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*Astérisque*, tome S131 (1985), p. 311-321
<http://www.numdam.org/item?id=AST_1985__S131__311_0>

Société Mathématique de France
Astérisque, hors série, 1985, p. 311-321

# BOUNDARY VALUE PROBLEMS FOR GROUP INVARIANT DIFFERENTIAL EQUATIONS

BY

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I shall begin this note by recalling several statements about boundary values of solutions of certain linear differential equations, some of them well-known or almost obvious, some not so well-known; common to all of them is the presence of a semisimple Lie group which acts transitively on the underlying manifold and preserves the differential equations. I shall then argue that these statements are special instances of a general phenomenon in representation theory.

A power series $\sum_{n\geq 0} a_n z^n$ is the Taylor series of an analytic function $f(z)$ on the unit disk $\Delta$ if and only if its coefficients satisfy the bound $|a_n| \leq Cr^n$ for every $r > 1$, with $C = C(r)$ depending on $r$. Almost as a matter of definition, the function $f(z)$ has hyperfunction boundary values on the unit circle $S^1$, represented by the Fourier series $\sum_{n\geq 0} a_n e^{in\theta}$. To make this concrete, one should observe that every real analytic function $\varphi(e^{i\theta})$ on $S^1$ extends complex analytically to some open anulus $\rho < |z| < \rho^{-1}$; consequently, the Fourier coefficients $c_n$ of $\varphi$ are bounded by $B(r)r^{-|n|}$, for some $r > 1$ — e.g., any $r$ between $\rho$ and 1. The constants $B(r)$ determine an inductive limit topology on the space of real analytic functions $C^\omega(S^1)$. Dually a formal series $\sum_n c_n e^{in\theta}$ represents a hyperfunction, i.e., a continuous linear functional on $C^\omega(S^1)$, whenever $|c_n| < C(r)r^{|n|}$ for every $r > 1$. In particular, the assignment

$$f(z) = \sum_{n\geq 0} a_n z^n \to \sum_{n\geq 0} a_n e^{in\theta}$$

(*) Supported in part by NSF grant DMS 8317436.

defines a linear isomorphism

$$(1)\qquad O(\Delta) \xrightarrow{\sim} \{\sum_n c_n e^{in\theta} \in C^{-\omega}(S^1) \mid c_n = 0 \text{ for all } n < 0\}\ ;$$

here $O(\Delta)$ denotes the space of holomorphic functions on $\Delta$, and $C^{-\omega}(S^1)$ the space of hyperfunctions on $S^1$. This map becomes a topological isomorphism when $O(\Delta)$ is given the topology of uniform convergence on compacta and $C^{-\omega}(S^1)$ the strong dual topology.

The isomorphism (1) has a direct analogue in the context of Cartan domains. If $D$ is such a domain, with automorphism group $G$, the isotropy subgroup $K$ at the origin acts transitively on the Shilov boundary $S$. The restrictions to $S$ of the polynomial functions on $D$ can be characterized among all $K$-finite (1) functions by the vanishing of certain Fourier coefficients [21]. *I* let $H^{-\omega}(S)$ denote the closure of the polynomials in $C^{-\omega}(S)$, the space of hyperfunctions on $S$, endowed with its natural Fréchet topology; equivalently, $H^{-\omega}(S)$ consists of those hyperfunctions which satisfy the same vanishing conditions on the Fourier coefficients. Then, just as in the case of the unit disk, the restriction of polynomials from $D$ to $S$ extends to a topological isomorphism

$$(2)\qquad O(D) \xrightarrow{\sim} H^{-\omega}(S)$$

between the space of holomorphic functions $O(D)$, topologized by locally uniform convergence, and $H^{-\omega}(S)$, with the topology inherited from $C^{-\omega}(S)$.

Maxwell's equations and the other zero rest mass equations of mathematical physics are invariant under the conformal group of the Minkowski inner product [4]. Consequently they extend to conformally compactified Minkowski space

$$M \cong U(2) \times U(2)/U(2).$$

The complexification

$$M_C \cong U(4)/U(2) \times U(2)$$

of $M$ contains two open $U(2,2)$-orbits $M_+$, $M_-$, both Cartan domains, which have $M$ as common Shilov boundary. According to a theorem of WELLS [23], every hyperfunction solution of the zero rest mass equations on $M$, of a given helicity $n \geq 0$, can be expressed uniquely as a difference $f_+ - f_-$, of boundary values $f_+$, $f_-$ of holomorphic solutions of the complexified equations on $M_+$ and $M_-$, respectively. Equivalently, there exists a topological isomorphism

$$(3)\qquad A_n(M_+) \oplus A_n(M_-) \xrightarrow{\sim} H_n^{-\omega}(M)$$

(1) a function is said to be $K$-finite if it lies in a finite dimensional, $K$-invariant subspace.

between the direct sum of the spaces of "future analytic" solutions on $M_+$ and "past analytic" solutions on $M_-$, and the space $H_n^{-\omega}(M)$ of hyperfunction solutions on $M$. The Penrose transform identifies the left hand side of (3) with the cohomology $H_P^2(CP^3, O(-2n-2))$ of the $(-2n-2)$nd power of the hyperplane bundle over $CP^3$, with support along the unique closed $U(2,2)$-orbit $P \subset CP^3$ [17,18]. Hence

$$H_P^2(CP^3, O(-2n-2)) \cong H_n^{-\omega}(M), \tag{4}$$

as topological vector spaces, which is an alternative statement of Wells theorem.

The isomorphism (1) can be generalized in a different direction. Taking real parts in and then complexifying, one obtains a topological isomorphism between $C^{-\omega}(S^1)$ and the space of complex valued harmonic functions on $\Delta$. With this example as motivation, HELGASON [11] formulated a conjecture, which was later established by six Japanese authors [13]. To explain the conjecture, I consider a non-compact Riemannian symmetric space $X$. The linear differential operators on $X$ which are invariant under the group of isometries $G$ constitute a commutative algebra $D(X)$ [10]. I fix a character $\chi$ of this algebra, and let $C^\infty(X)_\chi$ denote the corresponding space of joint eigenfunctions,

$$C^\infty(X)_\chi = \{f \in C^\infty(X) \mid Df = \chi(D)f, \text{ for } D \in D(X)\}. \tag{5}$$

The minimal boundary $B$ of $X$, which parametrizes the asymptotic directions of geodesics, is a compact homogeneous space for $G$ [15]. According to Helgason's conjecture, there exists a homogeneous line bundle (2) $L_\chi \to B$ and a $G$-invariant "Poisson transform"

$$P_\chi : C^\infty(L_\chi) \to C^\infty(X)_\chi,$$

which extends to a topological isomorphism

$$P_\chi : C^{-\omega}(L_\chi) \xrightarrow{\sim} C^\infty(X)_\chi \tag{6}$$

from the space of sections of $L_\chi$ with hyperfunction coefficients, in its natural topology, to $C^\infty(X)_\chi$, endowed with the $C^\infty$ topology. In the special case of the unit disk and the trivial character $\chi$, (4) reduces to the familiar isomorphism between $C^{-\omega}(S^1)$ and the space of harmonic functions on $\Delta$.

Certain representations of a semisimple Lie group $G$ can be realized geometrically as sheaf cohomology groups $H^p(G/V, O(L_\lambda))$ of homogeneous

(2) i.e., a line bundle to which the action of $G$ lifts.

holomorphic line bundles $L_\lambda$ over a quotient $G/V$ by a compact centralizer of a torus, equipped with a $G$-invariant complex structure [20]. For both technical and esthetic reasons, one would like to know that the $\bar{\partial}$ operator between the spaces of $L_\lambda$-valued $C^\infty$ $(0,p)$-forms,

$$\bar{\partial} : A^p(G/V, L_\lambda) \longrightarrow A^{p+1}(G/V, L_\lambda), \tag{7}$$

has closed range in the $C^\infty$ topology. To my knowledge, there are no useful criteria of a general nature which would imply the closure of the range for this topology. However, in the situation of homogeneous complex manifolds $G/V$ as above, one can establish a topological isomorphism

$$H^p(G/V, O(L_\lambda)) \xrightarrow{\sim} H^{-\omega}(S,U) \tag{8}$$

from the Dolbeault cohomology group $H^p(G/V, O(L_\lambda))$ into an appropriately defined space of hyperfunction sections $H^{-\omega}(S,U)$ of a homogeneous $C^\infty$ vector bundle $U \longrightarrow S$ over a compact quotient of $G$. Since $H^{-\omega}(S,U)$ is a Fréchet space, this implies the closed range property for the $\bar{\partial}$ operator.

The list of examples could be continued : there exist both non-elliptic and elliptic systems of invariant differential equations on homogeneous spaces, whose solutions have hyperfunction boundary values. Roughly speaking, any system with a semisimple symmetry group falls into this category, if it is "sufficiently determined".

To show how the isomorphisms (1-4, 6, 8) fit into a common pattern, I must recall HARISH-CHANDRA's construction of the infinitesimal representation attached to a global representation of a semisimple Lie group $G$. It will be convenient to assume that $G$ has finite center and a finite component group. I fix the choice of a maximal compact subgroup $K \subset G$, and denote the complexified Lie algebras by the lower case, boldfaced letters $\mathbf{g}, \mathbf{k}$. By a "representation" of $G$, I shall mean a continuous representation on a complete, locally convex Hausdorff space, of finite length — i.e., not containing an infinite chain of closed $G$-invariant subspaces — and "admissible", in the sense that any irreducible representation of $K$ occurs in it only finitely often. This latter assumption is automatically satisfied by irreducible unitary representations of $G$ [8,9]; one does not know at present whether it is also satisfied by all irreducible Banach representations.

If $(\pi, V_\pi)$ is a representation of $G$, the space $V$ of all $K$-finite vectors [3] in the representation space $V_\pi$ consists entirely of $C^\infty$ vectors : for every $v \in V$, $g \longrightarrow \pi(g)v$ is a $C^\infty$ map from $G$ into $V_\pi$ [7]. Moreover, $V_\pi$ contains $V$ as a dense subspace. The Lie algebra $\mathbf{g}$ acts on $V$ by differentiation, hence $V$ becomes a module for the universal enveloping algebra $U(\mathbf{g})$. In

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(3) equivalently, the linear span of the finite dimensional, $K$-invariant subspaces.

addition to $\mathbf{g}$, $K$ acts on $V$, but $G$ does not. The original assumptions about $\pi$, in particular admissibility and finite length, have the following algebraic consequences :

(9)
a) $V$ is finitely generated over $U(\mathbf{g})$;
b) as $K$-module, $V$ is a direct sum of irreducibles, each occuring with finite multiplicity;
c) the actions of $\mathbf{g}$ and $K$ are compatible.

By definition, $V$ is the Harish-Chandra module attached to the global representaion $\pi$. Unitary representations are completely determined by their Harish-Chandra modules [9]. On the other hand, the Harish-Chandra module of a non-unitary representation $\pi$ reflects only those properties which do not depend on the choice of a topology. The group $G = SU(1,1)$, for example, operates not only on the space $O(\Delta)$, but also on the related spaces of holomorphic functions with continuous, or $L^p$, or $C^\infty$ boundary values. Each of these carries a natural locally convex topology, which makes the action of $G$ continuous. The resulting representations all have the same Harish-Chandra module : holomorphic functions that transform finitely under the action of the maximal compact subgroup $K = U(1)$, i.e. polynomial functions, lie in the intersection of these spaces.

The passage from global representations to Harish-Chandra modules can be reversed. According to PRICHEPIONOK [19] and CASSELMAN [2], every simultaneous $\mathbf{g}$- and $K$-module $V$ arises as the space of $K$-finite vectors of a representation $(\pi, V_\pi)$, provided only $V$ satisfies the algebraic conditions (9a-c). If $V$ and $(\pi, V_\pi)$ are related in this manner, I call the latter a globalization of the Harish-Chandra module $V$.Globalizations are far from unique, unless $\dim V < \infty$; the example of $SU(1,1)$ acting on $O(\Delta)$ is quite typical of this phenomenon. Certain canonical globalizations do exist, however, and I shall describe these next.

The $K$-finite vectors in the algebraic dual $V^*$ of a Harish-Chandra module $V$ constitute another Harish-Chandra module, the dual module, which I denote by $V'$. If $(\pi, V_\pi)$ is any globalization of $V$, all linear functions $v' \in V'$ extend continuously from $V$ to $V_\pi$. One may therefore identify $V'$ with a subspace of the continuous dual $V'_\pi$ of $V_\pi$. Each pair of vectors $v \in V$, $v' \in V'$ determines a "matrix coefficient"

$$f_{v,v'}(g) = \langle v', \pi(g)v \rangle \qquad (g \in G). \tag{10}$$

These functions are real analytic, since they satisfy certain elliptic differential equations [9], and have Taylor series at the identity which can be calculated solely in terms of the $\mathbf{g}$-action on $V$. In particular, the matrix coefficients

$f_{v,v'}$ are invariants of the Harish-Chandra module $V$, despite their definition in terms of a particular globalization.

Now let $\{v'_1, v'_2, \ldots, v'_n\}$ be a finite set of $U(\mathbf{g})$-generators for $V'$. The assignment

$$(11a) \qquad v \longrightarrow (f_{v,v'_1}, f_{v,v'_2}, \ldots, f_{v,v'_n})$$

describes an inclusion

$$(11b) \qquad V \hookrightarrow C^\infty(G)^n,$$

which is $\mathbf{g}$-and $K$-equivariant with respect to the right actions on $C^\infty(G)$. Via this inclusion, $C^\infty(G)^n$ induces a topology on $V$. The completion of $V$ in the induced topology — alternatively, but less invariantly, the closure of the image in $C^\infty(G)^n$ — is a globalization of $V$. I shall refer to it as the maximal globalization ([4]) and denote it by $V_{\max}$. Any two generating sets for $V'$ are related by a matrix with entries in $U(\mathbf{g})$. Since $U(\mathbf{g})$ acts continuously on $C^\infty(G)$, neither the induced topology on $V$ nor the completion $V_{\max}$ depend on the initial choice of generators. In this sense, $V_{\max}$ is canonically attached to $V$. A variation on this argument shows that

$$(12) \qquad V \longrightarrow V_{\max}$$

is a functor from the category of Harish-Chandra modules to the category of global representations of $G$. If $(\pi, V_\pi)$ is any other globalization, the identity on $V$ extends uniquely to a continuous, $\mathbf{g}$- and $K$-equivariant inclusion from $V_\pi$ into $V_{\max}$, essentially because the definition (10) of the matrix entries $f_{v,v'}$ makes sense also for vectors $v \in V_\pi$. The existence of these inclusions justifies the terminology "maximal globalization".

The continuous dual of $(V')_{\max}$ equipped with the strong dual topology, constitutes another canonical and functorial globalization, the minimal globalization, $V_{\min}$. Just as $V_{\max}$ can be realized as a closed subspace of $C^\infty(G)$, $V_{\min}$ can be constructed as a quotient of $C_o^\infty(G)^N$. It injects continuously and equivariantly into any other globalization. Dualizing again, one gets back to $V_{\max}$ : the topological vector spaces $V_{\max}$, $V_{\min}$ inherit the property of being reflexive from $C^\infty(G)$ and $C_o^\infty(G)$. A different, but equivalent definition of a minimal globalization appears in papers of LITVINOV and ZHELOBENKO; PRICHEPIONOK [19] uses it to prove the existence of globalizations.

Harish-Chandra modules possess globalizations of Banach spaces and even on Hilbert spaces [2], though not, in general, unitary globalizations. To any

([4]) WALLACH [22] attaches a different meaning to the terms "maximal globalization" and "minimal globalization".

Banach globalization $(\pi, V_\pi)$, one can associate certain other globalizations, as follows. The subspace $V_\pi^\omega$ of analytic vectors — i.e., of those $v \in V_\pi$ for which $g \to \pi(g)v$ is a real analytic map — injects naturally into $C^\omega(G, V_\pi)$, the space of $V_\pi$-valued real analytic functions on $G$. All $K$-finite vectors in a Banach representation are analytic [9,16], hence $V \subset V_\pi^\omega$. The topology which $C^\omega(G, V_\pi)$ induces on $V_\pi^\omega$ turns the latter into a globalization of $V$. In the case of a reflexive Banach space, at least, one can introduce a space of hyperfunction vectors $V_\pi^{-\omega}$, namely the strong dual of the space of analytic vectors for the Banach dual $V_\pi'$ of $V_\pi$. This, too, is a globalization, as are the analogously defined spaces of $C^\infty$ and distribution vectors $V_\pi^\infty$, $V_\pi^{-\infty}$.

To illustrate these ideas, I return to the example of $G = SU(1,1)$, acting on $O(\Delta)$. As was remarked already, the corresponding Harish-Chandra module $V$ consists precisely of the polynomial functions. Evaluation at the origin defines a functional $v'$, which spans the dual module $V'$ over $U(\mathrm{g})$ : if a polynomial and all its infinitesimal translates vanish at the origin, then it vanishes identically. Since $K = U(1)$ acts trivially on $v'$, the functions in the image of the map (11), constructed in terms of the generating set $\{v'\} \subset V'$, drop to functions on $K \setminus G \cong \Delta$, and hence this map factors through the inclusion of $V$ into $C^\infty(\Delta)$. On $O(\Delta)$, the $C^\infty$ topology agrees with the topology of locally uniform convergence, which now implies the equality $V_{\max} = O(\Delta)$. The Hardy spaces $H^p(S^1)$, $1 \leq p < \infty$, provide Banach globalizations of $V$. One can check directly that the space of analytic vectors coincides with $H^\omega(S^1)$, the space of real analytic functions on $S^1$ whose negative Fourier coefficients vanish — regardless of $p$. Its strong conjugate-linear dual, the space $H^{-\omega}(S^1)$ of hyperfunctions with vanishing negative Fourier coefficients, is therefore the space of hyperfunction vectors for the Banach globalizations $H^p(S^1)$, $1 < p < \infty$. In view of these remarks, the isomorphism (1) identifies the intrinsically defined maximal globalization $V_{\max}$ with the extrinsically defined spaces of hyperfunction vectors of certain Banach globalizations.

As might be expected, the isomorphism in the preceeding example is merely a special case of a property of Harish-Chandra modules for an arbitrary semisimple Lie group $G$. To make this precise, I consider a Banach representation $(\pi, V_\pi)$ of $G$, which globalizes a particular Harish-Chandra module $V$. Then :

THEOREM. — *The natural inclusion of the minimal globalization* $V_{\min}$ *into the space of analytic vectors* $V_\pi^\omega$ *is an isomorphism of topological vector spaces. Dually, if* $V_\pi$ *is a reflexive Banach space, the space of hyperfunction vectors* $V_\pi^{-\omega}$ *is topologically isomorphic to the maximal globalization* $V_{\max}$.

Before giving and idea of the proof, I shall discuss various consequences; in particular, I want to indicate how (1-4,6,8) can be deduced from this theorem. For the isomorphism (1), I have already done so, and (2) can be treated by the same type of argument. To infer WELLS'theorem (3,4), it suffices to establish (3) on the level of the $K$-finite vectors : the Fréchet spaces $A_n(M^{\pm})$ coincide with the maximal globalizations of their own Harish-Chandra modules, for reasons similar to those in the case of $O(\Delta)$, and $H_n^{-\omega}(M)$ may be viewed as the space of hyperfunction vectors for the representation on the Hilbert space of (weak) $L^2$ solutions. To prove the $K$-finite version of (3) is essentially an algebraic problem; the preceeding theorem — or even its specialization to the situation (2) — then provides the analytic information implicit in (3). The details of this chain of reasoning can be found in the thesis of DUNNE [6], who also extends WELLS'theorem to a more general setting.

As for Helgason's conjecture, the left hand side of (6) is the space of hyperfunction vectors corresponding to the $G$-module of all $L^2$ sections of $L_\chi$, whereas the right hand side, by virtue of its definition (5) as a closed subspace of $C^\infty(G/K)$, is a maximal globalization. Here, too, the theorem above reduces an analytic problem to an algebraic one, namely the proof that the map (6) restricts to an isomorphism of the underlying Harish-Chandra modules. This algebraic analogue was first established by HELGASON [11]; a result of MILIČIĆ [14] makes it possible to simplify the argument considerably. The isomorphism (8), finally, depends on a corollary of the theorem. I shall return to it later.

The solutions of any $G$-invariant system of linear differential equations constitute a topological $G$-module. Under appropriate conditions on the system — i.e., it must be "sufficiently determined" —, the solution space will have finite length and be admissible. The space $V$ of all $K$-finite solutions then becomes a Harish-Chandra module, which can be embedded into an induced module $V_1$ [2]. By definition, $V_1$ consists of $K$-finite functions on a compact homogeneous space of $G$. Its completion in the hyperfunction topology contains any globalization of $V$, hence the full space of solutions. In this sense, all solutions have hyperfunction boundary values, subject only to a mild restriction on the original system of equations.

The theorem has applications beyond those already mentioned. By construction, the maximal globalization of a Harish-Chandra module is isomorphic to the closure of the image of the map (11). When this statement is dualized and combined with the theorem, one obtains the following slightly surprising conclusion, which bears a superficial resemblance to results of DIXMIER-MALLIAVIN [5] and ARNAL [1].

COROLLARY 1. — *Let $(\pi, V_\pi)$ be a Banach representation, $V$ its Harish-Chandra module,*

$$\{v_1, v_2, \ldots, v_n\} \subset V$$

*a set of $U(\mathfrak{g})$-generators. Then*

$$V_\pi^\omega = \{ \sum_{1\le i\le n} \int_G f_i(g)\pi(g)v_i \, dg \mid f_i \in C_\circ^\infty(G)\}.$$

According to a theorem of CASSELMAN and WALLACH, which has been announced in [22], the space of $C^\infty$ vectors for a Banach representation depends only on the underlying Harish-Chandra module, and not on the particular globalization. The analogous assertion about the space of analytic vectors follows from their result, and is also an immediate consequence of the theorem above :

COROLLARY 2. — *Any two Banach globalizations of a Harish-Chandra module have topologically isomorphic spaces of analytic vectors.*

From the definitions, it is not at all obvious whether the minimal and maximal globalizations are topologically exact functors ([5]), but the theorem, together with the preceeding corollary, settles this natural question. Any short exact sequence of Harish-Chandra modules $0 \to V' \to V \to V'' \to 0$ can be lifted to an exact sequence of Banach $G$-modules — for example, one may lift $V$ and complete $V'$, $V''$ in the induced topologies. The process of taking analytic vectors is exact, hence :

COROLLARY 3. — *The functors $V \longrightarrow V_{\max}$, $V \longrightarrow V_{\min}$ are exact in the topological sense.*

I can now sketch a proof of the isomorphism (8). For certain values of the parameter $\mu$, the cohomology of the line bundle $L_\mu$ occurs in only one dimension and is well understood [20]; in these cases, (8) follows directly from the definition of the maximal globalization. Any other homogeneous line bundle $L_\lambda$ can be reached from the $L_\mu$'s by a succession of the operations of tensoring with holomorphically trivial vector bundles, taking sub- and quotient bundles. The corollary ensures that such operations preserve the isomorphism (8), and thus implies the validity of (8) for arbitrary homogeneous line bundles.

To conclude this note, I shall describe the proof of the main theorem in broad outline. Details will appear elsewhere.

A simple, but ingenious argument of WALLACH [22] shows that it suffices to treat the case of a Harish-Chandra module $V$ induced from a — possibly reducible — finite dimensional representation of a minimal parabolic subgroup $P$, and its globalization as a space of vector valued $L^2$ functions

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([5]) The exactness, in the algebraic sense, of the minimal globalization is a formal consequence of its definition, but does not imply topological exactness.

on $G/P$. In this situation, the assertion of the theorem amounts to a particular lower bound on the matrix coefficients, in terms of the eigenvalues of the Casimir operator of $K$, acting on the left. For representations in the spherical principal series of groups of real rank one, the matrix coefficients are closely related to the hypergeometric function. HELGASON [12] used this fact to establish such lower bounds, which then imply his conjecture in the rank one case. The matrix coefficients of representations of a general group $G$ cannot be expressed as elementary functions, but they do satisfy certain ordinary differential equations with regular singular points, which have been studied by CASSELMAN-MILIČIĆ [3]. Upper bounds on the solutions are relatively easy to get, and they imply lower bounds because the determinant of the fundamental matrix can be written down explicitly.

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