenience we shall use the invariant measure $d\lambda = \phi dm$ as reference measure and relabel $[\phi(y)]^{-1}p(t,x,y)$, $[\phi(y)]^{-1}g_\alpha(x,y)$ by $p(t,x,y)$, $g_\alpha(x,y)$. Estimates (2.9) and (2.10) obviously hold for the new g_α. Equation (2.7) becomes

$$(2.7')\qquad g_\alpha(x,y) = \frac{1}{\Gamma(\alpha)} \int_0^\infty t^{\alpha-1}\{p(t,x,y)-1\}dt \quad ,$$

and (2.8) goes over into

$$(2.8')\qquad L_x g(x,y) = -\frac{1}{\phi(y)}\,\delta_y(x) + 1 \quad .$$

The new g_α form a semigroup with respect to $d\lambda$.

The Riesz operators $G_\alpha : L^2(M) \to L^2(M)$, $\alpha > 0$, are defined by

$$(2.12)\qquad (G_\alpha f)(x) = \int g_\alpha(x,y)f(y)d\lambda(y) \quad .$$

Equation (2.12) by the way defines also a semigroup of operators on $L^1(M)$. As $G = G_1$, we conclude from (2.10), that for $f \in L^1(M)$

$$(2.13)\qquad (\mathrm{grad}\ Gf)(x) = \int \mathrm{grad}_x g(x,y)f(y)d\lambda(y)$$

(in distribution sense). Also for $f \in L^1(M)$, we have from (2.8) that

$$(2.14)\qquad L(Gf) = -f + Sf$$

(in distribution sense). Recall that the identity

$$(2.15)\qquad \int fGfd\lambda = \frac{1}{2}\int |\mathrm{grad}\ Gf|^2 d\lambda,\ f \in L^2(M) \cup P(M) \ ,$$

implies positivity of G on $L^2(d\lambda)$.

If ν is a smooth measure on M, such that G is self-adjoint on $L^2(d\nu)$, then ν is proportional to λ. We denote by A* the (formal) adjoint of an operator A on $L^2(d\lambda)$. Obviously, $(G^*f)(x) = \int g^*(x,y)f(y)d\lambda(y)$, $f \in L^2(M)$, where $g^*(x,y) = g(y,x)$. According to [7], $L^* = \frac{1}{2}\Delta - V + \text{grad} \log \phi$. Moreover, L or equivalently G are self-adjoint on $L^2(d\lambda)$ iff V is a gradient field, in which case $V = \frac{1}{2}\text{grad} \log \phi$. Let $L^s = \frac{1}{2}(L + L^*) = \frac{1}{2}\Delta + \frac{1}{2}\text{grad} \log \phi$. Then $(L^s)^* = L^s$, and $L^s = L$ iff $L = L^*$. The transition semigroup generated by L^s has also λ as its invariant measure.

§3. THE SOBOLEV SPACES $H^\alpha(M)$, $\alpha \in R$

These spaces are defined invariantly in [8]. We shall use their real version, rather than the complex version of [8]. For $\alpha \geqslant 0$, the Hilbertian spaces $H^\alpha(M)$ form a monotone chain of linear subspaces of $H^0(M) = L^2(M)$. We have $H^{\alpha_1}(M) \subseteq H^{\alpha_2}(M)$ if $\alpha_1 > \alpha_2$, and $\bigcap_{\alpha \geqslant 0} H^\alpha(M) = C^\infty(M)$.

If $f \in H^2(M)$, we have $Lf \in L^2(M)$. Since the only solutions ϕ of $L\phi = 0$ are the constant functions, (2.14) implies for $f \in H^2(M)$

$$G(Lf) = -f + Sf \quad . \tag{3.1}$$

Notice that (2.14) also implies that the operator $G + S$: $L^2(M) \to L^2(M)$ is invertible. This by the way, implies invertibility of the operators $K_\alpha \underset{\text{def}}{=} G_{\alpha/2} + S$ on $L^2(M)$, since the K_α form a semigroup. Notice that $S(K_\alpha f) = Sf$ for $f \in L^2(M)$. We put $K_0 = I$. The operators K_α corresponding to L^s will be denoted by K_α^s. They form a semigroup of positive, self-adjoint operators on $L^2(d\lambda)$.

(3.2) THEOREM:

(a) <u>For all positive integers</u> k, K_{2k} <u>is a Hilbertian space isomorphism</u> K_{2k}: $L^2(M) \to H^{2k}(M)$.

(b) For all $\alpha > 0$, K_α^s is a Hilbertian space isomorphism $K_\alpha^s: L^2(M) \to H^\alpha(M)$.

Proof of (a): One proves that for every k, K_2 is a Hilbertian space isomorphism $K_2 : H^{2k} \to H^{2k+2}$. Firstly, the linear operator $T = L - S: H^{2k+2} \to H^{2k}$ is continuous. To show then that T is onto, let $f \in H^{2k}$. By (2.14) and the regularity theorem for elliptic differential equations with C^∞coefficients, we conclude that $Gf \in H^{2k+2}$, hence $-K_2 f \in H^{2k+2}$, and $T(-K_2 f) = f$. Moreover, T is invertible because (3.1) implies $-K_2 Tf = f$. By the open mapping theorem $T^{-1} = -K_2 : H^{2k} \to H^{2k+2}$ is also continuous.

Proof of (b): For $\alpha > 0$, we define the Hilbert spaces $\overline{H}^\alpha(M)$ by $\overline{H}^\alpha(M) = K_\alpha^s(L^2(M))$, endowed with the inner product $(K_\alpha^s u_1, K_\alpha^s u_2)_{\overline{H}^\alpha} = \int u_1 u_2 d\lambda$. To show (b), it suffices to show that the underlying Hilbertian spaces of $\overline{H}^\alpha(M)$ are $H^\alpha(M)$. By (a), this is true if $\alpha = 2k$. According to [8], H^α, $\alpha \in (2k, 2k+2)$, is the underlying Hilbertian space of the Hilbert space $Q_{k+1-\alpha/2}(\overline{H}^{2k+2}, \overline{H}^{2k})$, where Q_t denotes quadratic interpolation of Hilbert spaces. (This space does not depend on the particular choice of admissible inner products in the Hilbertian spaces H^{2k+2}, H^{2k}.) It therefore suffices to show that $\overline{H}^\alpha = Q_{k+1-\alpha/2}(\overline{H}^{2k+2}, \overline{H}^{2k})$. Letting $t = k + 1 - \alpha/2$, $H = Q_t(\overline{H}^{2k+2}, \overline{H}^{2k})$, we have for $f_1, f_2 \in \overline{H}^{2k}$, $(f_1, f_2)_{\overline{H}^{2k}} = (K_4^s f_1, f_2)_{\overline{H}^{2k+2}}$, and hence for $f_1, f_2 \in \overline{H}^{2k+2}$

$$(f_1, f_2)_H = (K_{4t}^s f_1, f_2)_{\overline{H}^{2k+2}} = (f_1, f_2)_{\overline{H}^\alpha} .$$

Now $\overline{H}^\alpha(M)$ is the completion realized in $L^2(M)$, of $C^\infty(M)$, hence of $\overline{H}^{2k+2}(M)$, with respect to $\| \ \|_{\overline{H}^\alpha}$, and H is defined as the completion, realized in $\overline{H}^{2k}(M) \subseteq L^2(M)$, of $\overline{H}^{2k+2}(M)$ with respect to $\| \ \|_H$. Part (b) follows.

It follows from the preceding theorem, that for $\alpha > 0$, $\beta > 0$, K_α^s defines a Hilbertian space isomorphism $K_\alpha^s : H^\beta(M) \to H^{\alpha+\beta}(M)$. Since $\bigcap_{\alpha>0} H^\alpha(M) = C^\infty(M)$, this implies in particular that $K_\alpha^s(C^\infty(M)) \subseteq C^\infty(M)$. As

$C^\infty(M) \subseteq K^s_{2k}(C^\infty(M))$ for all k, it follows that $K^s_\alpha(C^\infty(M)) = C^\infty(M)$ for all $\alpha > 0$.

Following [8], we let $H^{-\alpha}(M)$ = dual of $H^\alpha(M)$, $\alpha > 0$. As $C^\infty(M)$ is dense in $H^\alpha(M)$, we may also consider each $H^{-\alpha}(M)$ as a linear subspace of the dual of $C^\infty(M)$. The $H^{-\alpha}(M)$ are Hilbertian spaces. Clearly, $H^{-\alpha_1}(M) \subseteq H^{-\alpha_2}(M)$ for $\alpha_1<\alpha_2$. We shall from now on fix an admissible inner product in $H^0(M)$ by setting $H^0(M) = L^2(d\lambda)$. By doing so we may identify $H^0(M)$ with its dual $H^{-0}(M)$, and thereby "extend" the chain $\{H^\alpha(M), \alpha>0\}$ to a chain $\{H^\alpha(M), \alpha\in R\}$ of linear subspaces of the dual of $C^\infty(M)$. The semigroup K^s_α on $H^0(M)$ extends in a unique way to a semigroup on $\bigcup_{\beta\in R} H^\beta(M)$ if we require that for $\ell \in H^{-\alpha}$, $\alpha > 0$ we have $K^s_\alpha \ell \in H^0(M)$ and

$$\int K^s_\alpha \ell\cdot\psi\, d\lambda = \ell(K^s_\alpha\psi) \quad \text{for} \quad \psi \in C^\infty(M). \tag{3.3}$$

Notice that $K^s_\alpha : \bigcup_{\beta\in R} H^\beta(M) \to \bigcup_{\beta\in R} H^\beta(M)$ is invertible. Letting $K^s_{-\alpha} = (K^s_\alpha)^{-1}$, we have that for any $\alpha,\beta \in R$, K^s_α defines a Hilbertian space isomorphism

$$K^s_\alpha : H^\beta(M) \to H^{\alpha+\beta}(M). \tag{3.4}$$

The latter is a Hilbert space isomorphism if we endow as we shall all $H^\alpha(M)$ with the inner products carried over from $L^2(d\lambda)$ by the isomorphisms $K^s_\alpha : L^2(d\lambda) \to H^\alpha(M)$.

§4. PROOF OF THEOREM (1.9)

We may assume $\sigma_f^2 > 0$. It is well-known that for $f \in P(M)$, all $x \in M$, P^x-a.e. $(Gf)\circ X. \in C(R^+)$. It follows from (2.14) that for any probability measure ν on M, the process

$$M_t(f) \underset{\text{def}}{=} (Gf)(X_t) - (Gf)(X_0) + L_t(f) - tS(f), \quad t \geq 0 \tag{4.1}$$

is a continuous P^ν-martingale. Now (2.3) implies $\sup_{x\in M} E^x L_t^2(f) < \infty$ for $f \in P(M)$, and from (2.11) we conclude $|\text{grad } Gf|^2 \in P(M)$ for $f \in P(M)$. (Notice e.g. that by

(2.11) for $d > 3$, $\int[r(\xi_1,x)]^{-d+2}[r(\xi_2,x)]^{-d+1}[r(\xi_3,x)]^{-d+1}dm(x) \leq C\{[r(\xi_1,\xi_2)]^{-d+2} \cdot [r(\xi_2,\xi_3)]^{-d+2} + [r(\xi_1,\xi_2)]^{-d+2}[r(\xi_1,\xi_3)]^{-d+2}\})$. In any case, $M(f)$ is a square integrable P^ν-martingale, for all ν. The process

$$(4.2) \qquad \tau_t(f) = \int_0^t |\text{grad } Gf|^2(X_s)ds \ , \quad t \geq 0 \ ,$$

is its increasing process for all P^ν. By the ergodic theorem we have $P^\lambda\{\lim_{t\to\infty} \frac{\tau_t(f)}{t} = \sigma_f^2\} = 1$. Since $\{\ \}$ is shift-invariant, and since positive X-harmonic functions have to be constant, we obtain $P^x\{\lim_{t\to\infty} \frac{\tau_t(f)}{t} = \sigma_f^2\} = 1$, all $x \in M$, hence

$$(4.3) \qquad P^\nu\{\lim_{t\to\infty} \frac{\tau_t(f)}{t} = \sigma_f^2\} = 1, \quad \text{all } \nu.$$

Now, depending on f and ν, there exists a probability space (Ω',A',P') with a Wiener process $\{W_t, t\geq 0\}$ and a continuous time change $\{T_t, t\geq 0\}$ on it, such that the P^ν-law of $M(f)$ and the P'-law of $W\circ T$ are the same (see e.g. [6]). This implies that $W\circ T$ is a continuous L^2-martingale. Hence its increasing process is $\{T_t, t\geq 0\}$. It follows that the P'-law of T and the P^ν-law of $\tau(f)$ are the same.

<u>Proof of Theorem (1.9)(a):</u>

It is sufficient to prove that the P^ν-distribution of $t^{-1/2}M_t(f)$ converges weakly to $N(0,\sigma_f^2)$. For this it suffices to prove that with $a = \sigma_f^2$, for all $\varepsilon > 0$, $\lim_{t\to\infty} P'\{|\frac{W_{T_t}}{\sqrt{t}} - \frac{W_{at}}{\sqrt{t}}| > \varepsilon\} = 0$. So let $\varepsilon > 0$, $\eta \in (0,a)$. Then

$$(4.4) \quad P'\{|W_{T_t} - W_{at}| > \varepsilon\sqrt{t}\} \leq P'\{|W_{T_t} - W_{at}| > \varepsilon\sqrt{t}, \ |T_t - at| < \eta t\} + P'\{|\frac{T_t}{t} - a| > \eta\}.$$

As the P'-law of T and the P^ν-law of $\tau(f)$ coincide, we conclude from (4.3), that for all $\eta > 0$

$$(4.5) \qquad \lim_{t\to\infty} P'\{|\frac{T_t}{t} - a| > \eta\} = 0.$$

Moreover, the first term on the right side of (4.4) is majorized by

$$P'\{\sup_{0\leq s\leq \eta t}|W_{at+s}-W_{at}|>\varepsilon\sqrt{t}\}+P'\{\sup_{0\leq s\leq \eta t}|W_{at-s}-W_{at}|>\varepsilon\sqrt{t}\}.$$

Now the P'-laws of $\{W_{v+u}-W_v,\ u\geq 0\}$ and $\{W_u,\ u\geq 0\}$ coincide as do the P'-laws of $\{W_{v-u}-W_v, u\in[0,v]\}$ and $\{W_u, u\in[0,v]\}$. It follows that the first term on the right side of (4.4) is majorized by

$$8P'\{W_{\eta t}>\varepsilon\sqrt{t}\}=8P'\{\frac{W_{\eta t}}{\sqrt{\eta t}}>\frac{\varepsilon}{\sqrt{\eta}}\}=\frac{8}{\sqrt{2\pi}}\int_{\varepsilon/\sqrt{\eta}}^{\infty}e^{-\xi^2/2}d\xi.$$

This fact together with (4.5) implies that for all $\eta\in(0,a)$

$$\overline{\lim_{t\to\infty}}\ P'\{|W_{T_t}-W_{at}|>\varepsilon\sqrt{t}\}\leq\frac{8}{\sqrt{2\pi}}\int_{\varepsilon/\sqrt{\eta}}^{\infty}e^{-\xi^2/2}d\xi.$$

Letting $\eta\to 0$, we obtain $\lim_{t\to\infty}P'\{|W_{T_t}-W_{at}|>\varepsilon\sqrt{t}\}=0$, all $\varepsilon>0$.

<u>Proof of Theorem (1.9)(b):</u>

It suffices to prove that for all $x\in M$, P^x-a.e. $\overline{\lim_{t\to\infty}}\ \frac{M_t(f)}{\sqrt{2t\log\log t}}=1$. If (Ω',A',P'), W, T are as described above, corresponding to $\nu=\delta_x$, this is equivalent to showing that P'-a.e. $\overline{\lim_{t\to\infty}}\ \frac{W_{T_t}}{\sqrt{2t\log\log t}}=1$. The latter follows from the $\log_2$-law for W and (4.3).

§5. <u>PROOF OF THEOREMS (1.10) AND (1.14)</u>

We shall start with

(5.1) LEMMA: <u>If</u> $d\geq 2$, $\alpha>\frac{d}{2}-1$, <u>then for all</u> $t>0$

$$\sup_{x\in M}E^x\int d\lambda(\xi)\{\int_0^t d\sigma[r(\xi,X_\sigma)]^{-d+\alpha}\}^2<\infty \tag{5.2}$$

<u>and</u>

$$\sup_{\xi\in M}\int d\lambda(x)E^x\{\int_0^t d\sigma[r(\xi,X_\sigma)]^{-d+\alpha}\}^2<\infty\ . \tag{5.3}$$

Proof: Clearly

$$(5.4)\quad E^x\{\int_0^t d\sigma[r(\xi,X_\sigma)]^{-d+\alpha}\}^2 \leqslant 2\int\int d\lambda(y)d\lambda(z)\big[r(\xi,y)\big]^{-d+\alpha}\big[r(\xi,z)\big]^{-d+\alpha}$$

$$\int_0^t d\sigma_1 p(\sigma_1,x,y)\int_0^t d\sigma_2 p(\sigma_2,y,z) \quad .$$

From (2.3), applied to $\alpha = 1$, we have for $\delta > 0$,

$$\int_0^t d\sigma p(\sigma,x,y) \leqslant c(\delta)[r(x,y)]^{-d-\delta+2}$$

(for any dimension). We therefore have as a majorant for the right side of (5.4)

$$c\int dm(y)[r(\xi,y)]^{-d+\alpha}[r(x,y)]^{-d-\delta+2}\int dm(z)[r(\xi,z)]^{-d+\alpha}[r(y,z)]^{-d-\delta+2}.$$

If $\alpha > d - 2$, this expression is bounded in (ξ,x) for $\delta \in (0,\min\{2,\alpha-d+2\})$ by (2.11). If $\alpha \in (\frac{d}{2} - 1,\ d-2]$, $\delta \in (0,\ \min\{2,2(\alpha- \frac{d}{2} +1)\})$, we conclude from (2.11) that the expression is majorized by

$$c_1\int dm(y)[r(\xi,y)]^{-2d+2\alpha-\delta+2}[r(x,y)]^{-d-\delta+2} \leqslant c_2[r(\xi,x)]^{-2d+2\alpha-2\delta+4}.$$

Estimates (5.2) and (5.3) follow.

(5.5) COROLLARY: <u>If</u> $d \geqslant 2$ <u>and</u>

$$(5.6)\qquad \Omega_0 = \{\omega;\int dm(\xi)\{\int_0^t d\sigma[r(\xi,X_\sigma)]^{-d+\alpha}\}^2 < \infty,\ \text{all } t \geqslant 0,\ \alpha > \frac{d}{2} - 1\},$$

<u>then</u> $P^x(\Omega_0) = 1$ <u>for</u> $x \in M$.

<u>(5.7) Remark:</u> If $d = 1$, we define Ω_0 as the Ω-set where the local time L is defined as a continuous functional in (ξ,t).

We denote by g^s_α, $\alpha > 0$, the Riesz kernels associated with L^s and its invariant measure λ. Let $k^s_\alpha = g^s_{\alpha/2} + 1$. Then $(K^s_\alpha f)(x) = \int_M k^s_\alpha(x,y)f(y)d\lambda(y)$. As the k^s_α are bounded below, the processes

$$(5.8)\qquad A^\alpha_t(\xi) = \int_0^t k^s_\alpha(\xi,X_\sigma)d\sigma,\quad t \geqslant 0,\ \xi \in M$$

are well defined on Ω (possibly $+\infty$). Let $|A^\alpha(\xi)|_t = \int_0^t |k^s_\alpha(\xi,X_\sigma)|d\sigma$.

If we let for $\alpha > 0$, $\xi, y \in M$,

$$a_\alpha(\xi,y) = \{\Gamma(\alpha)\}^{-1}\int_1^\infty u^{\alpha-1}\{p^s(u,\xi,y) - 1\}du + 1 - \{\Gamma(\alpha+1)\}^{-1}$$

with p^s the transition densities with respect to λ, of the operator L^s, and define for $\alpha > 0$, $t \geq 0$

$$B_t^\alpha(\xi) = \{\Gamma(\alpha)\}^{-1}\int_0^t d\sigma\int_0^1 u^{\alpha-1}p^s(u,\xi,X_\sigma)du,$$

then $A_t^\alpha(\xi) = B_t^{\alpha/2}(\xi) + \int_0^t d\sigma a_{\alpha/2}(\xi,X_\sigma)$. Since for every α, a_α is bounded and for every α, $\delta > 0$, $B_t^{\alpha/2}(\xi) \leq C\int_0^t d\sigma[r(\xi,X_\sigma)]^{-d-\delta+\alpha}$ (for any dimension d, application of (2.3) to p^s), we have as corollary of (5.2) and (5.3)

(5.9) COROLLARY: Let $d \geq 2$. Then:

(5.10) For $\alpha > \frac{d}{2} - 1$, $t > 0$: $\sup_{x\in M} E^x\int d\lambda(\xi)|A^\alpha(\xi)|_t^2 < \infty$.

(5.11) For $\alpha > \frac{d}{2} - 1$, $t > 0$: $\sup_{\xi\in M} E^\lambda|A^\alpha(\xi)|_t^2 < \infty$.

(5.12) $\Omega_0 \subseteq \{\omega;\ \int d\lambda(\xi)|A^\alpha(\xi)|_t^2 < \infty \text{ all } t > 0, \text{ all } \alpha > \frac{d}{2} - 1\}$.

(5.13) Remark: If $d = 1$, then for all $t > 0$, $\sup_{x,\xi} E^x[L(t,\xi)]^2 < \infty$. (The proof follows from (2.3) and $E^x[L(t,\xi)]^2 \leq 2\phi^2(\xi)\int_0^t d\sigma_1\int_0^{t-\sigma_1} d\sigma_2 p(\sigma_1,x,\xi)\cdot p(\sigma_2,\xi,\xi)$).

Proof of theorem (1.10):

Let Ω_0 be as in (5.6). It remains to check (c) and (d). For any $f \in H^\alpha(M)$, $\alpha > 0$, we have $f = K_\alpha^s\bar{f}$, $\bar{f} \in L^2(M)$. If $f \in H^\alpha(M)$, $\alpha > \frac{d}{2} - 1$, $\omega \in \Omega_0$ we conclude from (5.12) that

$$\{\int d\lambda(\xi)|\bar{f}(\xi)||A^\alpha(\xi)|_t\}^2 \leq \int d\lambda(\xi)|\bar{f}(\xi)|^2\cdot\int d\lambda(\xi)|A^\alpha(\xi)|_t^2 < \infty.$$

This and Fubini's theorem imply that $f\circ X_\cdot(\omega) \in L^1_{loc}(R^+)$ and that

(5.14) $$\int d\lambda(\xi)\bar{f}(\xi)A_t^\alpha(\xi) = L_t(f,\omega) .$$

The latter equation also implies that $\lim_{n\to\infty} L_t(f_n,\omega) = L_t(f,\omega)$ if $\|f_n-f\|_{H^\alpha(M)} = \|\bar{f}_n-\bar{f}\|_{L^2(M)} \to 0$.

Proof of Theorem (1.14):

Part (a) follows immedately from Theorem (1.10),(d). For the proof of part (b), we notice that for $t \geq 0$, $\alpha > \frac{d}{2} - 1$, $\omega \in \Omega_0$ of (5.6) we have

(5.15) $$K_\alpha^s L_t = A_t^\alpha \quad .$$

In view of (3.3), this follows from (5.12) and

$$\int A_t^\alpha \cdot \phi d\lambda = L_t(K_\alpha^s \phi), \qquad \phi \in C^\infty(M),$$

which in turn follows from $K_\alpha^s \phi \in C^\infty(M)$, $L_t(K_\alpha^s\phi) = L_t(K_\alpha^s\phi)$ and (5.14). By (3.4) strong continuity of $L.(\omega)$ in $H^{-\alpha}(M)$ is equivalent to strong continuity of $A^\alpha.$ in $L^2(M)$. But if $\omega \in \Omega_0$ strong continuity of $A^\alpha.$ in $L^2(M)$ follows from (5.12). Obviously, $L_{s+t}(\omega) = L_t(\omega) + L_s(\theta_t\omega)$, $s,t \geq 0$, and F_t-measurability of L_t follows from F_t-measurability of $\omega \mapsto L_t(f,\omega)$ for all $f \in C^\infty(M)$.

§6. PROOF OF THEOREM (1.11)

For the proof of this theorem we need a well-known lemma for one-dimensional Brownian motion. This lemma follows e.g. from a result in [4], but we shall give here an independent proof for the reader's convenience.

(6.1) LEMMA: If $\{W_t, t\geq 0\}$ is a one-dimensional Brownian motion on (Ω', A', P'), then for all $p \geq 1$

$$E'\left[\sup_{t\geq e^2} \frac{|W_t|}{\sqrt{2t\log\log t}}\right]^p < \infty.$$

Proof: Letting $Y = \sup_{t\geq e^2} \frac{|W_t|}{\sqrt{2t\log\log t}}$, $b = e^2$, we obtain for $a > 0$ such that $a^2 > 2b$, $\frac{a\sqrt{\log 2}}{\sqrt{b}} \geq 1$, that

$$P'\{Y>a\} = P'\{\sup_{b\leq t<\infty} \frac{|W_t|}{\sqrt{2t\log\log t}} > a\} \leq \sum_{n=2}^{\infty} P'\{\sup_{b^{n-1}\leq s<b^n} \frac{|W_s|}{\sqrt{2b^{n-1}\log(\log b^{n-1})}} > a\}$$

$$\leq \sum_{n=2}^{\infty} 4P'\{\frac{W_{b^n}}{\sqrt{b^n}} > \frac{a\sqrt{2\log(\log b^{n-1})}}{\sqrt{b}}\} = \sum_{n=2}^{\infty} 4P'\{W_1 > \frac{a\sqrt{2}\sqrt{\log(2(n-1))}}{\sqrt{b}}\}$$

$$\leq \sum_{n=2}^{\infty} \frac{4}{\sqrt{2\pi}} e^{-a^2\log[2(n-1)]/b} = \sum_{n=2}^{\infty} \frac{4}{\sqrt{2\pi}} \frac{1}{[2(n-1)]^{a^2/b}}$$

$$\leq \frac{1}{2^{a^2/b}} \sum_{n=2}^{\infty} \frac{4}{\sqrt{2\pi}} \frac{1}{(n-1)^2} \leq \frac{c}{2^{a^2/b}} .$$

But $P'\{Y>a\} \leq \frac{c}{2^{a^2/b}}$ for large a, implies that $Y \in L^p(\Omega',P')$ for all $p \geq 1$.

In the following we shall consider the kernel k_α^s and the process $\{A_t(\xi),\ t \geq 0\}$ of §5. Even though in general for fixed $\xi \in M$, $k_\alpha^s(\xi,\cdot) \notin P(M)$, we were able to define $A(\xi)$ (possibly $+\infty$) on all of Ω, because k_α^s is bounded below. Recall that the set Ω_0 of (5.6) has P^x-measure 1, all $x \in M$. We shall prove the following

(6.2) LEMMA: Let $d \geq 2$. For all $\alpha > \frac{d}{2} - 1$, P^λ-a.e.

$$\sup_{t\geq e^2} \frac{\int d\lambda(\xi)[A_t^\alpha(\xi)-t]^2}{2t\log\log t} < \infty . \tag{6.3}$$

Proof: If we define the process $M^\alpha(\xi) = M(k_\alpha^s(\xi,\cdot))$ as in (4.1), i.e.

$$M_t^\alpha(\xi) = Gk_\alpha^s(\xi,\cdot)(X_t) - Gk_\alpha^s(\xi,\cdot)(X_0) + A_t^\alpha(\xi) - t \quad , \tag{6.4}$$

then $M^\alpha(\xi)$ is a square integrable P^λ-martingale with continuous sample paths and increasing process

$$\tau_t^\alpha(\xi) = \int_0^t |\operatorname{grad} Gk_\alpha^s(\xi,\cdot)|^2(X_\sigma)d\sigma \quad . \tag{6.5}$$

Recall that for all $t > 0$, $\sup_\xi E^\lambda[A_t^\alpha(\xi)]^2 < \infty$ for $\alpha > \frac{d}{2} - 1$ (5.11), and notice that for $\alpha > \max(0, \frac{d}{2} - 2)$

$$(6.6) \qquad \sup_{\xi} \int d\lambda(y)[Gk_\alpha^s(\xi,\cdot)]^2(y) < \infty, \quad \sup_{y} \int d\lambda(\xi)[Gk_\alpha^s(\xi,\cdot)]^2(y) < \infty .$$

For later use we remark, that if $\alpha > \frac{d}{2} - 1$, there exists $\delta > 0$ such that

$$(6.7) \qquad \sup_{\xi} E^\lambda [\tau_1^\alpha(\xi)]^{1+\delta} < \infty.$$

This follows from $\sup_{\xi} E^\lambda \int_0^1 |\mathrm{grad} Gk_\alpha^s(\xi,\cdot)|^{2+2\delta} < \infty$ for $\delta > 0$ satisfying $(2+2\delta)(d-\alpha-1) < d$, which in turn follows from (2.10) and (2.9) and (2.11). In view of (6.6), (6.3) is proved if we prove

$$(6.8) \qquad \sup_{t \geq e^2} \frac{\int d\lambda(\xi)[M_t^\alpha(\xi)]^2}{2t \log \log t} < \infty \quad P^\lambda\text{-a.e.}$$

But (6.8) is proved if we prove

$$(6.9) \qquad \sup_{\xi} E^\lambda \sup_{t \geq e^2} \frac{[M_t^\alpha(\xi)]^2}{2t \log \log t} < \infty.$$

For the proof of (6.9) we let (as in §4) W,T on (Ω',A',P') be Brownian motion and time change (depending on α,ξ) such that the P^λ-law of $M^\alpha(\xi)$ equals the P'-law of $W \circ T$. Then

$$(6.10) \qquad E^\lambda \sup_{t \geq e^2} \frac{[M_t^\alpha(\xi)]^2}{t \log \log t} = E' \sup_{t \geq e^2} \frac{W_{T_t}^2}{t \log \log t} .$$

Now

$$\sup_{t \geq e^2} \frac{W_{T_t}^2}{t \log \log t} \leq \sup_{t \leq e^2} W_t^2 + \sup_{t \geq e^2} \frac{W_t^2}{t \log \log t} \cdot \sup_{t \geq e^2} \frac{T_t \log^+ \log^+ T_t}{t \log \log t} .$$

Moreover, for any $\delta > 0$, $t \geq e^2$

$$\frac{T_t \log^+ \log^+ T_t}{t \log \log t} \leq \frac{T_t}{t} + \frac{T_t}{t} \log^+\left(\frac{T_t}{t}\right) \leq C_\delta\left\{1 + \left(\frac{T_t}{t}\right)^{1+\delta}\right\}.$$

We conclude from (6.10): For all $\delta > 0$, there exists C_δ (independent of ξ, α) such that

$$(6.11) \quad E^\lambda \sup_{t \geq e^2} \frac{[M_t^\alpha(\xi)]^2}{t \log \log t} \leq E' \sup_{t \leq e^2} W_t^2 + C_\delta E' \sup_{t \geq e^2} \frac{W_t^2}{t \log \log t} \cdot \left\{1 + \sup_{t \geq e^2}\left(\frac{T_t}{t}\right)^{1+\delta}\right\}.$$

Applying (6.1) and Hoelder's inequality, as well as $\{E|X|\}^{1/1+\delta} \leq 1 + E|X|$ and the fact that the P'-law of T equals the P^λ-law of $\tau^\alpha(\xi)$, we conclude that for all $\xi \in M$, $\delta > 0$

$$(6.12) \qquad E^\lambda \sup_{t \geq e^2} \frac{[M_t^\alpha(\xi)]^2}{t\log\log t} \leq C_\delta + C'_\delta\, E^\lambda \sup_{t \geq e^2} \left(\frac{\tau_t^\alpha(\xi)}{t}\right)^{1+\delta}.$$

Now, by the maximal ergodic inequality

$$E^\lambda \sup_{t \geq e^2} \left(\frac{\tau_t^\alpha(\xi)}{t}\right)^{1+\delta} \leq C''_\delta\, E^\lambda (\tau_1^\alpha(\xi))^{1+\delta}.$$

Choosing $\delta > 0$ such that (6.7) holds, we conclude (6.9).

For the proof of theorem (1.11), let $\{\overline{f}_n\}$ be a countable dense family in $L^2(M)$, $f_n = K_\alpha^s \overline{f}_n$. For $d \geq 2$, $\alpha > \frac{d}{2} - 1$, we consider the set

$$\Omega_\alpha = \Omega_0 \cap \{\omega;\ \underset{t\to\infty}{\text{cluster set}}\ \frac{L_t(f_n) - tS(f_n)}{\sqrt{2t\log\log t}} = [-\sigma_{f_n}, +\sigma_{f_n}] \text{ all } n\}$$

$$\cap\ \{\omega; \sup_{t \geq e^2} \frac{\int d\lambda(\xi)[A_t^\alpha(\xi) - t]^2}{2t\log\log t} < \infty\}.$$

From (1.9b), (5.5), (5.11), (6.2) we have $P^x\{\Omega_\alpha\} = 1$, $x \in M$. From (5.14), we obtain for $f = K_\alpha^s \overline{f}$, $\overline{f} \in L^2(M)$ and $\omega \in \Omega_0$ that

$$|L_t(f) - tS(f)| \leq \|\overline{f}\|_{L^2(d\lambda)} \int d\lambda(\xi)[A_t^\alpha(\xi) - t]^2 \quad .$$

Approximating $\overline{f} \in L^2(d\lambda)$ by functions $\{\overline{f}_n\}$ we obtain that for $\omega \in \Omega_\alpha$, $\underset{t\to\infty}{\text{cluster set}}\ \frac{L_t(f) - t\int f d\lambda}{2t\log\log t} = [-\sigma_f, +\sigma_f]$. Now consider Ω_{α_n} for $\alpha_n \downarrow \frac{d}{2} - 1$.

If $d = 1$, consider the square integrable P^λ-martingale

$$M_t(\xi) = g(\xi, X_t) - g(\xi, X_0) + [\phi(\xi)]^{-1} L(t,\xi) - t, \quad t \geq 0,$$

and prove with the preceding arguments that P^λ-a.e.

$$\sup_{t \geq e^2} \frac{\int dm(\xi)[L(t,\xi) - \phi(\xi)t]^2}{2t\log\log t} < \infty,$$

which together with (1.3) implies the assertion of the theorem for $d = 1$.

§7 Proof of Theorem (1.15)

We shall use in $H^{\alpha}(M)$ the inner product induced by $K^{s}_{\alpha}: L^{2}(d\lambda) \to H^{\alpha}(M)$, except in the proof of lemma (7.13). If we let for $d \geq 2$, $\alpha > \frac{d}{2} - 1$, $\omega \in \Omega_0$

(7.1) $$L^{\alpha}_{t} = K^{s}_{\alpha}A^{\alpha}_{t} = \int_0^t k^{s}_{2\alpha}(\cdot, X_{\sigma})d\sigma \quad ,$$

then $L^{\alpha}_{t} \in H^{\alpha}(M)$ and for $f \in H^{\alpha}(M)$

(7.2) $$L_t(f,\omega) - t\int f d\lambda = (L^{\alpha}_{t} - t, f)_{H^{\alpha}(M)}$$

by (5.14). If $d = 1$, $\alpha = 0$, $\omega \in \Omega_0$, we let

(7.3) $$L^{0}_{t}(\cdot\ ) = [\phi(\cdot\ )]^{-1}L(t,\cdot) \quad .$$

Then (7.2) holds with $\alpha = 0$. In any case we have from (5.15)

(7.4) $$L^{\alpha}_{t} = K^{s}_{2\alpha}L_t \quad .$$

Notice that

(7.5) $$L^{\alpha+\beta}_{t} = K^{s}_{2\beta}L^{\alpha}_{t}$$

for $d \geq 2$, $\alpha > \frac{d}{2} - 1$, $\beta > 0$, and if $d = 1$, $\alpha = 0$, it may be taken as the definition of L^{β}_{t}, $\beta > 0$.

(7.6) LEMMA: <u>If</u> $k : M\times M \to R \cup \{-\infty, +\infty\}$ <u>is measurable and if for some</u> $\delta > 0$, $\sup_y \int dm(x)|k(x,y)|^{1+\delta} < \infty$ <u>and</u> $\sup_x \int dm(y)|k(x,y)|^{1+\delta} < \infty$, <u>then the operator</u> K, <u>defined by</u> $(Kf)(x) = \int_M k(x,y)f(y)dm(y)$, <u>is a compact operator</u> $K : L^2(M) \to L^2(M)$.

<u>Proof</u>: Obviously $Kf \in L^1(M)$ for $f \in L^1(M)$. We show that K is a bounded operator on $L^2(M)$: Since $k' \underset{\text{def}}{=} \sup_x \int dm(y)|k(x,y)| < \infty$, $k'' \underset{\text{def}}{=} \sup_y \int dm(x)|k(x,y)| < \infty$, we have for the symmetric function $u(y,z) \underset{\text{def}}{=} \int dm(x)|k(x,y)|\cdot|k(x,z)|$ that

$\sup_{z\in M} \int dm(y)u(y,z) \leq k'\cdot k'' < \infty$. By Fubini's theorem and the Cauchy-Schwarz inequality we obtain for $f \in L^2(dm)$

$$\int dm(x)(Kf)^2(x) \leq (\iint dm(y)dm(z)f^2(y)u(y,z))^{1/2} \cdot (\iint dm(y)dm(z)f^2(z)u(y,z))^{1/2}$$
$$\leq k'k''\|f\|^2_{L^2(dm)} ,$$

which implies that K is a bounded operator on $L^2(dm)$ with

$$(*) \qquad \|K\| \leq (k'k'')^{1/2}.$$

But under the assumptions made, K is also a compact operator on $L^2(M)$. This is obvious for bounded k, since in this case K is even a Hilbert-Schmidt operator on $L^2(dm)$. In the general case we define the bounded kernels k_n by $k_n = k$ if $|k| \leq n$, $k_n = n$ if $k > n$, $k_n = -n$ if $k < -n$, and denote by K_n the corresponding compact operators $K_n : L^2(M) \to L^2(M)$. We conclude from (*) and the uniform integrability of $\{k(x,\cdot),\ x\in M\}$, $\{k(\cdot,y),\ y\in M\}$ that $\|K_n-K\| \to 0$. As the class of compact operators is closed with respect to convergence in the operator norm, K is compact.

As a consequence of the preceding lemma, the operators

$$G,G^*: L^2(M) \to L^2(M)$$

are compact, as are

$$K^s_\alpha : H^\beta(M) \to H^\beta(M)$$

for all $\alpha > 0$, $\beta \geq 0$.

We shall consider for $\alpha \geq 0$, the operator

$$(7.7) \qquad S_\alpha = K^s_{2\alpha}(G+G^*) : H^\alpha(M) \to H^\alpha(M).$$

For all $\alpha \geq 0$, the operator S_α is positive, self-adjoint and compact. Let B_α be the closed unit ball in $H^\alpha(M)$ and let

$$(7.8) \qquad \mathcal{K}_\alpha = S_\alpha^{1/2}B_\alpha \quad , \quad \alpha \geq 0 \quad .$$

Notice that $\mathcal{K}_\alpha$ is a norm-compact convex symmetric set in $H^\alpha(M)$.

(7.9) LEMMA: $\mathcal{K}_\alpha = K^s_{2\alpha}(G+G^*)^{1/2}B_0$

Proof: We introduce a CON system $\{\phi^\alpha_n,\ n\geq 0\}$ of eigenfunctions of $S_\alpha : H^\alpha(M) \to H^\alpha(M)$, and denote by λ^α_n the corresponding eigenvalues, letting

$\lambda_0^\alpha = 0$, $\phi_0^\alpha = 1$. The system $\{\phi_0^\alpha; (\lambda_n^\alpha)^{-1/2}(G+G^*)^{1/2}\phi_n^\alpha, n\geq 1\}$ is ON in $L^2(d\lambda)$. It is also complete in $L^2(d\lambda)$: If for $f \in L^2(M)$, $\int f\cdot(G+G^*)^{1/2}\phi_n^\alpha d\lambda = 0$ all $n \geq 1$, then $\int h(G+G^*)^{1/2} f d\lambda = 0$ all $h \in L^2(M)$, hence $\int f(G+G^*)f d\lambda = \int |\mathrm{grad} Gf|^2 d\lambda = 0$, hence $f = c$. We get $B_0 = \{\zeta_0 + \sum_{n=1}^{\infty} \zeta_n(\lambda_n^\alpha)^{-1/2}(G+G^*)^{1/2}\phi_n^\alpha; \sum_0^\infty \zeta_n^2 \leq 1\}$, from which the proof is easily concluded.

<u>(7.10) Remark</u>: The preceding lemma implies immediately that for $\alpha,\beta \geq 0$, $K_{\alpha+\beta} = K_{2\beta}^s K_\alpha$.

(7.11) LEMMA: <u>Let</u> $d \geq 2$, $\alpha > \frac{d}{2} - 1$ <u>or</u> $d=1$, $\alpha = 0$. <u>Then for all</u> $x \in M$, P^x-a.e., <u>the random set</u> $\{\frac{L_t^\alpha - t}{\sqrt{2t\log\log t}}, t \geq e^2\}$ <u>is relatively norm-compact in</u> $H^\alpha(M)$.

<u>(7.12) Remark</u>: In view of (7.5), relative norm-compactness in $H^\alpha(M)$ of $\{\frac{L_t^\alpha - t}{\sqrt{2t\log\log t}}, t\geq e^2\}$ implies for $\alpha' > \alpha$, relative norm-compactness in $H^{\alpha'}(M)$ of $\{\frac{L_t^{\alpha'} - t}{\sqrt{2t\log\log t}}, t\geq e^2\}$.

<u>Proof of Lemma (7.11):</u>

<u>Case 1</u>: $d \geq 2$, $\alpha > \frac{d}{2} - 1$.

It is sufficient to prove that P^λ-a.e. the random set $\{\frac{A_t^\alpha - t}{\sqrt{2t\log\log t}}, t\geq e^2\}$ is relatively norm-compact in $L^2(d\lambda)$. For this purpose let $\alpha_1 \in (\frac{d}{2} - 1, \alpha)$. Obviously,

$$\{\frac{A_t^\alpha - t}{\sqrt{2t\log\log t}}, t\geq e^2\} = K_{\alpha-\alpha_1}^s\{\frac{A_t^{\alpha_1} - t}{\sqrt{2t\log\log t}}, t\geq e^2\} ,$$

and relative $L^2(d\lambda)$-compactness of the left side follows from lemma (6.2) and compactness of the operator $K_{\alpha-\alpha_1}^s : L^2(d\lambda) \to L^2(d\lambda)$.

Case 2: $d = 1$, $\alpha = 0$.

The assertion follows in the same way from the following lemma, dealing with one-dimensional local time.

(7.13) LEMMA: If $d = 1$, then for all $x \in M$, P^x-a.e.:

(a) $L(t,\cdot) \in \bigcap_{0\le\alpha<1/2} H^{\alpha}(M)$ for all $t > 0$, and for every $\alpha \in [0,1/2)$, $t \mapsto L(t,\cdot)$ is a norm-continuous function from R^+ to $H^{\alpha}(M)$.

(b) $$\sup_{t\ge e^2} \frac{\|L(t,\cdot)-t\phi(\cdot)\|^2_{H^{\alpha}(M)}}{2t \log\log t} < \infty \text{ for all } \alpha \in [0,1/2).$$

Remark: Notice that (b) is equivalent to

(b') $$\sup_{t\ge e^2} \frac{\|K^s_{-\alpha}(L(t,\cdot)-t\phi)\|^2_{L^2(d\lambda)}}{2t \log\log t} < \infty \text{ for all } \alpha \in [0,1/2).$$

Proof: α) If $d = 1$, then $M = S^1$. We may assume w.l.o.g. that $m(S^1) = 1$. We introduce on S^1 the distance x from a fixed point 0 as coordinate. Then

$$\text{grad} = \frac{d}{dx}, \quad \Delta = \frac{d^2}{dx^2}, \quad L = \frac{1}{2}\frac{d^2}{dx^2} + v\frac{d}{dx}, \quad v \in C^{\infty}(S^1).$$

Let $\phi_0(x) = 1$, $\phi_{2n}(x) = 2 \sin 2\pi nx$, $\phi_{2n+1}(x) = 2\cos 2\pi nx$. Recall that $\Delta\phi_n = -\lambda_n\phi_n$, $\lambda_n \sim \pi^2 n^2$, and that $\{\phi_n,\ n\ge0\}$ is a CON system in $L^2(dm)$. Since $G\phi_n = \frac{2}{\lambda_n}\{\phi_n - \int\phi_n d\lambda + G(v\phi_n')\}$ for $n \ge 1$, we have for $n \ge 1$

(7.14) $$\|G\phi_n\|_{\infty} \le \frac{C}{n^2}, \quad \|(G\phi_n)'\|_{\infty} \le \frac{C}{n}.$$

As is well-known, for $\alpha \ge 0$

$$H^{\alpha}(S^1) = \{f\in L^2(S^1);\ \sum_0^{\infty} (1+n^2)^{\alpha}(f,\phi_n)^2_{L^2(dm)} < \infty\},$$

and $\|f\| = \{\sum_0^{\infty} (1+n^2)^{\alpha}(f,\phi_n)^2_{L^2(dm)}\}^{1/2}$ is an admissible norm in $H^{\alpha}(S^1)$, which we shall denote by $\|\ \|_{H^{\alpha}(S^1)}$ in this proof.

β) Also for the purpose of this proof only, we denote by $g(\cdot,\cdot)$ the continuous m-density of the operator G. Since $(g(x,\cdot),\phi_n)_{L^2(dm)} = (G\phi_n)(x)$, we conclude from (7.14), that $g(x,\cdot) \in \bigcap_{\alpha<3/2} H^\alpha(S^1)$ for all $x \in S^1$. Moreover, for $\alpha < 3/2$, $x \mapsto g(x,\cdot)$ is a norm-continuous function from S^1 to $H^\alpha(S^1)$, as $\sum_0^\infty (1+n^2)^\alpha \|G\phi_n\|_\infty^2 < \infty$.

γ) For $t \geq 0$, $\omega \in \Omega_0$, $\xi \in S^1$, let

$$M_t^0(\xi) = g(X_t,\xi) - g(X_0,\xi) + L(t,\xi) - t\phi(\xi). \tag{7.15}$$

Obviously $(M_t^0(\cdot),\phi_n)_{L^2(dm)} = M_t(\phi_n)$, $t \geq 0$, $\omega \in \Omega_0$, with

$$M_t(\phi_n) = (G\phi_n)(X_t) - (G\phi_n)(X_0) + L_t(\phi_n) - t\int\phi_n d\lambda$$

of §4. Recall that for all $x \in S^1$, $M_t(\phi_n)$ is a square integrable P^x-martingale with increasing process $\tau_t(\phi_n) = \int_0^t |(G\phi_n)'|^2(X_\sigma)d\sigma$. By the way, $M_\cdot(\phi_n)$ is continuous for all $\omega \in \Omega$. -If we let

$$\Omega_1 = \{\omega;\ \sum_0^\infty (1+n^2)^\alpha \sup_{t\leq T} (M_t^0,\phi_n)^2_{L^2(dm)} < \infty \text{ all } T > 0, \text{ all } \alpha \in [0,1/2)\},$$

then $P^x(\Omega_1) = 1$, $x \in S^1$, since for all $T > 0$, $x \in S^1$, $n \geq 1$

$$E^x \sup_{t\leq T}[M_t(\phi_n)]^2 \leq c\, E^x \int_0^T |(G\phi_n)'|^2(X_\sigma)d\sigma \leq \frac{CT}{n^2},$$

and therefore for all $T > 0$, $\alpha \in [0,1/2)$, $x \in S^1$

$$E^x[\sum_0^\infty (1+n^2)^\alpha \sup_{t\leq T}(M_t^0,\phi_n)^2_{L^2(dm)}] < \infty.$$

Moreover, if $\omega \in \Omega_1$, then $M_t^0(\cdot) \in \bigcap_{\alpha<1/2} H^\alpha(S^1)$ for $t \geq 0$, and for every $\alpha < 1/2$ the function $t \mapsto M_t^0(\cdot)$ is a norm-continuous function from R^+ to $H^\alpha(S^1)$. In view of (7.15) we have proved (a) for $L(t,\cdot) - t\phi$, rather than $L(t,\cdot)$. But as $\phi \in C^\infty(S^1) = \bigcap_{\alpha\geq 0} H^\alpha(S^1)$, (a) follows.

(δ) For the proof of (b), it suffices in view of (a), to show that for all $\alpha \in (0,1/2)$, P^λ-a.e.

$$\sup_{t \geq e^2} \frac{\|L(t,\cdot)-t\phi(\cdot)\|^2_{H^\alpha(S^1)}}{2t \log \log t} < \infty \quad ,$$

or rather

$$\sup_{t \geq e^2} \frac{\|M^0_t(\cdot)\|^2_{H^\alpha(S^1)}}{2t \log \log t} < \infty \quad . \tag{7.16}$$

For $\alpha \geq 0$, $t \geq 0$, $\omega \in \Omega$, $\xi \in S^1$, $N \geq 1$ let

$$M_t^{\alpha N}(\xi) = \sum_0^N (1+n^2)^{\alpha/2} [M_t(\phi_n)]\phi_n(\xi) \quad .$$

The process $M^{\alpha N}(\xi)$ is a continuous square integrable P^λ-martingale with increasing process

$$\tau_t^{\alpha N}(\xi) = \int_0^t ds \Big|\sum_0^N (1+n^2)^{\alpha/2} (G\phi_n)'(X_s)\phi_n(\xi)\Big|^2 \quad .$$

As in §6, we obtain for all $\alpha \geq 0$, $\xi \in S^1$, $N \geq 1$, $\delta > 0$

$$E^\lambda \sup_{t \geq e^2} \frac{|M_t^{\alpha N}(\xi)|^2}{2t \log \log t} \leq C_\delta\{1 + E^\lambda[\tau_1^{\alpha N}(\xi)]^{1+\delta/2}\} \tag{7.17}$$

$$\leq C_\delta\{1 + E^\lambda \int_0^1 ds \Big|\sum_0^N (1+n^2)^{\alpha/2} (G\phi_n)'(X_s)\phi_n(\xi)\Big|^{2+\delta}\} \quad .$$

By a version of the Hausdorff-Young Theorem in Harmonic Analysis and by (7.14), we conclude that for all $\alpha \in (0,1/2)$, $\delta \in (0, \frac{1-2\alpha}{\alpha})$, there exists $C_{\alpha,\delta}$ such that for all $x \in S^1$, $N \geq 1$

$$\int d\xi \Big|\sum_0^N (1+n^2)^{\alpha/2} (G\phi_n)'(x)\phi_n(\xi)\Big|^{2+\delta} \leq C_{\alpha,\delta} \quad .$$

From this estimate and (7.17) we conclude that for all $\alpha \in (0,1/2)$, there exists C_α such that for all $N \geq 1$

$$E^\lambda \sup_{t \geq e^2} \frac{\|M_t^{\alpha N}(\cdot)\|^2_{L^2(dm)}}{2t \log \log t} \leq C_\alpha \quad ,$$

or equivalently

$$E^{\lambda} \sup_{t \geq e^2} \frac{\| M_t^{ON}(\cdot) \|^2_{H^{\alpha}(S^1)}}{2t \log \log t} \leq C_{\alpha} \quad .$$

Letting $N \to \infty$, we obtain (7.16).

Just as in [3] we shall now prove the function space version of the $\log_2$-law from a $\log_2$-law for vector-valued functions and the compactness lemma (7.11). The following $\log_2$-law for vector-valued functions follows from the $\log_2$-law for real-valued functions (1.9b) by exactly the same argument which led to Theorem (4.1) in [3].

(7.18) THEOREM: <u>For all</u> $n \geq 1$, <u>all linearly independent functions</u> $f_1, \cdots, f_n \in P(M)$ <u>we have for all</u> $x \in M$, P^x-a.e. <u>that the</u> R^n-<u>cluster set as</u> $t \to \infty$ <u>of</u> $\frac{(L_t(f_1), \cdots, L_t(f_n))}{\sqrt{2t \log \log t}}$ <u>equals</u> $\{(\zeta_1, \cdots \zeta_n\} \in R^n; \sum_{i,j=1}^{n} a_{ij} \zeta_i \zeta_j \leq 1\}$ <u>where</u>

$a = \{(f_i, (G+G^*) f_j)_{L^2(d\lambda)}, i, j, \cdots, n\}^{-1}$.

Recall that $a_{ij} = \int \text{grad } Gf_i \cdot \text{grad } Gf_j \, d\lambda$, and that $G + G^*: L^2(d\lambda) \to L^2(d\lambda)$ is a positive self-adjoint compact operator. -We shall return to the eigenfunctions $\{\phi_n^{\alpha}, n \geq 0\}$ of S_{α}, introduced in the proof of lemma (7.9). Notice that

$(\phi_0^{\alpha}, f)_{H^{\alpha}(M)} = S(f)$ for $f \in H^{\alpha}(M)$, in particular $S(\phi_n^{\alpha}) = 0$ for $n \geq 1$. We denote by $\pi_N^{\alpha} : H^{\alpha}(M) \to H^{\alpha}(M)$ the projection on the subspace spanned by $\phi_0^{\alpha}, \phi_1^{\alpha}, \cdots, \phi_N^{\alpha}$. Obviously $\pi_0^{\alpha}(L_t^{\alpha} - t) = 0$ and for $N \geq 1$

(7.19) $$\pi_N^{\alpha}(L_t^{\alpha} - t) = \sum_{n=1}^{N} (L_t^{\alpha}, \phi_n^{\alpha})_{H^{\alpha}(M)} \phi_n^{\alpha} \quad .$$

(7.20) LEMMA: <u>Let</u> $d \geq 2$, $\alpha > \frac{d}{2} - 1$ <u>or</u> $d = 1$, $\alpha = 0$. <u>Then for all</u> $x \in M$, P^x-a.e. <u>for all</u> $N \geq 0$

(7.21) $$\| \quad \|_{H^{\alpha}(M)} - \underset{t \to \infty}{\text{cluster set}} \frac{\pi_N^{\alpha}(L_t^{\alpha} - t)}{\sqrt{2t \log \log t}} = \pi_N^{\alpha} \, K_{\alpha} .$$

Proof: It is sufficient to prove (7.21) for $N \geq 1$. From (7.2) and (7.19) we have

$$\pi_N^\alpha(L_t^\alpha - t) = \sum_{n=1}^{N} L_t(f,\omega)\phi_n^\alpha \quad .$$

As $\bigcup_{\alpha > \frac{d}{2}-1} H^\alpha(M) \subseteq P(M)$ for $d \geq 2$ and $H^0(M) \subseteq P(M)$ for $d = 1$, we conclude from Theorem (7.18) that for all $x \in M$, P^x-a.e. for all $N \geq 1$

$$\| \ \|_{H^\alpha(M)} - \underset{t\to\infty}{\text{cluster set}} \frac{\pi_N^\alpha(L_t^\alpha - t)}{\sqrt{2t\log\log t}} = \{\sum_1^N \zeta_n\phi_n^\alpha;\ \sum_{i,j=1}^N a_{ij}\zeta_i\zeta_j \leq 1\},$$

where $a = b^{-1}$, $b_{ij} = \int \phi_i^\alpha(G+G^*)\phi_j^\alpha d\lambda = (S_\alpha\phi_i^\alpha,\phi_j^\alpha)_{H^\alpha(M)} = \lambda_i^\alpha\delta_{ij}$. Obviously, $a_{ij} = (\lambda_i^\alpha)^{-1}\delta_{ij}$. On the other hand we have from the definition (7.8) of K_α, that

$$\pi_N^\alpha K_\alpha = \{\sum_1^N \zeta_n(\lambda_n^\alpha)^{1/2}\phi_n^\alpha;\ \sum_0^\infty \zeta_n^2 \leq 1\} = \{\sum_1^N \zeta_n(\lambda_n^\alpha)^{1/2}\phi_n^\alpha;\ \sum_1^N \zeta_n^2 \leq 1\}$$

$$= \{\sum_1^N \zeta_n\phi_n^\alpha;\ \sum_1^N(\lambda_n^\alpha)^{-1}\zeta_n^2 \leq 1\},$$

which proves (7.21).

The preceding lemma and lemma (7.11) imply

(7.22) LEMMA: Let $d \geq 2$, $\alpha > \frac{d}{2} - 1$ or $d = 1$, $\alpha = 0$. Then for all $x \in M$, P^x-a.e.

(7.23) $$\| \ \|_{H^\alpha(M)} - \underset{t\to\infty}{\text{cluster set}} \frac{L_t^\alpha - t}{\sqrt{2t\log\log t}} = K_\alpha \quad .$$

(7.24) Remark: In view of (7.5) and remark (7.10), the validity of (7.23) for an $\omega \in \Omega_0$, implies the validity of (7.23) for this ω and any $\alpha' > \alpha$.

Theorem (1.15) now follows from Lemma (7.11), Remark (7.12), Lemma (7.22), Remark (7.24), Lemma (7.9) and (7.4).

§8. PROOF OF THEOREM (1.16)

We shall start with three lemmas. In lemma (8.1) we shall take as version of the (weak) gradient of Gf the vector field

$$(\text{grad } Gf)(x) = \int \text{grad}_x g(x,y)f(y)d\lambda(y).$$

(8.1) LEMMA: Let $d \geq 2$, $\alpha > \frac{d}{2} - 1$ or $d = 1$, $\alpha = 0$. Then for all $f \in H^{\alpha}(M)$, Gf and grad Gf are continuous.

Proof: Under the assumptions made on α, we have for the kernels

$$h_{\alpha}(x,y) = \int g(x,z)k_{\alpha}^{s}(z,y)d\lambda(z)$$

and

$$h_{\alpha}^{*}(x,y) = \int \text{grad}_x g(x,z)k_{\alpha}^{s}(z,y)d\lambda(z) = \text{grad}_x h_{\alpha}(x,y)$$

that

$$(8.2) \qquad \sup_x \int |h_{\alpha}(x,y)|^2 d\lambda(y) < \infty, \quad \sup_x \int |h_{\alpha}^{*}(x,y)|^2 d\lambda(y) < \infty.$$

Let now $\bar{f} = K_{-\alpha}^{s} f$ and choose $\bar{f}_n \in C^{\infty}(M)$ such that $\|\bar{f}_n - \bar{f}\|_{L^2(d\lambda)} \to 0$. If we let $f_n = K_{\alpha}^{s}\bar{f}_n$, then $f_n \in C^{\infty}(M)$, hence Gf_n and grad $Gf_n \in C^{\infty}(M)$. From (8.2) and Cauchy-Schwarz we obtain $\|Gf-Gf_n\|_{\infty} = \|GK_{\alpha}^{s}(\bar{f}-\bar{f}_n)\|_{\infty} \to 0$, $\|\text{grad } Gf - \text{grad } Gf_n\|_{\infty} \to 0$, which concludes the proof.

(8.3) LEMMA: Let $d \geq 2$, $\alpha > \frac{d}{2} - 1$ or $d = 1$, $\alpha = 0$. Let $\{\phi_n^{\alpha}, n \geq 0\}$ be a CON system in $H^{\alpha}(M)$. Then the series $\sum_0^{\infty} |G\phi_n^{\alpha}|^2$ and $\sum_0^{\infty} |\text{grad } G\phi_n^{\alpha}|^2$ converge uniformly on M.

Proof: We shall use (8.2) of the preceding proof. Clearly

$$(K_{-\alpha}^{s}\phi_n^{\alpha}, h_{\alpha}(x,\cdot))_{L^2(d\lambda)} = (G\phi_n^{\alpha})(x) \text{ and } (K_{-\alpha}^{s}\phi_n^{\alpha}, \text{grad}_x h_{\alpha}(x,\cdot))_{L^2(d\lambda)} = (\text{grad } G\phi_n^{\alpha})(x).$$

This together with (8.2) implies that for all $x \in M$

$$\sum_0^\infty |(G\phi_n^\alpha)(x)|^2 = \int |h_\alpha(x,y)|^2 d\lambda(y)$$

and

$$\sum_0^\infty |\text{grad } G\phi_n^\alpha|^2(x) = \int |\text{grad}_x h_\alpha(x,y)|^2 d\lambda(y).$$

Now the argument that led to (8.2) for our choice of α can be strengthened to give continuity of the functions $\int |h_\alpha(x,y)|^2 d\lambda(y)$, $\int |\text{grad}_x h_\alpha(x,y)|^2 d\lambda(y)$. The assertion then follows from lemma (8.1) and Dini's Theorem.

(8.4) LEMMA: Let $d \geq 2$, $\alpha > \frac{d}{2} - 1$ or $d = 1$, $\alpha = 0$. Then $S_\alpha = K_{2\alpha}^s(G+G^*): H^\alpha(M) \to H^\alpha(M)$ is of trace class.

Proof: Consider the kernel of S_α i.e.

$$s_\alpha(x,y) = \begin{cases} \int k_{2\alpha}^s(x,z)[g(z,y) + g(y,z)]d\lambda(z) & \text{if } \alpha > 0 \\ g(x,y) + g(y,x) & \text{if } \alpha = 0 \end{cases} .$$

Under the assumptions made on α it follows from (2.9) and (2.11) that s_α is continuous on $M \times M$. Approximating S_α by finite-dimensional operators, we obtain for the eigenvalues λ_n^α of S_α that $\sum_0^\infty \lambda_n^\alpha \leq \int s_\alpha(x,x)d\lambda(x)$.

For the proof of theorem (1.16), notice that for $f_1, f_2 \in H^\alpha(M)$

$$\big((G+G^*)f_1, f_2\big)_{L^2(d\lambda)} = (S_\alpha f_1, f_2)_{H^\alpha(M)} \qquad (8.5)$$

Since $S_\alpha : H^\alpha(M) \to H^\alpha(M)$ is of trace class, there is a Gaussian measure μ_α on $H^{-\alpha}(M)$ with mean 0 such that (1.18) holds (See e.g. [9]). This measure is unique. Its characteristic functional

$$\Psi(f) = \int_{H^{-\alpha}(M)} e^{i\ell(f)} d\mu_\alpha(\ell), \quad f \in H^\alpha(M),$$

is given by $\Psi(f) = e^{-(S_\alpha f,f)_{H^\alpha(M)}/2}$, $f \in H^\alpha(M)$.

We denote by μ_t^α the P^ν-distribution on $H^{-\alpha}(M)$ of $t^{-1/2}\{L_t - tS\}$ and by $\{\phi_n^\alpha, n \geq 0\}$ a CON system in $H^\alpha(M)$ such that $\phi_0^\alpha \equiv 1$. In order to show that $\mu_t^\alpha \to \mu_\alpha$ weakly, it is sufficient to show that

$$(1) \qquad \lim_{t\to\infty} \int_{H^{-\alpha}(M)} d\mu_t^\alpha(\ell) e^{i\ell(f)} = e^{-(S_\alpha f, f)_{H^\alpha(M)}/2}, \quad f \in H^\alpha(M)$$

and

$$(2) \qquad \lim_{N\to\infty} \sup_{t\geq 1} \int_{H^{-\alpha}(M)} d\mu_t^\alpha(\ell) \sum_{n=N}^{\infty} [\ell(\phi_n^\alpha)]^2 = 0$$

(see e.g. [9]).

Now (1) is equivalent to the weak convergence for all $f \in H^\alpha(M)$ of the P^ν-distribution of $t^{-1/2}(L_t(f) - t\int f d\lambda)$ to the normal distribution with mean 0 and variance $\big((G+G^*)f, f\big)_{L^2(d\lambda)}$, and the latter follows from Theorem (1.9a). Furthermore, as $\int \phi_n^\alpha d\lambda = 0$ for $n \geq 1$, (2) is equivalent to

$$(2') \qquad \lim_{N\to\infty} \sup_{t\geq 1} E^\nu \sum_{n=N}^{\infty} \Big(\frac{L_t(\phi_n^\alpha)}{\sqrt{t}}\Big)^2 = 0 \quad ,$$

and the latter is verified as follows: We consider again the processes

$$M_t(\phi_n^\alpha) = (G\phi_n^\alpha)(X_t) - (G\phi_n^\alpha)(X_0) + L_t(\phi_n^\alpha), \; t \geq 0, \; n \geq 1$$

of §4. We recall that for all $n \geq 1$, $M_t(\phi_n^\alpha)$ is a square integrable P^ν-martingale with increasing process

$$\tau_t(\phi_n^\alpha) = \int_0^t |\text{grad } G\phi_n^\alpha|^2(X_s)ds.$$

It follows that for $n \geq 1$

$$E^\nu[L_t(\phi_n^\alpha)]^2 \leq 3E^\nu[(G\phi_n^\alpha)(X_t)]^2 + 3E^\nu[(G\phi_n^\alpha)(X_0)]^2 + 3E^\nu\int_0^t |\text{grad } G\phi_n^\alpha|^2(X_s)ds,$$

therefore,

$$E^\nu \sum_{n=N}^{\infty} \Big(\frac{L_t(\phi_n^\alpha)}{\sqrt{t}}\Big)^2 \leq 6 \, \Big\| \sum_{n=N}^{\infty} |G\phi_n^\alpha|^2 \Big\|_\infty + 3 \, \Big\| \sum_{n=N}^{\infty} |\text{grad } G\phi_n^\alpha|^2 \Big\|_\infty \quad ,$$

from which (2') follows by lemma (8.3).

REFERENCES

[1] J.R. Baxter and G.A. Brosamler, Energy and the Law of the Iterated Logarithm, Math. Scand. 38(1976), 115-136.

[2] E. Bolthausen, On the Asymptotic Behaviour of the Empirical Random Field of Brownian Motion, to appear.

[3] G.A. Brosamler, Laws of the Iterated Logarithm for Brownian Motions on Compact Manifolds, to appear in Z. Wahrscheinlichkeitstheorie verw. Gebiete.

[4] J. Hoffmann-Jϕrgensen, Sums of Independent Banach Space Valued Random Variables, Stud. Math. 52(1974/75), 159-186.

[5] S. Minakshisundaram, Eigenfunctions on Riemannian Manifolds, J. Indian Math. Soc. 17(1953), 159-165.

[6] I. Monroe, On Embedding Right Continuous Martingales in Brownian Motion, Ann. Math. Stat. 43(1973), 1293-1311.

[7] E. Nelson, The Adjoint Markov Process, Duke Math. J. 25(1958), 671-690.

[8] R.S. Palais, Seminar on the Atiyah-Singer Index Theorem, Princeton University Press (1965).

[9] Y. Prokhorov, Convergence of Random Processes and Limit Theorems in Probability Theory, Theory Prob. Appl. 1(1956), 157-214.

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