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ALMOST HOMOGENEOUS SPACES

by

Ernst A. Ruh

1. Introduction

Homogeneous spaces serve as models in the study of general riemannian manifolds. It is natural to start with manifolds whose local structure does not differ much from that of one of the models. For a general account of this point of view we refer to Cheeger and Ebin [2], Buser and Karcher [1], and, for a very brief survey of recent results, to Hirzebruch [4]. The purpose of this paper is first to introduce the notion of almost homogeneous space, and second to give a new proof of the theorem of Gromov [3] and Ruh [9] on almost flat manifolds.

Let $\overline{M} = G/K$ be a homogeneous space and $\overline{\omega} : TG \to \mathfrak{g}$ the Maurer-Cartan form of the Lie group G. In order to compare a general manifold M with $\overline{M}$ it is natural to assume that M, exactly as $\overline{M}$, is the base space of a principal K-bundle $P \to M$. The role of $\overline{\omega}$ in the model will be played by a Cartan connection $\omega : TP \to \mathfrak{g}$. M will be called almost homogeneous, see Definition 3 of section 2, if the curvature $\Omega = d\omega + [\omega,\omega]$ is suitably small. The motivation is the Maurer-Cartan equation $\overline{\Omega} = 0$.

There are several advantages in working with Cartan connections instead of connections in the usual sense. One is that Cartan connections make it easy to treat a large number of comparison theorems simultaneously, see [7]. Another advantage is that, because the definition of curvature is relative to a model space it can be changed without a change in the connection form. This gives us a better chance for constructing a connection with vanishing curvature, often the main task of a proof. In view of a result of Kobayashi [5], the restriction to Cartan connections is not unduly severe. [5, Th 2], at least for weakly reductive homogeneous spaces as models, shows how to convert a given connection into a Cartan connection.

Whenever the model space has non-trivial deformations, the notion of Cartan connections is extremly useful.

2. Definitions and results

Let G denote a Lie group with Lie algebra $\mathfrak{g}$, $K \subset G$ a subgroup, and $P \to M$ a principal bundle with structure group K. In addition, assume that the dimensions of G and P coincide.

Definition 1. A $\mathfrak{g}$-valued 1-form $\omega : TP \to \mathfrak{g}$ is called a Cartan connection form of type (G,K) if the following assertions hold.

(i) $\omega(A^*) = A$ for all $A \in \mathfrak{k}$, where $\mathfrak{k}$ is the Lie algebra of K and A^* is the fundamental vector field on P defined by the action of expt A.

(ii) $R_a\omega = \mathrm{ad}(a^{-1})\omega$ for all $a \in K$, where R_a is the action (from the right) of a on P.

(iii) $\omega(X) = 0$ for $X \in TP$ implies $X = 0$.

Definition 2. Let ω denote a Cartan connection (form). The 2-form $\Omega = d\omega + [\omega,\omega]$ on P is called the Cartan curvature (form) of ω. Here, the value of $[\omega,\omega]$ on a pair of tangent vectors (X,Y) is equal to $[\omega(X),\omega(Y)]$ and $[\ ,\ ]$ is the Lie bracket of $\mathfrak{g}$.

If $\bar{M} = G/K$ is a homogeneous space, then the Maurer-Cartan form $\bar{\omega}$ is a Cartan connection of type (G,K) with vanishing curvature. ($\bar{\Omega} = 0$ is the Maurer-Cartan equation.) The following converse is well known: If the Cartan curvature of a Cartan connection ω on P vanishes, then ω defines a locally homogeneous structure on $M = P/K$. This is the motivation for the next definition.

Let $\mathfrak{g}$ denote a Lie algebra with scalar product $\langle\ ,\ \rangle$ normalized such that the Lie bracket $[\ ,\ ] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ has operator norm bounded by 1. By assertion (iii) of Definition 1, ω defines a riemannian metric on P. Let d denote the diameter of P.

Definition 3. A compact manifold M is called ε-almost homogeneous of type (G,K) if M is the base space of a principal K-bundle P with Cartan connection form ω of type (G,K) whose Cartan curvature Ω satisfies

$$\|\Omega\|\, d < \varepsilon ,$$

where d is the diameter of P and $\|\ \|$ is the L_∞-norm.

This definition was inspired by Gromov's definition of almost flat manifolds. The first step of the proof in [9] shows that M ε-almost flat implies M ε'-almost homogeneous of type $(\mathbb{R}^n, e)$ with $\varepsilon' = c(n)\varepsilon^{1/2}$, $c(n)$ a constant depending on the dimension n of M only, and e the identity in the abelian group $\mathbb{R}^n$.

The main question now is whether for a suitably small ε a given ε-almost homogeneous structure can be deformed into a locally homogeneous one. This question has been settled in a number of cases. The easiest cases to deal with are manifolds modelled on riemannian symmetric spaces of compact type. Here is a somewhat more general result:

Theorem 1. *Let* G *denote a compact semi-simple Lie group, and* $K \subset G$ *a closed subgroup. There exists* $\varepsilon > 0$ *such that any* ε-*almost homogeneous space of type* (G,K) *is diffeomorphic to a locally homogeneous space of type* (G,K).

This theorem is an immediate consequence of the theorems of [7]. As one expects in this case, it is not necessary to scale with the diameter. The corresponding result holds, with some exceptions, if G is semi-simple and non-compact and K is a maximal compact subgroup, compare [8]. It is unknown whether scaling is necessary in this case. In the above theorems the type of the resulting homogeneous space is known a priori. The next theorem, where $G = \mathbb{R}^n$ and K consists of the identity alone, treats a more delicate problem.

Theorem 2. *There exists* $\varepsilon = \varepsilon(n) > 0$ *such that any* ε-*almost homogeneous space of type* $(\mathbb{R}^n, e)$ *is diffeomorphic to a locally homogeneous space of type* (N,e) *with* N *nilpotent*.

In view of the fact, proved in [9, step 1], that ε-flat implies ε'-almost homogeneous of type $(\mathbb{R}^n, e)$, Theorem 2 implies the well known theorem of Gromov [3] on almost flat manifolds in the stronger version proved in [9]. The purpose of the above formulation is to suggest that both Theorem 1 and Theorem 2 may be special cases of a more general result. It is not known what the obstruction is for the deformation of an almost homogeneous structure to a locally homogeneous one. Surprisingly, no counter example seems to be known if we don't insist in fixing the type of the resulting locally homogeneous space.

3. Proof of Theorem 2

For the proof it is convenient to generalize the concept of Cartan connection. Instead of assuming that the connection form ω is Lie algebra valued we only assume that its values lie in a vector space endowed with a skew symmetric product $[\ ,\ ] : V \times V \to V$, i.e., $[\ ,\ ]$ need not satisfy the Jacobi identity. Except for this change, the definition of the generalized Cartan curvature is the same as Definition 2. It is important to note that vanishing generalized Cartan curvature implies the Jacobi identity of the skew product $[\ ,\ ]$. To prove this let X and Y be vector fields on P with $\omega(X)$ and $\omega(Y)$ constant.
$\Omega(X,Y) = X\omega(Y) - Y\omega(X) - \omega([X,Y]) + [\omega(X),\omega(Y)]$, and since $X\omega(Y) = Y\omega(X) = 0$, $\Omega = 0$ implies that ω is an isomorphism of the Lie algebra of constant vector fields with the vector space V endowed with the skew product $[\ ,\ ]$. Therefore, any generalized flat Cartan connection is a flat Cartan connection in the strict sense.

For convenience we will normalize the diameter d of P to $d = 1$. In order not to obscure the main lines of the proof, we assume a bound not only on Ω but also on some of its derivatives. The assumption

$$(1) \qquad \|\Omega\|_{2,q} < \Lambda ,$$

where $\|\ \|_{2,q}$ denotes the Sobolev norm, see definition (10), involving up to second derivatives measured in L_q and q large enough to imply the Sobolev inequality

$$(2) \qquad \|\Omega\|_{1,\infty} < c\|\Omega\|_{2,q}$$

is convenient. This can be achieved for any $\Lambda > 0$ by smoothing with an appropriate kernel while choosing ε of Theorem 2 sufficiently small.

The proof consists of constructing a sequence of generalized Cartan connections converging to a flat Cartan connection. As is well known, a flat Cartan connection defines a locally homogeneous structure on the base manifold M. First some preparations.

Let $\omega : TP \to \mathbb{R}^n$ denote a generalized Cartan connection and $[\ ,\ ] : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ a skew product. By Definition 2, assertion (iii), ω is non-degenerate and defines a riemannian metric on P. Let $\{X_i\}$ denote the basis of orthonormal vector fields mapped by ω onto the standard basis $\{e_i\}$. Let $\beta^i_{1\ldots p} = \beta^i(X_1,\ldots,X_p)$ denote the coordinate functions of the p-form $\beta = \sum \beta^i e_i$.

We define

(3) $$d'\beta(X_o,\dots,X_p) = \sum_{j=o}^{p} (-1)^j X_j \beta(X_o,\dots,\hat{X}_j,\dots,X_p) ,$$

(4) $$\delta'\beta(X_2,\dots,X_p) = \sum_{i=1}^{n} X_i \beta(X_i,X_2,\dots,X_p) , \text{ and}$$

(5) $$\Delta'\beta(X_1,\dots,X_p) = \Delta\, \beta^i_{1\dots p} ,$$

where Δ is the Laplace operator on functions $(\Delta = \frac{-\partial^2}{\partial x^2}$ on $\mathbb{R}$). In addition, we will need the average $\overline{\beta}$ of a differential form $\beta = \sum \beta^i e_i$ defined by

(6) $$\overline{\beta}(X_1,\dots,X_p) = \sum \overline{\beta}^i_{1\dots p} ,$$

where $\overline{\beta}^i_{1\dots p}$ is the average of the corresponding coordinate function over the manifold P.

In the following iteration scheme the vector fields $\{X_i\}$ will change in each step. Let ω^i, $[\ ,\ ]^i$, and Ω^i denote connection form, skew product and curvature form of the i^{th} step. We define

(7) $$\omega^o = \omega ,$$

with ω the Cartan connection form of Theorem 2,

(8) $$[\ ,\ ]^i = \overline{d\omega^i} ,$$

the average of the exterior form $d\omega^i$ according to (6), and

(9) $$\omega^{i+1} = \omega^i + \delta'\beta ,$$

where β is the unique solution of $\Delta'\beta = -\Omega^i$ with $\overline{\beta} = 0$. (By definition, Ω^i has average zero.)

To establish convergence to a flat Cartan connection we need the following lemmas. First we define the Sobolev norm. Let $\exp: T_x P \to P$ denote the exponential map.

(10) $$\|\beta\|_{s,q} = \sup_{x\in P,i,j_1<j_2\dots j_p} \left(\int \sum_{\mu=o}^{s} \left|\frac{\partial}{\partial x^\mu}\beta^i_{j_1\dots j_p}(\exp(y))\right|^q dy\right)^{1/q} ,$$

where B is a ball of radius 1 and center $0 \in T_x P$, and $\frac{\partial}{\partial x^\mu}$, in standard multi index notation, is a derivative of order $|\mu|$.

Lemma 1. For $\| [\ ,\ ]^i \|$ and $\| \Omega^i \|_{2,q}$ sufficiently small, there exists c_1 such that

$$\| \omega^{i+1} - \omega^i \|_{2,q} < c_1 \| \Omega^i \|_{2,q} . \tag{11}$$

Proof. $\| [,]^i \|$ and $\| \Omega^i \|_{2,q}$ sufficiently small, the Ricci curvature of the metric of P induced by ω^i has norm smaller than $\frac{\pi^2}{4(n-1)}$, and an estimate of Li and Yau [6] on the first eigenvalue of the Laplace operator yields an estimate for the inverse in L_2 of the Laplacian on functions which are perpendicular to constants. By (8) the components of Ω^i are perpendicular to constants and we can apply this estimate to obtain $\| \beta \|_{o,2} < c \| \Omega \|_{o,2}$. An interior regularity estimate for the Laplacian implies

$$\| \beta \|_{3,q} < \| \Omega^i \|_{2,q} \tag{12}$$

and Lemma 1 is proved.

Lemma 2. Let $\| [\ ,\ ]^i \| + \| \Omega^i \|_{2,q} < 1$. There exists c_2 such that

$$\| d'\Omega \|_{1,q} < c_2 (\| [\ ,\ ]^i \| + \| \Omega^i \|_{2,q}) \| \Omega^i \|_{2,q} . \tag{13}$$

Proof. Define ϕ^s_{jk} by $[X_j, X_k] = \sum \phi^s_{jk} X_s$, where $[X_j, X_k]$ is the vector field bracket of the vector fields mapped under ω^i to e_j and e_k of the standard basis in $\mathbb{R}^n$. Let $\overline{\phi}^s_{jk}$ denote the average of ϕ^s_{jk} over P. By (8), $\{\overline{\phi}^s_{jk}\}$ is the coordinate expression of $[\ ,\ ]^i$. The Jacobi equation for the vector fields X_j, X_k, X_ℓ implies

$$X_j \phi^t_{k\ell} + X_k \phi^t_{\ell j} + X_\ell \phi^t_{jk} = - \sum (\phi^s_{k\ell} \phi^t_{js} + \phi^s_{\ell j} \phi^t_{ks} + \phi^s_{jk} \phi^t_{\ell s}) . \tag{14}$$

Note that $\phi^t_{k\ell} - \overline{\phi}^t_{k\ell}$ is the coordinate expression for Ω^i and, because $X_j \overline{\phi}^t_{k\ell} = 0$, the left hand side of (14) is the coordinate expression for $d'\Omega^i$. Substitute $\phi = (\phi - \overline{\phi}) + \overline{\phi}$ into (14) and note that except for the term

$$\sum (\overline{\phi}^s_{k\ell} \overline{\phi}^t_{js} + \overline{\phi}^s_{\ell j} \overline{\phi}^t_{ks} + \overline{\phi}^s_{jk} \overline{\phi}^t_{\ell s}) \tag{15}$$

the right hand side of (14) satisfies inequality (13). Because (15) by definition is constant, it suffices to estimate the global scalar product of terms of the form Xf, where X is one of the vector fields $\{X_j\}$ and f is a function on P, with the constant 1. Let σ denote the volume form. We have

$$\int (Xf)\sigma = \int X(f\sigma) - \int f \operatorname{div} X\sigma = - \int f \operatorname{div} X\sigma = - \int (f - \overline{f}) \operatorname{div} X\sigma .$$

Now, div X , for $X = X_j$, can be estimated in terms of $\|[\ ,\ ]^i\| + \|\Omega^i\|_{2,q}$, and the substitution $f = \phi^s_{k\ell}$ yields Lemma 2.

Lemma 3. For $\|[\ ,\ ]^i\|$ and $\|\Omega^i\|_{2,q}$ sufficiently small, there exists c_3 such that

$$(16) \qquad \|\Omega^{i+1}\|_{2,q} < c_3(\|[\ ,\ ]^i\| + \|\Omega^i\|_{2,q})\|\Omega^i\|_{2,q} .$$

Proof. Let L denote half the right hand side of (16). First we claim

$$(17) \qquad \|\Omega^{i+1}\|_{2,q} \leq \|d\omega^{i+1} - [\ ,\ ]^i\|_{2,q} < \delta'd'\beta + L .$$

The first inequality holds because exchanging the average $\overline{d\omega}^{i+1} = [\ ,\ ]^{i+1}$ in the definition of Ω^{i+1} for $[\ ,\ ]^i$ can only increase the norm. To obtain the second inequality we utilize (12) and compute the differences $d-d'$ and $\Delta' - (d'\delta'+\delta'd')$ respectively. A routine computation shows that (12) implies

$$(18) \qquad \|\Delta'\delta'd'\beta\|_{o,q} < L .$$

The reason for the gain of one derivative in this inequality is that $d'd'$ and $\delta'\delta'$ are operators of order one only. To finish the proof we observe that the scalar product of the coordinates of $\delta'd'\beta$ with the constant 1 is bounded by L . The argument is the same as in the proof of Lemma 2. On the other hand, as shown in the proof of Lemma 1, the inverse of Δ' on the space of forms with average zero is bounded. Therefore, (17) and (18) together with an interior regularity estimate for the Laplacian prove Lemma 3.

In the conclusion of the proof of Theorem 2 we proceed as if the norms would not depend on the iteration step. The following estimates show that a suitable curvature bound Λ in (1) implies that the change is small and can be disregarded.

Let $M < 1$ denote a bound which is small in the sense of Lemmas 1 and 3. The next goal is to prove that a suitable choice of Λ implies

$$(19) \qquad \|[\ ,\ ]^i\| + \|\Omega^i\|_{2,q} < \min(M, \frac{1}{2c_3}) = \Lambda_1$$

for all $i \in \mathbb{N}$. Let $k \in \mathbb{N}$ be such that (19) holds for all $i < k$. Lemmas 1 and 3 imply

$$(20) \qquad \|\Omega^k\|_{2,q} < (1/2)^k\Lambda \ , \quad \|\omega^k-\omega^o\|_{2,q} < 2c_1\Lambda \ ,$$

and because of (8),

$$(21) \qquad \| [\ ,\ ]^k \| + \| \Omega^k \|_{2,q} < 4c_1\Lambda \quad .$$

Therefore, a suitable choice of Λ implies that (19) holds for $i = k$ and by induction for all $i \in \mathbb{N}$.

Lemma 1 grants the existence of $\lim \omega^k$, (20) shows that the limit is nondegenerate, and again by (20), is a Cartan connection with vanishing curvature. This proves that the manifold of Theorem 2 is locally homogeneous of type (G,e). To show that G is nilpotent we appeal, as in [9, p. 13], to a theorem of Zassenhaus and Kazdan-Margulis.

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