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CARTAN CONNECTIONS AND CR STRUCTURES WITH NON-DEGENERATE LEVI-FORM

BY

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A CR structure, a pseudo-conformal structure of É. CARTAN, on a manifold M is a subbundle T'' of the complex tangent vector bundle $\mathbf{C}TM$ of M satisfying the following conditions :

1) If X and Y are sections of T'', so is $[X, Y]$.
2) $T'' \cap T' = \{0\}$ where T' is the complex conjugate of T''.
3) $S_{T''} = \mathbf{C}TM/(T'' + T')$ has fiber complex dimension one.

If these are the case, we have Levi-form $c_{T''}(X,Y) = -i[X, \overline{Y}] \pmod{T' + T''}$. It has value in $S_{T''}$ for sections X, Y of T''.

If M is a real hypersurface of codimension one in a complex manifold, the set of complex tangent vectors, which are of type $(0,1)$ when written in terms of a complex chart of the ambient complex manifold, is a CR structure. When its Levi-form is definite, the induced CR structure determines the ambient complex manifold near M. Thus CR structures are intimately connected with complex manifolds with boundary. Viewed as a system of differential equations : $Xf = 0$ for all X in T'', they also furnish interesting examples of partial differential equations.

The fundamental work of É. CARTAN [2] on the geometry of CR structures with non-degenerate Levi-form (when M is of dimension 3) was generalized by TANAKA [7, 8] and CHERN-MOSER [4] for general dimensions. The work of TANAKA treats the cases more general than the case we consider here and includes some of non-integrable CR structures. The geometry was further enriched by the discovery of C. FEFFERMAN [6] which relates it with the conformal geometry. Another conformal structure was given by BURNS-DIEDERICH-SHNIDER [1] in terms of the above work of CHERN-MOSER, and they showed that it agrees with the Fefferman metric. In this talk we would like to look at these works from the viewpoint of the generalized space of É. CARTAN and also add a little more information on the CR structure.

The generalized space may be considered as a synthesis of Riemann geometry and F. KLEIN's notion of geometry. As in KLEIN we start with a homogeneous space, say G/H. To be precise, we consider the space of cosets gH and g_1 in G operates on G/H by sending gH to g_1gH. To define a generalized space structure (patterned after G/H) on a manifold M we start as in C. EHRESMAN [5] by attaching G/H to each point p of M. Namely, we consider a set A_p of diffeomorphisms of open neighborhoods of 0, the coset H, into M sending 0 to p. For our geometry, however, we consider attaching maps up to an equivalence. And this equivalence should have the following property :

(*) *If f_1 and f_2 are attaching maps at a point of the structure, there is a unique h in H such that f_1 and $f_2 \circ h$ are equivalent.*

When such a set of equivalence classes of attaching maps are given, we say that we have a pre-generalized space. We may call an equivalence class of the attaching maps a frame of the structure. The above property (*) implies that the set of the frames forms a principal bundle over M, say E, with the structure group H. The fiber over p will be denoted by E_p.

The sets A_p and A'_p, $p \in M$, are considered to define the same structure when $A_p \cup A'_p$ still defines a structure. We say that the structure is flat when, for any attaching map and for any g in G sufficiently near the identity, $f \circ g$ is also an attaching map at $f(g(0))$.

Now our geometry is viewed as the study of properties of M expressible in terms of attaching maps which are independent of the choice of the employed attaching map.

The model homogeneous space G/H is considered as a flat generalized space. Namely, the operations of elements in G are considered as attaching maps.

In the case of Riemann manifold, G/H is $\mathbf{R}^m$ viewed as the quotient of the group of Euclidean motions by the orthogonal group. A diffeomorphism of an open neighborhood of 0 in $\mathbf{R}^m$ into M is an attaching map when its differential at 0 is an isometry. Attaching maps are considered equivalent when the value at 0 as well as the differentials at 0 agree. In this context the positivity of the metric does not play any role and we might as well start with any constant non-degenerate symmetric quadratic form

$$(dx, dx) = \mathbf{g}_{jk}\, dx^j\, dx^k$$

on $\mathbf{R}^m$, and call it the standard metric.

In the case of the conformal geometry we still consider $\mathbf{R}^m$ and the conformal structure induced by the standard metric. However, we have to enlarge it to have a homogeneous space. Namely, as is well known, we consider $\mathbf{R}^{m+2}$ with general element $\xi = (\xi^0, \dots, \xi^{m+1})$ and consider

$$\Phi(\xi, \xi) = (\xi', \xi') - 2\xi^0 \xi^{m+1}$$

where $\xi' = (\xi^1, \dots, \xi^m)$ and $(\xi', \xi') = \mathbf{g}_{jk}\xi^j\xi^k$. Denote by $\widetilde{G}$ the subgroup of $SL(\mathbf{R}, m+2)$ which leaves the quadratic form Φ invariant. On

$$\widetilde{Q}_{\mathbf{R}} = \{\xi \in \mathbf{R}^{m+2} : \Phi(\xi, \xi) = 0 \text{ and } \xi \neq 0\}$$

we have a degenerate metric $\Phi(d\xi, d\xi)$ and $\widetilde{G}$ acts effectively, transitively, and as isometry on $\widetilde{Q}_{\mathbf{R}}$. $\mathbf{R}^*$, the group of non-zero real numbers, acts on $Q_{\mathbf{R}}$ by the scalar multiplication. We set

$$\overline{Q}_{\mathbf{R}} = \widetilde{Q}_{\mathbf{R}}/\mathbf{R}^*.$$

The action of $\widetilde{G}$ on $\widetilde{Q}_{\mathbf{R}}$ induces its action on $\overline{Q}_{\mathbf{R}}$. When we denote by K the subgroup of elements on $\widetilde{G}$ which induce the identity map on $\overline{Q}_{\mathbf{R}}$, the group $G = \widetilde{G}/K$ acts effectively and transitively on $\overline{Q}_{\mathbf{R}}$ making it a homogeneous space G/H. When m is odd, K consists of the identity element. When m is even, K consists of two elements containing the multiplication by -1. We denote the group by $\mathbf{Z}_2$.

The metric $\Phi(d\xi, d\xi)$ on $\widetilde{Q}_{\mathbf{R}}$ induces a conformal structure on $\overline{Q}_{\mathbf{R}}$ and G is the group of conformal automorphism of $\overline{Q}_{\mathbf{R}}$. We have an embedding

$$\mathbf{R}^m \ni y \to \left(1, y, \frac{1}{2}(y, y)\right) \quad (\text{mod } \mathbf{R}^*) \in \overline{Q}_{\mathbf{R}}.$$

The image is an open submanifold $Q_{\mathbf{R}}$ of $\overline{Q}_{\mathbf{R}}$ and the induced conformal structure is the conformal structure of the standard metric of $\mathbf{R}^m$.

For simplicity we assume that $m \geq 3$. Let M be a conformal structure such that its constant conformal structure on the tangent vector space is isomorphic to the standard conformal structure on $\mathbf{R}^m$. A diffeomorphism of an open neighborhood $Q'_{\mathbf{R}}$ of 0 into M sending 0 to p is an attaching map of the conformal structure when the induced structure on $Q'_{\mathbf{R}}$ and the standard structure on $Q'_{\mathbf{R}}$ agree at 0 up to order one. Attaching maps are considered equivalent when they agree up to order 2 (i.e. as 2-jets) at 0. We can show that attaching maps exist and the equivalence satisfies the condition (*).

To see that we can view a CR structure with non-degenerate Levi-form as a pre-generalized space, we first describe (following TANAKA and CHERN-MOSER) the homogeneous space G/H after which we pattern our geometry. We start with a constant non-degenerate hermitian quadratic form

$$\langle z, z\rangle = \mathbf{h}_{jk} z^j \overline{z^k}$$

where $z = (z^1, \ldots, z^{n-1})$ denotes a general element in $\mathbf{C}^{n-1}$. We consider $\mathbf{C}^{n+1}$ with general element $\varsigma = (\varsigma^0, \ldots, \varsigma^n)$ and let

$$\Psi(\varsigma, \bar{\varsigma}) = i(\bar{\varsigma}^0 \varsigma^n - \varsigma^0 \bar{\varsigma}^n) + \langle \varsigma', \varsigma' \rangle \tag{1}$$

where $\varsigma' = (\varsigma^1, \ldots, \varsigma^{n-1})$. We denote by $\widetilde{G}$ the subgroup of elements in $SL(\mathbf{C}, n+1)$ which leave Ψ invariant. $\widetilde{G}$ acts transitively on

$$\widetilde{Q} = \{\varsigma \in \mathbf{C}^{n+1} : \Psi(\varsigma, \bar{\varsigma}) = 0 \text{ and } \varsigma \neq 0\}.$$

The group $\mathbf{C}^*$ of non-zero complex numbers also acts on $\widetilde{Q}$ by the scalar multiplication and we set

$$\overline{Q} = \widetilde{Q}/\mathbf{C}^*.$$

Since $\overline{Q}$ is a real hypersurface (defined by $\Psi = 0$) in the complex projective space, $\overline{Q}$ has the induced CR structure. The kernel of the induced action of $\widetilde{G}$ on $\overline{Q}$ is $\mathbf{Z}_{n+1}$, the group of the multiplication operator by $e^{i\theta}$ with θ in $(2\pi/(n+1))\mathbf{Z}$. Thus Q is a homogeneous space G/H where $G = \widetilde{G}/\mathbf{Z}_{n+1}$.

$$Q = \{\varsigma \bmod \mathbf{C}^* \text{ in } \overline{Q} : \varsigma^0 \neq 0\}$$

is an open submanifold with the global standard chart $(z, x) \in \mathbf{C}^{n+1} \times \mathbf{R}$ given by

$$j : (z, x) \to \left(1, z, x + \frac{i}{2}\langle z, z\rangle\right) \bmod \mathbf{C}^* \in Q.$$

Thus the CR structure on Q is induced by $\mathbf{C}^n$ when Q is regarded as real hypersurface

$$\operatorname{Im} w = \frac{1}{2}\langle z, z\rangle$$

in $\mathbf{C}^n$ with general element (z, w). As such it is defined by a system of equations

$$\theta_Q = \omega_Q^1 = \cdots = \omega_Q^{n-1} = 0, \quad \omega_Q^j = d2^j,$$
$$\theta_Q = dx - \frac{i}{2}\langle dz, z\rangle + \langle z, dz\rangle = j^*\partial\big(w - \overline{w} - i\langle z, z\rangle\big).$$

We denote the CR structure on Q by $T''Q$. Note that

$$d\theta_Q = i\langle\omega_Q,\omega_Q\rangle = i\mathbf{h}_{j\overline{k}}\omega^j_Q \wedge \overline{\omega^k_Q}. \tag{2}$$

On the other hand for $X,Y \in T''Q$

$$d\theta_Q(X,\overline{Y}) = -\theta_Q([X,\overline{Y}])$$

and θ_Q is a generator of the annihilator of $T'Q + T''Q$ (i.e. a generator of the dual space of $S_{T''Q}$). Hence the above formulas show that the Levi-form of $T''Q$ is given by the matrix $(\mathbf{h}_{j\overline{k}})$ for a suitable choice of a base.

Let M be a CR structure such that its Levi-form is given by the matrix $(\mathbf{h}_{j\overline{k}})$ for a suitable choice of a base. We regard M as a pre-generalized space as follows : a map

$$f : (Q',0) \to (M,p),$$

where Q' is an open neighborhood of 0 in Q, is an attaching map when

1) $(df)_0$ sends T''_0Q onto T''_pM and preserves the Levi-form;

2) the bull back by f of the dual space of $S_{T''}$ and the dual space of $S_{T''Q}$, both considered as a subspace of one-forms, have contact of order 1 at 0.

Attaching maps f_1 and f_2 are considered equivalent when their differentials agree at 0 and $(f_1^{-1})^*\theta_Q \equiv (f_2^{-1})^*\theta_Q \pmod{m_p^2}$, where m_p is the maximal ideal in the algebra of the germs at p of C^∞ functions on M. We find by writing down explicitly the transformations in H that our definition of equivalence satisfies the condition (*) making CR structures with non-degenerate Levi-form pre-generalized spaces, provided such attaching maps exist.

We construct an attaching map as follows : by the first step in the construction of CHERN-MOSER's normal defining equation, an open neighborhood M' of any point p in M can be embedded in $\mathbf{C}^n$ (with general element (z,w)) sending p to 0 such that :

1) the induced CR structure by the embedding and the given CR structure on M' have contact of order 2;

2) the image of the embedding is given by

$$\operatorname{Im} w = \frac{1}{2}\langle z, z'\rangle + N, \tag{3}$$

where $N \equiv 0 \pmod{(z,\overline{z},x)^4}$. Thus (z,x), where $x = \mathcal{R}w$ may be considered as a chart of M and define a map $f : (Q',0) \to (M,p)$. f is an attaching map of the structure.

Going back to the general situation, the homogeneous spaces attached to points of M are so far unrelated. The central point of É. CARTAN's idea in constructing the geometry of generalized space [3] is to connect these homogeneous spaces. The consideration leads to the notion of Cartan connection on the principal bundle of frames. Namely, denote by $\mathbf{g}$ (resp. by $\mathbf{h}$) the Lie algebra of left invariant 1-forms on G (resp. on H). Then a Cartan connection on E is a 1-form ω on E valued in $\mathbf{g}$ satisfying the following conditions :

(1) for each frame $\mathbf{f}$ the restriction $\omega_{\mathbf{f}}$ of ω to $T_{\mathbf{f}}E$ is an isomorphism of vector spaces;

(2) if $R_h : E \to E$ for h in H is the right operation of h,

$$R_h^*\omega = \mathrm{Ad}(h^{-1})\omega,$$

where $\mathrm{Ad}(h)$ is the automorphism of the Lie algebra $\mathbf{g}$ induced by the automorphism $g \to hgh^{-1}$ of G;

(3) for $\xi \in \mathbf{h}$ let $\xi(t)$ be the one-parameter subgroup of H generated by ξ, and for arbitrary frame $\mathbf{f}$ let $X_{\mathbf{f}}$ be the tangent vector at $t = 0$ of the curve $t \to R_{\xi(t)}(\mathbf{f})$. Then $\omega(X_{\mathbf{f}}) = \xi$.

Different choices of Cartan connections lead to different geometry. Thus we define a generalized space of É. CARTAN as a pre-generalized space together with a choice of a Cartan connection on E. However, in the cases we mentioned above there is a unique Cartan connection which is particularly simple. For the geometry of such pre-generalized space we use this particular Cartan connection. In the following we outline how this is done in the three cases mentioned above. A recent work of TANAKA [9] shows that the situation remains true for more general cases of homogeneous spaces.

Looking at the definition of the Cartan connection we see that there are many Cartan connections on a principal bundle. In fact, if $U \times H \to E$ is a local trivialization of E where U is an open subset of M, a Cartan connection ω on U has an expression

$$\omega = \mathrm{Ad}(h^{-1})(\alpha + \omega_H), \tag{4}$$

where ω_H is the left invariant 1-form of H and α is an arbitrary $\mathbf{g}$-valued 1-form on U. Thus the set of Cartan connections is locally parameterized by an arbitrary $\mathbf{g}$-valued 1-form.

Note that there is a canonical isomorphism

$$T_0(G/H) \to \mathbf{g}/\mathbf{h}.$$

Assume that we have a pre-general space. Denote by $\rho_E : E \to M$ the projection. We say that a Cartan connection ω on E is admissible, when for any frame $\mathbf{f}$ at p represented by an attaching map f, $\omega_{\mathbf{f}}$ modulo $\mathbf{h}$ is equal to $\left(df^{-1}\right)_p \cdot d\rho_E$ (in terms of the above canonical isomorphism). Since in our examples a frame at p prescribes $(df)_p$ for any representative attaching map f, this definition makes sense. In terms of a chart expression of ω the admissibility prescribes α modulo $\mathbf{h}$. Thus the set of admissible Cartan connections on U is parametrized by an arbitrary smooth 1-form valued in $\mathbf{h}$.

The curvature of a Cartan connection, K_ω, is defined as

$$K_\omega = d\omega + \frac{1}{2}[\omega, \omega],$$

where $[\omega, \omega]$ is defined in terms of the Lie bracket in $\mathbf{g}$. In terms of the chart expression we find that

$$K_\omega = \mathrm{Ad}(h^{-1})(d\alpha + \frac{1}{2}[\alpha, \alpha]). \tag{5}$$

We now say that a Cartan connection ω is quasi-normal when it is admissible and

$$K_\omega \equiv 0 \pmod{[\mathbf{h}, \mathbf{h}]}.$$

In the case of Riemann geometry, $[\mathbf{h}, \mathbf{h}] = \mathbf{h}$ and there is a unique quasi-normal Cartan connection, i.e. the Levi-Civita connection. To see what happens in the conformal structures (Case I) and in the CR structures with non-degenerate Levi-form (Case II) we have to describe in a little more detail the structure of $\mathbf{g}$. In both of these cases the group G is a group of matrices. There is a natural splitting

$$\mathbf{g} = \mathbf{m} + \mathbf{h} \tag{6}$$

where $\mathbf{m}$ consists of matrices of the form

$$\begin{pmatrix} 0 & \dots & 0 \\ \xi & 0 & \vdots \\ \nu & \eta & 0 \end{pmatrix}.$$

In case I ξ runs over all m dimensional real column vectors, $\nu = 0$, and η is ξ^*, i.e. $\eta_j = \mathbf{g}_{jk}\xi^k$. In case II ξ runs over all $(n-1)$-dimensional complex column vectors z, ν is a real number x, and η is z^*, i.e. $\eta_j = \mathbf{h}_{jk}\overline{z^k}$. Thus they are determine by (ξ, ν). The corresponding element in $\mathbf{m}$ will be denoted by $\lambda(\xi, \nu)$. In case I $\mathbf{h}$ has a system of generators

$$\pi_{\mathbf{R}} = \begin{pmatrix} 1 & 0 & \dots & . & 0 \\ 0 & & & & . \\ . & 0 & & & . \\ \vdots & \vdots & \ddots & & \vdots \\ . & . & \dots & . & 0 \\ 0 & . & \dots & 0 & -1 \end{pmatrix}$$

$$t_{\mathbf{h}} = \begin{pmatrix} 0 & \dots & 0 \\ \vdots & t & \vdots \\ 0 & \dots & 0 \end{pmatrix} \quad \text{with } t \in so(m),$$

where the orthogonality is with respect to the matrix $(\mathbf{g}_{jk})$, and

$$\beta_{\mathbf{h}} = \begin{pmatrix} 0 & \beta^* & 0 \\ \vdots & 0 & \beta \\ 0 & \dots & 0 \end{pmatrix} \quad \text{with } \beta \in \mathbf{R}^m.$$

$[\mathbf{h}, \mathbf{h}]$ is generated by $t_{\mathbf{h}}$ and $\beta_{\mathbf{h}}$.

In case II $\mathbf{h}$ is generated by

$$\pi = \begin{pmatrix} 1 & 0 & \dots & . & 0 \\ 0 & & & & . \\ . & 0 & \dots & . & . \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ . & . & \dots & . & 0 \\ 0 & . & \dots & 0 & -1 \end{pmatrix}, \qquad \mu = \begin{pmatrix} 1 & 0 & \dots & & 0 \\ 0 & & & & \vdots \\ \vdots & & -\frac{2i}{n-1} I & & \vdots \\ & & & & 0 \\ 0 & & \dots & 0 & i \end{pmatrix},$$

where I denotes the identity $(n-1) \times (n-1)$ matrix,

$$u_{\mathbf{h}} = \begin{pmatrix} 0 & \dots & 0 \\ \vdots & u & \vdots \\ 0 & \dots & 0 \end{pmatrix} \quad \text{with } u \in su(n-1),$$

where unitarity is with respect to the matrix $(\mathbf{h}_{j\bar{k}})$,

$$v_{\mathbf{h}} = \begin{pmatrix} 0 & -iv^* & 0 \\ \vdots & 0 & v \\ 0 & \dots & 0 \end{pmatrix} \quad \text{with } v \in \mathbf{C}^n,$$

and

$$\psi = \begin{pmatrix} 0 & \dots & 1 \\ \vdots & 0 & \vdots \\ 0 & \dots & 0 \end{pmatrix}.$$

$[\mathbf{h}, \mathbf{h}]$ is generated by $u_{\mathbf{h}}$, $v_{\mathbf{h}}$ and ψ.

A local trivialization of the frame bundle E is given by assigning a frame $\mathbf{f}_p$ at p in U induced by an attaching map f_p depending smoothly on p. Then the chart is given by

$$(p, h) = R_h \mathbf{f}_p.$$

In case I for the standard chart $y = (y^1, \dots, y^m)$ of Q_R we set

$$(\omega_0^s)_p = (f_p^{-1})^*(dy^s)_0.$$

$\omega_0^1, \dots, \omega_0^m$ form a base of 1-forms on U. Denote by ω_0 the column vector of 1-forms whose s-th component is ω_0^s. Then ω is admissible if and only if

$$\alpha = \lambda(\omega_0, 0) + \alpha_{\mathbf{h}},$$

where $\alpha_{\mathbf{h}}$ is an arbitrary $\mathbf{h}$-valued 1-form on U. Similarly, in Case II we have the standard chart (z, x) of Q and set

$$(\omega_0^j)_p = (f_p^{-1}) * (dz^j)_0,$$
$$(\theta_0)_p = (f_p^{-1}) * (dx)_0.$$

$\omega_0^1, \dots, \omega_0^{n-1}, \overline{\omega_0^1}, \dots, \overline{\omega_0^{n-1}}, \theta_0$ form a complex base of $\mathbf{C}TU$ and the admissibility of ω is given by

$$\alpha = \lambda(\omega_0, \theta_0) + \alpha_{\mathbf{h}},$$

where $\alpha_{\mathbf{h}}$ is an arbitrary $\mathbf{h}$-valued 1-form on U.

In view of (5) ω is quasi-normal when it is admissible and

$$d\alpha + \frac{1}{2}[\alpha, \alpha] \equiv 0 \pmod{[\mathbf{h}, \mathbf{h}]}.$$

Hence the equation is a linear equation in the unknown $\alpha_{\mathbf{h}}$. In fact, the choice of $[\mathbf{h}, \mathbf{h}]$ in the definition of the quasi-normality is intended to have this effect. This is a differential equation in $\alpha_{\mathbf{h}}$. However, in our cases the space A of the unkown $\alpha_{\mathbf{h}}$ has a decomposition $A = A^1 + \cdots + A^l$ satisfying the following condition :

Let $\alpha_{\mathbf{h}} = \alpha^1 + \cdots + \alpha^l$ with $\alpha^\nu \in A^\nu$. Then the equation is a sum of equations $F^1, \dots, F^l$ where F^ν is a linear algebraic equation in $\alpha^1, \dots, \alpha^{\nu-1}$, their derivatives, and in α^ν. Moreover, for any solution $(\alpha^1, \dots, \alpha^{\nu-1})$ of $F^1 \cup \cdots \cup F^{\nu-1}$ the equation F^ν has a solution in α^ν.

Thus starting from α^1 we can write down all the solutions without solving a differential equation. To solve the equation is actually not quite trivial. There are obstructions for the existence of solutions. However these obstructions vanish. In case I we write

$$\alpha_{\mathbf{h}} = \alpha^\pi \pi_{\mathbf{R}} + (\alpha^{so})_{\mathbf{h}} + (\alpha^{\mathbf{R}^m})_{\mathbf{h}},$$

where α^{so} (resp. $\alpha^{\mathbf{R}^m}$) is a $so(m)$-valued (resp. $\mathbf{R}^m$-valued) 1-form on U. Write further

$$(\alpha^{\mathbf{R}^m})^s = Y_t^s \omega_0^t.$$

Then for arbitrary choice of α^π and of symmetric matrix valued function (S_{st}) on U, there is a unique quasi-normal Cartan connection such that

$$\mathbf{g}_{us}(Y_t^s - \mathbf{g}^{sr}S_{rt})$$

is anti-symmetric in (u,t). In case II we write $\alpha_{\mathbf{h}}$ similarly using the above mentioned generators of $\mathbf{h}$. Then for arbitrary α^π and α^ψ both modulo $\omega_0^1, \ldots, \omega_0^{n-1}, \overline{\omega_0^1}, \ldots, \overline{\omega_0^{n-1}}, \omega_0^{n-1}$, there is a unique quasi-normal Cartan connection. Thus in this case the set of quasi-normal Cartan connection depends locally on two real-valued arbitrary functions.

When ω is a quasi-normal Cartan connection, the Bianchi identity : $dK_\omega + [\omega, K_\omega] = 0$ imposes conditions on its curvature. When we consider the condition modulo $[\mathbf{h},\mathbf{h}]$, we obtain conditions which do not involve the derivatives of K_ω. The conditions appear as symmetric conditions on the coefficients in K_ω when expressed in terms of a base of $\mathbf{C}T^*M$. The total information thus obtained in the case II is the neatly expressed condition of CHERN-MOSER [4]. To write them down, note that K_ω takes value in $[\mathbf{h},\mathbf{h}]$ and hence can be written as

$$K_\omega = (K^{su})_{\mathbf{h}} + (K^{\mathbf{C}^{n-1}})_{\mathbf{h}} + K^\psi\psi,$$

where K^{su} (resp. $K^{\mathbf{C}^{n-1}}, K^\psi$) is a $su(n-1)$-valued (resp. $\mathbf{C}^{n-1}$-valued, real valued) 2-form on U. Set

$$\lambda(\Omega,\Theta) = \text{the projection of } \omega \text{ to } \mathbf{m},$$

(*cf.* (6)).Then we can write

$$\begin{aligned}
(K^{su})_k^j &= S^j_{kl\bar{m}}\Omega^l \wedge \overline{\Omega^m} + \Theta \wedge (V^j_{k\bar{l}}\overline{\Omega^l} + \overline{V^{\bar{j}}_{\bar{k}l}}\Omega^l),\\
(K^{\mathbf{C}^{n-1}})^j &= V^j_{k\bar{l}}\Omega^k \wedge \overline{\Omega^l} + \Theta \wedge (P^j_k\Omega^k + Q^j_{\bar{k}}\overline{\Omega^k}),\\
K^\psi &= -iP_{j\bar{k}}\Omega^j \wedge \overline{\Omega^k} + \Theta \wedge (R_k\Omega^k + \overline{R_k\Omega^k}),
\end{aligned}$$

where the above coefficients satisfy the following conditions :

For each frame $\mathbf{f}$, there is homogeneous form of type (2.2), say $S(\mathbf{f})$, in a complex variable ς, $\bar{\varsigma}$ ($\varsigma \in \mathbf{C}^{n-1}$) such that

$$S^j_{kl\bar{m}} = \mathbf{h}^{j\bar{i}}\frac{\partial^4}{\partial\overline{\varsigma^i}\partial\overline{\varsigma^m}\partial\varsigma^k\partial\varsigma^l}S(\mathbf{f}).$$

Similarly, $V^i_{k\bar{l}}$ (resp. p^j_k, $Q^i_{\bar{k}}$, R_k) comes from a homogeneous form of type (1.2) (resp. of type (1.1), (0.2), (1.0)).

In case I we say that ω is normal when

$$\left(K^{so}_{rs}\right)^{r}_{t} = 0 \quad \text{for all } s, t,$$

where $K^{so} = K^{so}_{rs}\Omega^r \wedge \Omega^s$ with $\lambda(\Omega, 0)$ the projection of ω to $\mathbf{m}$. In case II we say that ω is normal when

$$\mathbf{h}^{j\bar{k}} P_{j\bar{k}} = 0.$$

Using Bianchi-identity we find that there is a unique normal Cartan connection for arbitrary choice of α^π (resp. $\alpha^\pi \pmod{\omega_0^1, \dots, \omega_0^{n-1}, \overline{\omega_0^1}, \dots, \overline{\omega_0^{n-1}}}$) in Case I (resp. in Case II). As noted by TANAKA all normal Cartan connections are isomorphic. Thus there is a unique normal Cartan connection up to isomorphisms. In the case of the conformal structure it is customary to assign a particular choice of α^π and call it the normal Cartan connection.

For case II we also assign, following CHERN-MOSER, a particular choice of $\alpha^\pi \pmod{(\omega_i, \overline{\omega_i})}$ and call it the normal Cartan connection. Note that so far we did not use to its full extent the information stored in the definition of frames. We only used the first order information of attaching maps. The assignment of α^π is given by using the second order information stored in the definition of frames. Namely, in case II, le f be an attaching map at p. Then by condition 2) in the definition of attaching maps, we can write

$$(7) \qquad (\rho_E)^* (f^{-1}) * \theta_Q \equiv \kappa \Theta \pmod{(z, \bar{z}, x)^2}.$$

Now we define the real part of the upper left side corner component of $(\omega)_{\mathbf{f}}$ to be $-(1/2)(d\kappa)_{\mathbf{f}}$. Actually this assignment has the effect of determining α^π and is consistent with the above construction of normal Cartan connections. Note by (2) and (7) that

$$d\Theta \equiv i\langle \Omega, \Omega \rangle + \Theta \wedge d\kappa \pmod{m_{\mathbf{f}}},$$

where $m_{\mathbf{f}}$ is the maximal ideal in the algebra of germs of smooth functions at $\mathbf{f}$.

If we assign for $\mathbf{f}$ in E over p a triple

$$\left(\left((f^{-1})^*\theta_Q\right)_p,\ \left((f^{-1})^*\omega_Q\right)_p,\ \left((d\kappa)_{\mathbf{f}}\right)\right),$$

where f is an attaching map representing $\mathbf{f}$, we have a chart of E (valued in 1-forms). In the tautology form approach of CHERN-MOSER for the construction of the normal Cartan connection, we use this chart. We see easily that the normal Cartan connection is preserved under isomorphisms of structures in Cases I and II.

For a function N in variables $(z,\overline{z},x)$, we write N in Taylor series in $(z,\overline{z})$ centered at $(0,x)$. We denote by $N_{(a,b)}$ the type (a,b)-part of the series. We also define

$$\operatorname{Tr} N = \mathbf{h}^{j\bar{k}} \frac{\partial^2}{\partial z^j \partial \overline{z^k}} N.$$

Let f be the attaching map obtained by using an embedding given in (3). We consider the case when the equation (3) is in Chern-Moser form. This means in particular that $N_{(0,a)} = N_{(1,a)} = 0$ and

$$\operatorname{Tr} N_{(2,2)} = (\operatorname{Tr})^2 N_{(2,3)} = (\operatorname{Tr})^3 N_{(3,3)} = 0$$

for N given in (3). Let $\mathbf{f}$ be the frame given by f. Then by solving the equation for $\alpha_{\mathbf{k}}$ for the normal Cartan connection, we find an explicit expression of the curvature in terms of N :

$$\begin{aligned}
S(\mathbf{f}) &= -8N_{(2,2)}\,, \quad V(\mathbf{f}) = \frac{4i}{n+1}\operatorname{Tr} N_{(2,3)}\,,\\
P(\mathbf{f}) &= \frac{2}{n(n+1)}\big((\operatorname{Tr})^2 N_{(3,3)} - \operatorname{Tr}(N,N)_{(2,2)}\big)\\
&\qquad + \frac{2}{(n-1)n(n+1)}\big((\operatorname{Tr})^2 (N,N)_{(2,2)}\big)\langle\varsigma,\varsigma\rangle\,,\\
Q(\mathbf{f}) &= \frac{4}{n(n+1)}\big((\operatorname{Tr})^2 N_{(2,4)} - \operatorname{Tr}(N,N)_{(1,3)}\big)\,,\\
R(\mathbf{f}) &= \frac{2}{(n-1)n(n+1)}\big((\operatorname{Tr})^3 N_{(4,3)} - (\operatorname{Tr})^2 (N,N)_{(3,2)}\big)\,,
\end{aligned}$$

where we replace z by ς, set $x=0$, and

$$\begin{aligned}
(N,N) &= \mathbf{h}^{j\bar{k}}\mathbf{h}^{p\bar{q}}(Z_j\overline{Z}_q N)(Z_p\overline{Z}_k N),\\
Z_j &= \frac{\partial}{\partial z^j} + \frac{i}{2} z_j^* \frac{\partial}{\partial x}, \qquad z_j^* = \mathbf{h}_{j\bar{k}}\overline{z^k}.
\end{aligned}$$

We next describe BURNS-DIEDERICH-SHNIDER's description of FEFFERMAN's conformal structure associated with a CR structure with non-degenerate Levi-form. Following W. HUBER we rewrite $\Psi(\varsigma,\overline{\varsigma})$ in the definition of $\widetilde{Q}$ (*cf.* (1)) in a real chart $\xi = (\xi^0,\ldots,\xi^{2n+1})$ where

$$\varsigma^\alpha = \xi^{2\alpha} + i\xi^{2\alpha+1} \qquad (\alpha = 0,\ldots,n).$$

Then we find that

$$\Psi(\varsigma,\overline{\varsigma}) = (\xi',\xi') - 2\xi^0\xi^{2n+1},$$

where $\xi' = (\xi^1, \ldots, \xi^{2n})$ and

$$(\xi', \xi') = 2\xi^1\xi^{2n} + \langle \xi', \xi' \rangle.$$

Now we see by the description of the flat conformal structure that $\widetilde{Q}/\mathbf{R}^*$ has the flat conformal structure induced by $\Psi(d\varsigma, d\bar{\varsigma})$.

We describe the conformal structure in terms of G. Let $\widetilde{H}$ be the inverse image of H under the quotient map $\widetilde{G} \to G = \widetilde{G}/\mathbf{Z}_{n+1}$. Actually, $\widetilde{H}$ is the group of all matrices in $\widetilde{G}$ such that their first column vectors are of the form $(a, 0, \ldots, 0)^{\mathrm{tr}}$, where a may be considered as a representation of $\widetilde{H}$ in $\mathbf{C}^*$. Set

$$\widetilde{H}_1 = \text{the kernel of } a, \quad \widetilde{H}_r = \{h \in \widetilde{H} : a(h) \in \mathbf{R}\}.$$

Then clearly $\widetilde{H}_1$ is the isotropy group in $\widetilde{G}$ of $(1, 0, \ldots, 0)$ in $\widetilde{Q}$ and hence

$$\widetilde{Q} = \widetilde{G}/\widetilde{H}_1, \qquad \widetilde{Q}/R^* = \widetilde{G}/\widetilde{H}_r.$$

Observe now that $G = \widetilde{G}/\mathbf{Z}_{n+1}$, $\widetilde{H}_1 \cap \mathbf{Z}_{n+1} = I$, and the operation of $\mathbf{Z}_{n+1}$ preserves Ψ. Hence, when we denote by H_1 (resp. by H_r) the image of $\widetilde{H}_1$ and $\widetilde{H}_r$ under the canonical map $\widetilde{G} \to G$, the conformal structure on $\widetilde{Q}/\mathbf{R}^*$ induces a conformal structure on

$$G/H_r = (\widetilde{Q}/R^*)/(\mathbf{Z}_{n+/1}/(\widetilde{H}_r \cap \mathbf{Z}_{n+1})).$$

Note that $\widetilde{H}_r \cap \mathbf{Z}_{n+1} = I$ (resp. $\widetilde{H}_r \cap \mathbf{Z}_{n+1} = \mathbf{Z}_2$) when n is even (resp. when n is odd). Hence $\widetilde{Q}/R^*$ is a $(n+1)$-sheets (resp. $((n+1)/2)$-sheets) covering of G/H_r when N is even (resp. when n is odd). Since the metric is obviously invariant under the operation of G, it is expressible in terms of Mauer-Cartan form of G. When we denote by $(\omega_0^0, \Omega^1, \ldots, \Omega^{n+1}, \Theta)^{\mathrm{tr}}$ the first column vector of the Mauer-Cartan form on G (viewed as a matrix valued form in terms of the embedding $\mathbf{g} \subseteq gl(n+1, \mathbf{C})$), the metric turns out to be

$$2(\operatorname{Im}\omega_0^0)\Theta + \mathbf{h}_{j\bar{k}}\Omega^j\overline{\Omega^k}. \tag{8}$$

When the structure is the model homogeneous space G/H the frame bundle E (resp. Mauer-Cartan form) is identified with G (resp. with the normal Cartan connection). Hence the above observation suggests that we may use (8) (where ω_0^0, Ω, Θ are defined in terms of the normal Cartan connection) to define a conformal structure on

$$M_F = E/H_r.$$

Actually we can justify the construction. This is the Burns-Diederich-Shnider conformal structure. We denote by $G_{\mathbf{R}}/H_{\mathbf{R}}$ the model homogeneous space for the conformal structure. $G_{\mathbf{R}}$ is the quotient group of the group $\widetilde{G}_{\mathbf{R}}$ of the automorphism over $\mathbf{R}$ of $\widetilde{Q}$ by its center (which is $\mathbf{Z}_2$ since the dimension of $Q_{\mathbf{R}}$ is even). Since an automorphism over $\mathbf{C}$ is also an automorphism over $\mathbf{R}$ we have the canonical injection

$$\widetilde{G} \to \widetilde{G}_{\mathbf{R}}. \tag{9}$$

We have the frame bundle E_F over M_F of the conformal structure as well as the frame bundle E over M of the CR structure. To relate the two principal bundles denote by $\widetilde{H}_{\mathbf{R}}$ the inverse image of $H_{\mathbf{R}}$ under the canonical map $\widetilde{G}_{\mathbf{R}} \to G_{\mathbf{R}}$. $\widetilde{H}_{\mathbf{R}}$ is the conformal case analogy of $\widetilde{H}$ and, in particular, as in $\widetilde{H}$ we have a representation $a : \widetilde{H}_{\mathbf{R}} \to \mathbf{R}^*$. $\widetilde{H}$ does not map into $\widetilde{H}_{\mathbf{R}}$ under the injection (9). However, $\widetilde{H}_r$ is mapped into $\widetilde{H}_{\mathbf{R}}$, and we find that (9) thus induces a canonical injection

$$H_r \to H_{\mathbf{R}}.$$

The principal bundle E over $M = E/H$ with the structure group H can be also viewed as a principal bundle over $M_F = E/H_r$ with the structure group H_r, which we denote by (M_F, H_r, E).

We claim that there is a canonical embedding

$$(M_F, H_r, E) \to E_F$$

compatible with the injection $H_r \to H_{\mathbf{R}}$. Namely, we first note that any frame $\mathbf{f}_0$ at p_0 of the CR structure is represented by an attaching map f_0 obtained by embedding M locally as a real hypersurface (3), where

$$\begin{aligned} N &\equiv 0 \pmod{(z,\overline{z},x)^4}, \\ \Delta_b N &\equiv 0 \pmod{(z,\overline{z},x)^3}, \\ \Delta_b^2 N &\equiv 0 \pmod{(z,\overline{z},x)^2}, \end{aligned} \tag{10}$$

where $\Delta_b = \mathbf{h}^{j\overline{k}} Z_j \overline{Z_k}$. We may assume also that the CR structure of M and the induced structure by the embedding agree at p_0 up to order 5. We now construct $f_q : (Q', q) \to (M, f_0(q))$ which satisfies the conditions of the attaching maps at q, depending smoothly on q. Moreover we may construct such f_q so that $f_q(q') = f_0(q') + R_q(q - q')$ with $R_q(\eta) \equiv 0 \pmod{(z(q), \overline{z}(q), x(q))^3}$. Then we have a map

$$\begin{aligned} &(Q_{\mathbf{R}})' \ni g \pmod{H_r} \longrightarrow \\ &(\text{the frame at } g(0) \text{ represented by the} \\ &\text{attaching map } f_{g(0)} \circ g) \pmod{H_r} \in M_F \end{aligned}$$

(where g in G). By writing down ω_0^0 explicitly we find that the map is an attaching map of the conformal structure on M_F. This a consequence of the condition (10) on N. We check further that a different choice of f_0 in the equivalence class of f_0 but satisfying the conditions mentioned above gives rise to an equivalent attaching map for M_F.

Now we are in a position to state a theorem of BURNS-DIEDERICH-SHNIDER on the relation between the two Cartan connections. Note first that the canonical injection (9) induces a canonical injection

$$\mathfrak{g} \to \mathfrak{g}_{\mathbf{R}}. \tag{11}$$

Then the theorem asserts that the diagram

$$\begin{array}{ccc} T(E) & \longrightarrow & T(E_{\mathbf{F}}) \\ \downarrow & & \downarrow \\ \mathfrak{g} & \longrightarrow & \mathfrak{g}_{\mathbf{R}} \end{array}$$

is commutative, where the vertical arrows are the normal Cartan connections and the horizontal arrows are the canonical injections. The outline of the proof is as follows : by the local trivialization of E and E_F in terms of the above f_q, the attaching map for the conformal structure induced by f_q, and by the expression (4) of the normal Cartan connection ω of our CR structure, we may reinterpret ω as a Cartan connection ω_F of our conformal structure. Then we check that ω_F is actually the normal Cartan connection for our conformal structure by explicitly writing down the injection (9).

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