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## **Poisson structures and Lie algebras**

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## POISSON STRUCTURES AND LIE ALGEBRAS

BY

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### 0. Introduction

The topic which I would like to survey in this talk is related not only to Élie CARTAN's well known interests in Lie groups, normal forms, and integral invariants, but also to a subject in which CARTAN's work is much less known to mathematicians — the stability of equilibrium fluid flows [CA1] [CA2].

Poisson structures were first introduced by Sophus LIE [L] in connection with his study of Lie groups and their realization by canonical transformations. Their importance was rediscovered in the 1960's and 1970's in connection with linear representation theory and as a setting for generalizations of hamiltonian mechanics. My interest in Poisson structures first arose because, in infinite dimensions, they provide hamiltonian formulations of equations of motion in physical variables for conservative fluid and plasma dynamics. Subsequently, I found many interesting questions in pure mathematics related to these structures, and it is these which I want to emphasize here. For further details on the applications of Poisson structures in mechanics, see [MA] or [MA-W2] and the papers cited therein.

From the beginning, my work on Poisson structures has involved collaboration with Jerry MARSDEN. In addition to him, I wish to thank many others for conversations and correspondence about this subject. Among them are

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## 1. Definition and first properties

A *Poisson structure* on a manifold  $P$  is a Lie algebra operation  $\{\cdot, \cdot\}$  on the space  $C^\infty(P)$  which satisfies the derivation law  $\{FG, H\} = F\{G, H\} + \{F, H\}G$ .<sup>(1)</sup> The operation  $\{\cdot, \cdot\}$  determines a contravariant antisymmetric tensor  $\pi(\cdot, \cdot)$  such that  $\{F, G\} = \pi(dF, dG)$ . A Poisson structure may also be defined by such a tensor; the Jacobi identity is then a first-order quadratic partial differential equation which must be satisfied by  $\pi$ .

For each  $H \in C^\infty(P)$ , the derivation  $\{\cdot, H\}$  corresponds to a vector field  $\xi_H$  which we call the *hamiltonian vector field* with *hamiltonian function*  $H$ . The derivative  $\dot{F}$  of any  $F \in C^\infty(P)$  along the trajectories of  $\xi_H$  is then given by  $\dot{F} = \{F, H\}$ , a formula which is known to physicists as “Hamilton’s equations in Poisson bracket form.” (When quantized, the formula becomes Heisenberg’s equation of motion  $\dot{\mathbf{F}} = \frac{i}{\hbar}[\mathbf{F}, \mathbf{H}]$ , where  $\mathbf{F}$  and  $\mathbf{H}$  are now self-adjoint operators,  $\hbar$  is Planck’s constant, and  $[\cdot, \cdot]$  is the commutator bracket.)

The Jacobi identity implies that  $\xi_H$  is an infinitesimal automorphism of the Poisson structure and also that the map  $H \rightarrow \xi_H$  is a homomorphism of Lie algebras from  $C^\infty(P)$  to the vector fields on  $P$  (with the sign of the bracket suitably chosen). Thus the one parameter (pseudo) group  $\exp(t\xi_H)$  consists of mappings  $f : P \rightarrow P$  for which  $f^* : C^\infty(P) \rightarrow C^\infty(P)$  is a Lie algebra homomorphism. Such a mapping between Poisson manifolds is called, in general, a *Poisson mapping*.

A homomorphism  $\mathfrak{p}$  from a Lie algebra  $\mathfrak{g}$  to  $C^\infty(P)$  is called a *Poisson action* of  $\mathfrak{g}$  on  $P$ . The mapping  $J : P \rightarrow \mathfrak{g}^*$  associated with  $\mathfrak{p}$  by the definition.  $J(x)(v) = \mathfrak{p}(v)(x)$  is called a *momentum mapping*. We will see shortly that  $\mathfrak{g}^*$  carries a natural Poisson structure for which momentum mappings are Poisson mappings.

Associated with the tensor  $\pi$  is a bundle map  $\tilde{\pi} : T^*P \rightarrow TP$  in terms of which we can write each  $\xi_H$  as  $\tilde{\pi} \circ dH$ . If  $\tilde{\pi}$  is an isomorphism, we call the Poisson structure nondegenerate or *symplectic*. The inverse  $\tilde{\pi}^{-1} : TP \rightarrow T^*P$

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<sup>(1)</sup> More generally, one may define a Poisson algebra as a vector space with both a commutative associative algebra structure and a Lie algebra structure made compatible by the derivation law. A somewhat different algebraic generalization of Poisson structures has been used by Gelfand and Dorfman [G-DO2], who take the de Rham complex  $\Omega(P)$  rather than the multiplication of  $C^\infty(P)$  to “represent” the manifold  $P$ .

then gives a 2-form  $\omega$  which turns out to be closed; conversely, if  $\omega$  is any symplectic structure in the usual sense (i.e. a closed nondegenerate 2-form), it is connected in this way with a Poisson structure.

In terms of local coordinates  $(x_1, \dots, x_m)$ , the components of  $\pi$  are just the brackets  $\pi_{ij} = \{x_i, x_j\}$ , so a Poisson structure may be specified by giving the matrix  $\pi_{ij}$  of functions of  $x$  (subject to the differential equations of the Jacobi identity). If the structure is symplectic, DARBOUX's theorem tells us that local coordinates  $q_1, \dots, q_n, p_1, \dots, p_n$  can be found so that their brackets are constant :  $\{q_i, p_j\} = \delta_{ij}$ ,  $\{q_i, q_j\} = \{p_i, p_j\} = 0$ . Such coordinates are called *canonical*. More generally, it was shown by LIE [L] that coordinates can be found to make the  $\pi_{ij}$ 's constant in the neighbourhood of any *regular* point where  $\pi_{ij}$  has locally constant rank. In this case, coordinates  $(q_1, \dots, q_n, p_1, \dots, p_n, c_1, \dots, c_l)$  can be found for which  $(q_1, \dots, q_n, p_1, \dots, p_n)$  have the canonical brackets, and the  $c_i$ 's Poisson-commute with everything.

The most important singular (i.e. not regular) Poisson structures are associated with Lie algebras. If  $\mathfrak{g}$  is any Lie algebra, its dual  $\mathfrak{g}^*$  carries the *Lie-Poisson structure*

$$\{F, G\}(\mu) = \langle \mu, [\frac{\delta F}{\delta \mu}, \frac{\delta G}{\delta \mu}] \rangle.$$

Here,  $\delta F/\delta \mu$  and  $\delta G/\delta \mu$  are the differentials of  $F$  and  $G$  considered as maps into  $\mathfrak{g}$  rather than  $\mathfrak{g}^{**}$ , and  $\langle \ , \ \rangle$  is the pairing of  $\mathfrak{g}^*$  with  $\mathfrak{g}$ . (In infinite dimensions, there are often delicate problems of functional analysis which must be settled if the Lie-Poisson structure is to be more than a "formal" construction.) If  $X_1, \dots, X_n$  form a basis for  $\mathfrak{g}$  and  $x_1, \dots, x_n$  are the corresponding coordinate functions on  $\mathfrak{g}^*$ , the basic bracket relations are

$$(*) \quad \{x_i, x_j\} = \sum_k c_{ijk} x_k,$$

where the  $c_{ijk}$ 's are the structure constants of  $\mathfrak{g}$ . Conversely, any Poisson structure of the form (\*) arises in this way from a Lie algebra (the algebra of linear functions). Such structures were introduced and studied by LIE [L].

A splitting theorem proved in [W1] shows that, near any point  $x$  of a (finite-dimensional) Poisson manifold  $P$  there are local coordinates

$$(q_1, \dots, q_n, p_1, \dots, p_n, y_1, \dots, y_l)$$

such that

$$\begin{aligned} \{q_i, p_j\} &= \delta_{ij}, \quad \{q_i, q_j\} = \{p_i, p_j\} = 0, \quad \{q_i, y_j\} = \{p_i, y_j\} = 0, \\ \{y_i, y_j\} &\text{ is a function } \Phi_{ij} \text{ of the } y_i \text{'s alone, and } \Phi_{ij}(x) = 0. \end{aligned}$$

Thus, every Poisson manifold is locally the product of a symplectic manifold and one which resembles to a certain extent a neighbourhood of zero in some  $\mathfrak{g}^*$  with its Lie-Poisson structure (see § 3). The Poisson structure  $\{y_i, y_j\} = \Phi_{ij}(y)$  is determined up to isomorphism by the structure on  $P$  near  $x$ ; we call it the *transverse* Poisson structure at  $x$ .

## 2. Symplectic leaves and Casimir functions

The image of  $\tilde{\pi} : T^*P \rightarrow TP$  is integrable in the sense that through every  $x \in P$  there is a unique maximal connected submanifold  $\mathcal{S}(x)$  such that, for each  $y \in \mathcal{S}(x)$ ,  $T_y\mathcal{S}(x) = \tilde{\pi}(T_y^*P)$ . The submanifold  $\mathcal{S}(x)$  has a symplectic structure for which the inclusion in  $P$  is a Poisson mapping, and the relation  $\mathcal{S} = \{(x, y) \mid y \in \mathcal{S}(x)\}$  is an equivalence relation. We call the manifolds  $\mathcal{S}(x)$  the *symplectic leaves* of  $P$ ; they form a foliation on the open dense set of regular points, but elsewhere the dimension of the symplectic leaves will vary. For the Lie-Poisson structure on  $\mathfrak{g}$ , the symplectic leaves are just the orbits of the coadjoint representation of a connected Lie group whose Lie algebra is  $\mathfrak{g}$ ; the symplectic structure on each leaf is the one defined by KIRILLOV, KOSTANT and SOURIAU.

The kernel of  $\tilde{\pi}$  is a closed subset of  $T^*P$ . The functions whose differentials lie in this kernel are just those which are constant on each symplectic leaf; they form the center of  $C^\infty(P)$ . We call these functions Casimir functions because, in the case  $P = \mathfrak{g}^*$ , they are closely related to the Casimir operators in the universal enveloping algebra  $\mathcal{S}(\mathfrak{g})$ . (The Poisson algebra of polynomials on  $\mathfrak{g}^*$  is a “simplification” of the algebra  $\mathcal{S}(\mathfrak{g})$ ; see [BE].)

The Casimir functions define a second equivalence relation

$$\mathcal{C} = \{(x, y) \mid C(x) = C(y) \text{ for all Casimir functions } C\}$$

on  $P$  which is no finer than  $\mathcal{S}$ , and is generally strictly coarser. A useful fact for stability theory (see § 7) is that the differentials of local Casimir functions at a point  $x \in P$  span the kernel of  $\tilde{\pi}$  at  $x$  if and only if  $x$  is regular. To separate the singular symplectic leaves, one may sometimes find “subcasimir” functions; i.e. Casimirs defined on a Poisson submanifold of  $P$ . For example, “color” is such a subcasimir for spinless elements in the dual of the Poincaré Lie algebra (see [SO]).

### 3. Linearization

What kind of local normal form can be expected for Poisson structures of variable rank? By the splitting theorem at the end of § 1, we can confine our attention to structures on  $\mathbf{R}^m$  near a point where  $\pi = 0$ . Writing  $\pi_{ij}(x) = \sum_k c_{ijk}x_k + 0(x^2)$ , we may verify that the equations  $\{x_i, x_j\}_0 = \sum c_{ijk}x_k$  define a Lie-Poisson structure which we call the *linear approximation* to  $\pi$  at 0. The associated  $\mathfrak{g}$  will be called the tangent Lie algebra; when we apply this construction to a transverse Poisson structure on a general Poisson manifold, we call  $\mathfrak{g}$  the transverse Lie algebra. (It may be naturally identified with the conormal space to the symplectic leaf.)

It is natural to ask when the term  $0(x^2)$  above can be eliminated by a change to new coordinates  $y_i = x_i + 0(x^2)$ ; i.e. when a Poisson structure is locally isomorphic to its linear approximation. Such a structure will be called *linearizable*.

The linearization question has different answers when posed in the categories of formal,  $C^\infty$ , and analytic mappings. Very simple examples (e.g.  $\{x_1, x_2\} = x_1^2 + x_2^2$ ) show that some structures are not even formally linearizable, but it is not hard to prove (see [W1]) that a structure is formally linearizable when the tangent Lie algebra is semisimple. Recently, J. CONN [CO1] has proven the linearizability of analytic Poisson structures with semisimple tangent Lie algebra. On the other hand, an example is given in [W1] of a non linearizable  $C^\infty$  structure whose tangent Lie algebra is  $sl(2; \mathbf{R})$ . A reasonable conjecture is thus that a  $C^\infty$  structure is  $C^\infty$  linearizable if its tangent Lie algebra is of compact type. P. DAZORD [DA] has proven this for the case of  $so(3)$ , and J. CONN [CO2] has just obtained a proof for the general case. The situation in general for semisimple algebras of noncompact type is still not well understood.

### 4. Transverse structures to coadjoint orbits

At the time of writing [W1], I thought that non linearizable structures might be hard to find "in nature". I "proved" (THEOREM 3.1 of [W1]) that the transverse structure to each symplectic leaf in a Lie-Poisson manifold is linearizable. In fact, as was pointed out to me by A. GIVENTAL [GI] and P. MOLINO [MO], the proof is incorrect; GIVENTAL even provided counterexamples which I shall describe here.

The transverse Lie algebra at a point  $\mu \in \mathfrak{g}^*$  is the coadjoint isotropy algebra  $\mathfrak{g}_\mu$ . Molino noted that my *proof* of the linearizability of the transverse structure was valid only when  $\mathfrak{g} = \mathfrak{g}_\mu \oplus \mathfrak{m}$  with  $[\mathfrak{g}_\mu, \mathfrak{m}] \subseteq \mathfrak{m}$ , i.e. when the coadjoint orbit  $G \cdot \mu$  is a reductive homogeneous space. There are still many

cases, though, where the transverse structure is linearizable without this assumption.

As the simplest of a class of examples involving “subregular nilpotent elements”, GIVENTAL considers the neighbourhood of

$$\mu = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

in  $sl(3; \mathbf{R})$ . (If  $\mathfrak{g}$  is semisimple, it is convenient to identify  $\mathfrak{g}$  with  $\mathfrak{g}^*$  by the Killing form, so that  $\mathfrak{g}$  itself becomes a Lie-Poisson manifold in which the symplectic leaves are the orbits under the adjoint representation.)

The adjoint orbit of  $\mu$  is 4-dimensional; the transverse Lie algebra is the centralizer  $\mathfrak{g}_\mu$  of  $\mu$ , which consists of the matrices

$$\begin{pmatrix} \alpha & \beta & \gamma \\ 0 & -2\alpha & \delta \\ 0 & 0 & \alpha \end{pmatrix}.$$

Thinking of  $(\alpha, \beta, \gamma, \delta)$  as a basis for  $\mathfrak{g}_\mu^*$ , we may take the dual basis  $(a, b, c, d)$  as coordinates on  $\mathfrak{g}_\mu^*$ ; the bracket relations for the Lie-Poisson structure are  $\{a, c\} = \{b, c\} = \{d, c\} = 0$ ,  $\{a, b\} = 3b$ ,  $\{a, d\} = -3d$ ,  $\{b, d\} = c$ , and the Casimirs are generated by  $c$  and  $bd + \frac{1}{3}ac$ . Thus, in the linearized transverse structure, the  $\mathcal{C}$  equivalence class of 0 (common zeros of the Casimirs) is given by the equations  $c = 0$  and  $bd = 0$  and so consists of two planes intersecting along a line. (Each point of the line is a symplectic leaf; the two planes must break into four leaves.)

To find the  $\mathcal{C}$  equivalence class of  $\mu$  in the true transverse structure, one may intersect the class of  $\mu$  in  $sl(3; \mathbf{R})$  with any submanifold which cuts the orbit  $G \cdot \mu$  transversely in a point, for instance the affine space of matrices

$$\begin{pmatrix} p & 0 & 1 \\ q & -2p & 0 \\ r & s & p \end{pmatrix}.$$

The Casimirs for  $sl(3; \mathbf{R})$  (i.e. ad-invariant polynomials) are generated by the traces of the second and third exterior powers. In the coordinates  $(p, q, r, s)$  on our manifold, these Casimirs have the form  $-3p^2 - r$  and  $-2p^3 + qs + 2pr$ ; their common zero set is given by the equations  $r = -3p^2$  and  $qs - 8p^3 = 0$ , which determine a surface with an isolated singularity (of type  $A_2$  in ARNOL'D'S classification [A2]) at the origin. The difference between the transverse structure and its linear approximation is now apparent.

GIVENTAL's example raises several interesting questions. The first one concerns a theorem of DUFLO and VERGNE [DU-V] which may be stated in the following form : if the transverse structure at  $\mu \in \mathfrak{g}^*$  is trivial, then  $\mathfrak{g}_\mu$  is abelian. The converse of this theorem would be a consequence of my "THEOREM" 3.1. Using results from BOURBAKI [BO], MEDINA [ME] has proven this converse for the case where  $\mathfrak{g}$  is semisimple. On the other hand, DUFLO [DU] has given a simple counterexample for the general case. Namely, one defines on  $\mathbf{R}^4$  a structure by the rules  $\{x, y\} = z$ ,  $\{t, x\} = x$ ,  $\{t, y\} = y$ ,  $\{t, z\} = 2z$ . The regular orbits are the open sets  $z > 0$  and  $z < 0$ , but the points with  $z = 0$  and  $x^2 + y^2 > 0$  have abelian isotropy. The transverse structure is given by the rule  $\{\alpha, \beta\} = -2\beta^2$ .

The nature of the Casimirs for the transverse structure at  $\mu \in \mathfrak{g}^*$  is related to the algebraic geometry of the coadjoint orbits. As a generalization of the converse of the Duflo-Vergne theorem in the semisimple case, KOSTANT [KO] has asked whether the differentials at  $\mu$  for the Casimirs in the transverse structure always span the center of  $\mathfrak{g}_\mu$  (in which they are necessarily contained). KOSTANT has proven the conjecture whenever the closure of the complexified orbit  $G_{\mathbf{C}} \cdot \mu \subseteq \mathfrak{g}_{\mathbf{C}}$  is a normal variety. This includes the case of  $sl(n; \mathbf{R})$  for all  $n$ ; in the special of GIVENTAL's example, it is easy to verify the conjecture directly.

Leaving the special class of Lie-Poisson structures, I would like to close this section with the question of deciding when a Poisson structure is determined by its jet of some finite order at a point. We have seen that, in the semisimple case, 1-jets are "determining" in various senses. Perhaps the order of determination in general may be related to the structure as an algebraic variety of the  $\mathcal{C}$ -equivalence class of the point in question.

## 5. Reduction and dual pairs

Non-symplectic Poisson structures often arise as the phase spaces of mechanical systems when one starts with a variational, or symplectic, description and then ignores certain variables connected with internal ("gauge") or external symmetries. The process of passing to a quotient in symplectic or Poisson geometry is called *reduction*.

A foliation  $\mathcal{R}$  on a symplectic manifold  $P$  will be called a *Poisson foliation* if the functions constant on leaves form a Lie subalgebra of  $C^\infty(P)$ . If  $P/\mathcal{R}$  is a manifold, it inherits a Poisson structure such that the projection  $P \rightarrow P/\mathcal{R}$  is a Poisson mapping; we call  $P/\mathcal{R}$  the *reduced* Poisson manifold.

For example, if a group  $G$  acts as automorphisms of  $P$ , with a single orbit type, then the orbit manifold  $P/G$  becomes a Poisson manifold in a natural way, since the  $G$ -invariant functions form a Lie subalgebra of  $C^\infty(P)$ . If  $H$  is

a  $G$ -invariant function on  $P$ , it induces a hamiltonian  $H_G$  on  $P/G$ , and the trajectories of  $\xi_H$  project to the trajectories of  $\xi_H$ . This important fact makes it possible to “reduce” the study of  $G$ -invariant hamiltonian systems on  $P$  to the study of hamiltonian systems on  $P/G$ . (Actually, recent experience shows that some problems are simplified by “unreducing” them from  $P/G$  up to  $P$  [K-K-S] [F-W].)

When the cotangent bundle  $T^*G$  with its usual symplectic structure is reduced by the action of  $G$  by lifts of left [right] translations,  $T^*G/G$  is isomorphic to  $\mathfrak{g}^*$  with the usual Lie-Poisson structure [or its negative]. Hamiltonian systems on  $\mathfrak{g}^*$ , identified with  $G$ -invariant systems on  $T^*G/G$ , are called *Euler equations*, since the Euler equations for a free rigid body, an incompressible fluid, and many other physically interesting systems are of this form. (See [A3] or [MA-W2].)

Other Poisson structures on  $\mathfrak{g}^*$  may be obtained if we change the symplectic structure on  $T^*G$  by adding the pullback of a closed left [or right]  $G$ -invariant 2-form  $\theta$  on  $G$ . The structures coming from  $\theta_1$  and  $\theta_2$  turn out to be isomorphic by a translation of  $\mathfrak{g}^*$  if and only if  $\theta_2 = \theta_1 + d\Phi$  for some invariant 1-form  $\Phi$ .

We have seen that the Lie-Poisson structure on  $\mathfrak{g}^*$  entered the study of  $G$  actions on Poisson manifolds through the momentum mappings  $J : P \rightarrow \mathfrak{g}^*$ . Namely, an action of  $G$  on  $P$  is called a Poisson action if the corresponding Lie algebra homomorphism from  $\mathfrak{g}$  to vector fields on  $P$  is provided with a lift to a Poisson action from  $\mathfrak{g}$  to  $C^\infty(P)$ . Now if the action of  $G$  on a symplectic manifold  $P$  has a single orbit type,  $J$  has constant rank, and the symplectic leaves in  $P/G$  turn out to be isomorphic to the *reduced symplectic manifolds*  $J^{-1}(\mu)/G_\mu$  for  $\mu \in J(P) \subseteq \mathfrak{g}^*$ .

This relation between  $P \rightarrow P/G$  and  $P \xrightarrow{J} \mathfrak{g}^*$  is an example of a general structure called a *dual pair*, following HOWE [HO], who introduced the term for a similar idea in a closely related context. If

$$P_1 \xleftarrow{\pi_1} P \xrightarrow{\pi_2} P_2$$

is a diagram of constant rank Poisson mappings, with  $P$  symplectic, we call it a *dual pair* if either of the following equivalent conditions is satisfied :

- (i)  $\pi_1^*C^\infty(P_1)$  and  $\pi_2^*C^\infty(P_2)$  are mutual centralizers in  $C^\infty(P)$ ;
- (ii) at each  $x \in P$ , the tangent subspaces  $\ker T_x\pi_1$  and  $\ker T_x\pi_2$  are symplectic annihilators of one another in  $T_xP$ .

In this situation, the intersection  $\pi_1^*C^\infty(P_1) \cap \pi_2^*C^\infty(P_2)$  is the common center of the two algebras, and so the Poisson submanifolds  $\pi_1(P) \subset P_1$  and  $\pi_2(P) \subset P_2$  have naturally isomorphic algebras of Casimirs. In fact, it is proven in [W1] that, for each  $x \in P$ , the transverse structures to  $\pi_1(x)$  and

$\pi_2(x)$  in  $\pi_1(P) \subset P_1$  and  $\pi_2(P) \subset P_2$  are isomorphic. A basic example of all this is the dual pair

$$\mathfrak{g}_+^* \xleftarrow{\pi_1} T^*G \xrightarrow{\pi_2} \mathfrak{g}_-^* ,$$

where  $\pi_1$  and  $\pi_2$  are given by right and left translation, and  $\mathfrak{g}_+^*$  and  $\mathfrak{g}_-^*$  carry the standard Lie-Poisson structure and its negative. (Applications of this idea to the role of semidirect products in mechanics are given in [MA-R-W].)

The reduction process can be attempted in cases where the Poisson action of  $G$  on  $P$  has variable orbit types. Since the momentum map  $J : P \rightarrow \mathfrak{g}^*$  does not have constant rank and  $J^{-1}(\mu)/G_\mu$  is not a manifold, we reduce algebraically instead, dividing  $C^\infty(P)$  by the ideal  $J_\mu$  generated by the components of  $J_\mu$  and then taking the  $G$ -invariant elements in  $C^\infty(P)/J_\mu$ . (See [GO][SN-W].) The resulting object is a Poisson algebra which may even contain nilpotent elements. A natural problem in this context is to decide whether the reduced algebra is always symplectic in the sense that its only Casimirs are the constants. (To eliminate trivial counterexamples, one should assume that the action of  $G$  is proper.)

Reduction has been an important tool in developing hamiltonian formulations of equations of motion in plasma and fluid dynamics. Typically, one starts with a lagrangian description of a system in terms of particle positions, vector potentials, and their conjugate momenta. This description is symplectic. Reducing by a group of symmetries, including particle relabelling and gauge transformations, one obtains a formulation in terms of "physical" variables (eulerian velocity, electromagnetic field, etc). See [MA-R-W] [MA-W1] [MA-W2]. It remains an interesting problem to understand to what extent this program is meaningful when the potentials carry non-trivial physical information (Bohm-Aharonov effect, non-abelian gauge theories).

## 6. Integrable systems

One of the most exciting mathematical developments of the 1970's and 1980's has been the discovery that many interesting dynamical systems are completely integrable, and that their integrability is linked to such diverse areas of mathematics as algebraic geometry and inverse scattering theory. An important method in the study of these integrable systems in their realization as hamiltonian flows on (symplectic leaves in) Poisson manifolds. We refer to [R] and [T-F] for surveys of the use of Poisson structures in the theory of integrable systems. For this lecture, I wish only to make a few small remarks on the subject.

One basic technique for finding commuting functions on a Lie-Poisson manifold is the translation theorem of MISHCHENKO and FOMENKO, and

related theorems of KOSTANT, SYMES, and RATIU (see [R] for a discussion of all of them). For instance, the Mishchenko-Fomenko theorem [MI-F] states that, if  $F$  and  $G$  are Casimirs on  $\mathfrak{g}^*$  and  $\nu \in \mathfrak{g}^*$ , then the functions  $F_{a,\nu}(\mu) = F(\mu + a\nu)$  and  $G_{b,\nu}(\mu) = G(\mu + b\nu)$  Poisson commute for any real numbers  $a$  and  $b$ . The proof of this fact involves a simple algebraic manipulation, but it seems to me that the *geometry* of this and related involution theorems has still not been made sufficiently transparent.

Another method for establishing integrability involves the use of a pair of symplectic structures. Poisson structures are relevant here because they come into the following definition: two symplectic structures  $\omega_0$  and  $\omega_1$  on a manifold  $P$  are *compatible* if the sum of the corresponding Poisson structures satisfy the Jacobi identity. (Note that the sum of two closed forms is always closed, but the Jacobi identity is a nonlinear condition on contravariant 2-tensors.)

If symplectic structures  $\omega_0$  and  $\omega_1$  are both invariant under the flow of a vector field  $\xi$ , then one can construct many constants of motion for  $\xi$ . Denoting by  $\tilde{\omega}_i : TP \rightarrow T^*P$  the bundle map associated with  $\omega_i$ , we see that the automorphism  $R = \tilde{\omega}_0^{-1} \circ \tilde{\omega}_1$  of  $TM$  is invariant under  $\xi$ , so its eigenvalues (or coefficients of its characteristic polynomial) are functions on  $P$  which are constant along the flow of  $\xi$ . It is not clear what conditions would insure that these functions are in involution.

If  $\omega_0$  and  $\omega_1$  are compatible, one can go further [G-DO1] [M]. Since the operators  $\tilde{\omega}_0^{-1}$ ,  $\tilde{\omega}_1^{-1}$ , and  $\tilde{\omega}_0^{-1} + \tilde{\omega}_1^{-1}$  all give Poisson structures, and the Jacobi identity is quadratic, it follows that  $\tilde{\omega}_0^{-1} + \lambda\tilde{\omega}_1^{-1}$  is a Poisson structure for all real  $\lambda$ . Whenever  $\tilde{\omega}_0^{-1} + \lambda\tilde{\omega}_1^{-1}$  is invertible (e.g. for small  $\lambda$ ),  $(\tilde{\omega}_0^{-1} + \lambda\tilde{\omega}_1^{-1})^{-1}$  corresponds to a *closed* form. Expanding

$$(\tilde{\omega}_0^{-1} + \lambda\tilde{\omega}_1^{-1})^{-1} = \tilde{\omega}_0(I + \lambda R^{-1})^{-1}$$

in powers of  $\lambda$  and using the fact that being a closed form is a linear condition, we find that the operators  $\tilde{\omega}_{-n} = \tilde{\omega}_0 R^{-n}$  all correspond to closed forms. Reversing the roles of  $\omega_0$  and  $\omega_1$ , we find that the  $\tilde{\omega}_n = \tilde{\omega}_0 R^n$  all correspond to closed forms as well, so we have an infinite sequence

$$\dots \omega_{-2}, \omega_{-1}, \omega_0, \omega_1, \omega_2, \dots$$

of symplectic structures, all invariant under the flow of  $\xi$ . An easy computation shows that the hamiltonians  $H_n$  defined (at least locally) by  $\xi \lrcorner \omega_n = dH_n$  all Poisson commute with respect to any  $\omega_k$ .

It has been suggested that there is *always* such a sequence of hamiltonian structures underlying any integrable system (see [G-DI] for one case), but it is not obvious how to formulate this idea precisely. The statement is trivial

in some sense, if one looks in action-angle variables. Presumably, the idea is that  $\omega_0$  and  $\omega_1$  should have a relatively simple form in the “natural” coordinates of most integrable systems.

A small final remark is that the  $-\lambda$ 's for which  $\tilde{\omega}_0^{-1} + \lambda\tilde{\omega}_1^{-1}$  is *not* invertible are just the previously mentioned eigenvalues of  $R$ , but the significance of this fact is not clear to me.

## 7. Stability

We shall refer to any reflexive relation  $\mathcal{D}$  on a set  $P$  as a *dynamical system*. Writing  $\mathcal{D}(x)$  for  $\{y \mid (x, y) \in \mathcal{D}\}$ , we call  $x$  an equilibrium point if  $\mathcal{D}(x) = x$ , and we call  $A \subset P$  an invariant set if  $\mathcal{D}(A) = A$ . If  $P$  carries a topology, the invariant set  $A$  is  *$\mathcal{D}$ -stable* if, for every neighbourhood  $\mathcal{V}$  of  $A$  there is a neighbourhood  $\mathcal{U}$  of  $A$  such that  $\mathcal{D}(\mathcal{U}) \subset \mathcal{V}$ . If  $P$  is a metric space, and we take  $\mathcal{U}$  and  $\mathcal{V}$  to be  $\epsilon$  and  $\delta$  neighbourhoods of  $A$ , then we call  $A$  *Lipschitz  $\mathcal{D}$ -stable* if  $\epsilon$  can be chosen so that the ratio  $\delta/\epsilon$  remains bounded as  $\delta \rightarrow 0$ .

Let  $P$  be a Poisson manifold and  $H : P \rightarrow \mathbf{R}$  a hamiltonian. If we define the dynamical system  $\mathcal{D}_H$  to be

$$\{(x, y) \mid y = \exp(t\xi_H)(x) \text{ for some } t\},$$

then  $\mathcal{D}_H$ -stability is just the usual notion of Liapunov stability for  $\xi_H$ . (For dissipative systems, it is appropriate to use the dynamical system in which  $t$  is restricted to positive values.) Notice that the symplectic leaf relation  $S$  introduced in § 2 is just the union of the  $\mathcal{D}_H$  for all functions  $H$ ; in particular, for any  $H$  we have  $\mathcal{D}_H \subset S \subset \mathcal{C}$ , so  $\mathcal{C}$ -stability or  $S$ -stability implies  $\mathcal{D}_H$ -stability.

In a Lie-Poisson manifold  $\mathfrak{g}^*$ , the origin is  $S$ -stable (or  $\mathcal{C}$ -stable) if and only if  $\mathfrak{g}^*$  admits a positive-definite quadratic Casimir function. This applies, for example, to the phase space  $so(3)^*$  of the Euler equations for a free rigid body, where it implies the stability of the origin for any hamiltonian system.

In general, to determine the stability of a hamiltonian system on a Poisson manifold it is usually necessary to use the given hamiltonian in addition to the Casimir functions. In the symplectic case, one has the sufficient condition (Lagrange, Dirichlet) that a fixed point  $x$  of  $\mathcal{D}_H$  is (Lipschitz) stable if the hessian  $D_x^2 H$  is positive or negative definite. In general, a fixed point of  $\mathcal{D}_H$  is not even a critical point, but one has the following criterion due to ARNOL'D (see [A3]) :

**STABILITY CRITERION.** — *A point  $x$  in the Poisson manifold  $P$  is an equilibrium of  $\mathcal{D}_H$  if and only if the restriction of  $H$  to  $S(x)$  has a critical point at  $x$ . If  $x$  is a regular point of  $P$ , a sufficient condition for (Lipschitz)*

$D_H$ -stability of  $x$  is that the hessian at  $x$  of  $H | S(x)$  be positive or negative definite.

An instructive proof of the stability criterion is based on the fact that if a regular point  $x$  is a critical point of  $H | S(x)$ , then there is a Casimir function  $C$  such that  $H - C$  has a critical point at  $x$ . The whole argument breaks down when  $x$  is a singular point (see the remarks in the next to last paragraph of § 4); there are also examples where one has stability but not Lipschitz stability (see [W2]). For applications to continuum mechanics, definiteness of a hessian at  $x$  alone is usually not sufficient, for functional-analytic reasons. Instead one needs a convexity condition, as introduced by ARNOL'D [A1]. The systematic application of these ideas to a variety of fluid and plasma systems is carried out in [H-M-R-W]. We hope eventually to apply these methods to the problem which interested Cartan — the stability of self-gravitating rotating fluids [CA1] [CA2].

NOTE ADDED IN PROOF (17 February, 1985). — By generalizing DIRAC's constraint procedure, T. COURANT and R. MONTGOMERY have obtained a simple algorithm for computing transverse structures to symplectic leaves (see § 1). Using this algorithm, Y.-G. OH has shown that that transverse structure at  $\mu$  in the Lie-Poisson space  $\mathfrak{g}^*$  (see § 4) is at most quadratic if  $\mathfrak{g}$  is the direct sum of  $\mathfrak{g}_\mu$  and another subalgebra of  $\mathfrak{g}$ . There is some hope that this result may lead to a better geometric understanding of the involution theorems mentioned in § 6.

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