PETER LI

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Riemannian Manifolds

by

Peter Li*

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§0 Introduction

Let M be a complete noncompact Riemannian manifold without boundary. The trace of the Hessian operator defined on C^2 functions on M is known as the Laplacian and is denoted by Δ. Our purpose is to discuss uniqueness of solutions to the Laplace equation and the heat equation restricted to suitable function spaces. Typically, and in this case, our function spaces to be considered are the spaces of L^p functions defined on M, denoted by $L^p(M)$ for $p \in (0,\infty]$.

§1 The Laplace Equation

The Laplace equation is the equation for harmonic functions given by

$$\Delta f(x) = 0, \quad \text{for all } x \in M. \tag{1}$$

We will insist upon the function f be in $L^p(M)$, ie. the $p^{\underline{th}}$ power of the absolute value of f is integrable with respect to the Riemannian measure induced by the given Riemannian metric. We should point out that all constant functions are harmonic, moreover they are in $L^\infty(M)$. For $p < \infty$, the situation divides into two cases. The first case is when M has finite volume, then all constant functions are in $L^p(M)$ for any $p \in (0,\infty)$. The second case is when M has infinite volume, then all of the constant functions, but zero, are not in $L^p(M)$ for any $p \in (0,\infty)$.

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For the sake of simplicity, we say M satisfies property $\mathcal{H}^p$ if all L^p harmonic functions on M are constants (in the appropriate sense described above). We also say M satisfies property $\mathcal{S}^p$ if all nonnegative L^p subharmonic functions on M are constants. Since the absolute value of a harmonic function is subharmonic, we observe that M satisfying $\mathcal{S}^p$ implies it also satisfies $\mathcal{H}^p$.

The first result towards finding when does a complete manifold satisfy $\mathcal{S}^p$ or $\mathcal{H}^p$ was due to Greene-Wu [5] in 1974. They established that if the sectional curvature of M is nonnegative, then M satisfies $\mathcal{S}^p$ for $p \in [1,\infty)$.

Later in 1976, Yau [12] showed that when $p \in (1,\infty)$, completeness of M automatically implies property $\mathcal{S}^p$ without any extra geometric condition. He also proved that there does not exist any nonconstant nonnegative L^p harmonic function on a complete manifold for $p \in (0,1)$.

For the case when $p = \infty$, in another article of Yau [11], he showed that if the Ricci curvature, Ric_M, of M is nonnegative, then M satisfies property $\mathcal{H}^\infty$. This hypothesis on the Ricci curvature is sharp, since on a simply connected manifold with -1 curvature there exists abundance of bounded harmonic functions.

As it turned out, in an unpublished paper of Chung [2], he gave an example of a complete manifold which does not satisfy $\mathcal{H}^1$. Recently, Sullivan also produced examples of manifolds which do not satisfy $\mathcal{H}^p$ for p sufficiently small $(p < 1)$. These examples showed that unlike those cases when $p \in (1,\infty)$, extra geometric assumption must be imposed on M to ensure property $\mathcal{H}^p$ for $p \in (0,1]$.

Last year, in Garnett's thesis [4], she proved that if M has bounded geometry, then M satisfies $\mathcal{H}^1$. However, up to that point, it is almost certain that both theorems of Greene-Wu and Garnett for the case $p = 1$ are not optimal.

In recent work of R. Schoen and the author [8], optimal curvature assumptions on M was derived to ensure property $\mathcal{H}^p$ for any $p \in (0,1]$. To summarize the situation, we separate the cases when $p = 1$ and $p < 1$ into the following theorems.

Theorem 1: Let M be a complete Riemannian manifold. Suppose $r_0(x)$ is the distance function from x_0. If there exists positive constants C and α such that

$$\mathrm{Ric}_M(x) \geq -C(1+r_0(x)^2)\left[\log(1+r_0(x))^2\right]^{-\alpha}$$

for all $x \in M$, then M satisfies property $\mathcal{S}^1$.

Theorem 2: Let M be a complete Riemannian manifold of dimension n. There exists a constant $\delta(n)$ depending on n, such that if

$$\mathrm{Ric}_M(x) \geq -C\, r_0(x)^{-2}$$

for some $C \leq \delta(n)$, then M satisfies $\mathcal{S}^p$ for all $p \in (0,1)$.

We remark that Theorem 2 is best possible. In fact, in [8] we followed Sullivan's construction to produced examples of manifolds with sectional curvature behave like

$$K(x) \sim -\beta\left[(1-\beta)r_0(x)\right]^{-2},$$

for $\beta < 1$ as $r_0(x) \to \infty$, which does not satisfy property $\mathcal{H}^p$ for $p < 2\beta - 1$.

For the case when $p = 1$, we constructed an example of a manifold whose curvature behaves like

$$K(x) \sim -C\, r_0(x)^{2+\varepsilon},$$

for positive C and ε, which does not satisfy property $\mathcal{H}^1$. This indicates that Theorem 1 is almost best possible, except when $\mathrm{Ric}_M(x)$ behaves exactly like $-C\, r_0(x)^2$, which falls outside the scoop of both the theorem and the example. However, all evidence points toward the validity of property $\mathcal{H}^1$ for this critical situation. We have also provided an example of a complete manifold which possesses a nonconstant nonnegative L^1 harmonic function. This implies Yau's theorem cannot be generalized to the critical case of $p = 1$.

The method of proof for both Theorem 1 and 2 also gives the following:

Theorem 3: Let M be a complete Riemannian manifold. M satisfies property $\mathcal{S}^p$ for $p \in (0,1]$, provided that one of the following conditions is fulfilled:

(i) M is simply connected with nonpositive sectional curvature.

(ii) M has Ricci curvature bounded from below by a (possibly negative) constant and the volume of any unit geodesic ball is bounded from below by a positive constant independent of the center point.

Observing that if f is a function on M satisfying

$$\Delta f = \lambda f, \quad \lambda > 0,$$

then its absolute value, $|f|$, is a nonnegative subharmonic function. We have the next corollary.

Corollary 1: If M satisfies either the hypothesis of Theorem 1 or Theorem 3, then there does not exist any nontrivial L^1 function satisfying $\Delta f = \lambda f$ for $\lambda > 0$.

§2 The Heat Equation

We will now discuss the heat equation

$$(\Delta - \partial/\partial t)\ F(x,t) = 0 \tag{2}$$

for all $x \in M$ and $t \in (0,\infty)$, with initial condition

$$F(x,0) = F_0(x). \tag{3}$$

One constructs a minimal fundamental solution for the heat equation on M as follows:

Consider $\{D_n\}$ be a compact exhaustion of M. Let $H_n(x,y,t)$ be the fundamental solution for the heat equation on D_n with Dirichlet boundary condition. One argues that the sequence of fundamental solutions $H_n(x,y,t)$ must converge uniformly on any compact subdomain (see [3,10]), hence produces a fundamental solution on M defined by

$$H(x,y,t) = \lim_{n\to\infty} H_n(x,y,t).$$

Observing that since $\int_{D_n} H_n(x,y,t)\ dy < 1$ for all $x \in D_n$ and $t > 0$, $H(x,y,t)$ must satisfy

$$\int_M H(x,y,t)\ dy \le 1$$

for all $x \in M$ and $t > 0$. Moreover, $H(x,y,t)$ is the minimal positive fundamental solution for the heat equation on M. Using $H(x,y,t)$, one defines a semi-group P_t on $L^p(M)$ for all $p \in [1,\infty]$.

On the other hand, another semi-group can be defined abstractly in the sense of Strichartz [10]. We will outline the construction as follows: The domain $\mathscr{D}$ of the Laplacian Δ is defined to be the set of functions f such that each is a L^2 limit of some sequence of functions f_i in $C_c^\infty(M)$, and also Δf_i converges to the distribution Δf in L^2. It is known that Δ is essentially self-adjoint on $\mathcal{D}$. By the spectral theorem, we can form the operator $e^{\Delta t}$. Strichartz proved that $e^{\Delta t}$ is a strongly continuous contractive semi-group defined on $L^p(M)$, for $1 < p < \infty$, with Δ as its infinitesimal generator. Moreover, he showed that $e^{\Delta t}$ is the unique semi-group on $L^p(M)$ which is strongly continuous, contractive, and satisfying the heat equation, for $p \in (0,\infty)$. In particular, it implied that $P_t = e^{\Delta t}$ on $L^p(M)$ for $p \in (0,\infty)$.

It is our main purpose here to discuss the cases when $p = \infty$ and $p = 1$. In the first case, since $C_c^\infty(M)$ is not dense in $L^\infty(M)$, the existence of $L^\infty(M)$ semi-group with Δ as infinitesimal generator is impossible. However, one can still discuss, uniqueness of solutions for the heat equation in $L^\infty(M)$ in the following setting. Assuming $F(x,t)$ is a solution of (2), with initial data

$$\lim_{t\to 0} F(x,t) = F_0(x)$$

in $L^\infty(M)$. In particular, we are assuming that $F(x,t) \in L^\infty(M)$ for all $t \in [0,\varepsilon)$. Then we ask if $F(x,t)$ is determined by its initial condition $F_0(x)$.

In [7], L. Karp and the author succeeded in deriving sharp geometric conditions on M to ensure uniqueness of (2) in $L^\infty(M)$ (as in the sense described above).

<u>Theorem 4</u>: Let M be a complete Riemannian manifold. Suppose $x_0 \in M$ and $B_R(x_0)$ is the geodesic ball of radius R centered at x_0 in M. If

$$\mathrm{Vol}(B_R(x_0)) \leq e^{CR^2}$$

for some positive constant C, then for any L^∞ solution $F(x,t)$ of (2) with

$$\lim_{t\to 0} F(x,t) = F_0(x)$$

in $L^2_{loc}(M)$, $F(x,t)$ is uniquely determined by $F_0(x)$.

In particular, (2) has unique solution in $L^\infty(M)$ since $L^\infty(M) \subseteq L^2_{loc}(M)$.

Our assumption of the volume growth of M is best possible due to an example of Azencott [1], which is a simply connected manifold with negative curvature but possesses nonunique solutions for (2) on $L^\infty(M)$. Particularly, Azencott's example has sectional curvature behave like $-C\, r_0(x)^{2+\varepsilon}$ for some positive constants C and ε.

Using a standard comparison theorem argument, one checks that

$$\mathrm{Vol}(B_R(x_0)) \geq e^{CR^{2+\varepsilon}},$$

hence confirm the sharpness of Theorem 4.

We should point out that uniqueness on $L^\infty(M)$ of (2) is equivalent to the equation

$$\int_M H(x,y,t)dy \equiv 1,$$

for all $x \in M$ and all $t \in [0,\infty)$. In particular, since $H(x,y,t)$ is the minimal positive heat kernel, we deduced the following corollary.

<u>Corollary 2</u>: Let M be complete, and assume that for some $x_0 \in M$

$$\mathrm{Vol}(B_R(x_0)) \leq e^{CR^2}.$$

If $K(x,y,t)$ is a positive fundamental solution of (2) with $\int_M K(x,y,t)dy \leq 1$ for all $x \in M$ and $t \in [0,\infty)$, then $H(x,y,t) \equiv K(x,y,t)$ and $\int_M K(x,y,t)dy \equiv 1$. In particular, the kernel corresponding to the heat semi-group $e^{\Delta t}$ constructed by Strichartz is the same as $H(x,y,t)$, hence $e^{\Delta t} = P_t$.

We will now consider the case $p = 1$. Following Strichartz's argument in [10], and applying Corollary 1 and Theorem 4, the following theorem was proved in [8].

<u>Theorem 5</u>: Let M be a complete Riemannian manifold satisfying the hypothesis of Theorem 1. Then $e^{\Delta t}$ is a strongly continuous contractive semi-group on $L^1(M)$ with Δ as infinitesimal generator. Moreover $e^{\Delta t}$ is the unique such semi-group on $L^1(M)$.

Other than the L^p spaces on M, another natural function space is $C_0(M)$, the space of continuous functions on M which vanish at infinity. This space introduces the Dirichlet boundary condition on M. By the maximum principle, $e^{\Delta t} = P_t$ is the unique solution on $C_0(M)$. However, it is desirable that $e^{\Delta t}$ is a strongly continuous semi-group on $C_0(M)$. It turns out that the only condition one needs to verify for $e^{\Delta t}$ being a strongly continuous semi-group on $C_0(M)$ is the preservation of $C_0(M)$ by $e^{\Delta t}$, ie. $e^{\Delta t}(C_0(M)) \subseteq C_0(M)$. In [7], Karp and the author also proved:

Theorem 6: Let M be a complete Riemannian manifold. Suppose

$$\mathrm{Ric}_M(x) \geq -C\, r_0(x)^2$$

for some positive constant C, then $e^{\Delta t}(C_0(M)) \subseteq C_0(M)$. In particular $e^{\Delta t}$ is a strongly continuous semi-group on $C_0(M)$.

In [7], we also gave examples of manifolds with curvature satisfying

$$K(x) \sim -C\, r_0(x)^{2+\varepsilon},$$

and $e^{\Delta t}(C_0(M)) \not\subseteq C_0(M)$. This implies the sharpness of Theorem 6.

We remark that the above mentioned example is of finite volume, hence probabilistically complete. We also feel that may be one can strengthen Theorem 6 by imposing assumptions on the volume decay rate of geodesic annuli on M rather than on the Ricci curvature. A natural guess is that if $\mathrm{Vol}(B_R(x_0) - B_{R-1}(x_0)) \geq e^{-CR^2}$ then $e^{\Delta t}(C_0(M)) \subseteq C_0(M)$. However the authors of [7] were unsuccessful in proving this statement.

We shall point out that recently Seeley [9] studied uniqueness type properties for the operator $\Delta + V + q - \partial/\partial t$ on a complete Riemannian manifold, where V is a vector field on M and q is a function defined on M. His results were derived under the assumptions of the existence of some exhaustion functions on M. Though the point of view he took was different than ours, but there are interesting overlaps in some cases.

In [13] and [3], Yau and Dodziuk studied $e^{\Delta t}$ and P_t respectively under the assumption that M has Ricci curvature bounded from below. In [13], Yau proved that with the Ricci curvature assumption, $e^{\Delta t}\cdot 1 \equiv 1$ and $e^{\Delta t}(C_0(M)) \subseteq C_0(M)$. While in [3], Dodziuk proved that $e^{\Delta t} = P_t$ and uniqueness of solution of (2) on $L^\infty(M)$ when Ricci curvature has lower bound.

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Peter Li
Department of Mathematics
Purdue University
West Lafayette, IN 47907
U.S.A.