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1. Friedlander and Lagarias [2] considered the problem of estimating the number $\psi(X,Z,Y)$ of integers in the interval $(X-Y,X]$ having no prime factor $>Z$. Especially they defined $f(\alpha)$ as the infimum of the values of θ for which for all $\alpha'>\alpha$ one had $\psi(X,X^{\alpha'},X^{\theta})>0$ for sufficiently large X and they proved for $0<\alpha\leqq\frac{1}{2}$

(1) $$f(\alpha)\leqq 1-2(1-2^{-[\frac{1}{\alpha}]})\alpha .$$

The proof was based on a simple combinatorial conscruction, a special case of it was discovered independently by Balog and Sárközy [1]. Our aim is to develop an alternative method. Instead of $\psi(X,Z,Y)$ itself we investigate a weighted sum by analytic arguments originated from Heath-Brown and Iwaniec [3]. We have

THEOREM: For $0<\alpha\leqq 1$

(2) $$f(\alpha)\leqq\frac{1}{2} .$$

The theorem is a simple consequence of our main lemma

LEMMA: Let $k\geqq 1$ be an integer, $\frac{1}{8k}\geqq\delta>0$, $X>X_0$ be real numbers, $|a_m|\leqq 1$ be arbitrary complex coefficients and we define $M=X^{1/2-1/4k}$, $Y=X^{1/2+1/8k+\delta}$, finally

(3) $$d_n=\sum_{\substack{m_1m_2|n\\ M<m_i\leqq 2M}} a_{m_1}a_{m_2} .$$

For any $A>0$ we have

(4) $$\sum_{X-Y<n\leq X} d_n = Y\Big(\sum_{M<m\leq 2M} \frac{a_m}{m}\Big)^2 + O\Big(\frac{Y}{\log^A X}\Big) .$$

2. For a given $\varepsilon>0$ and $0<\alpha\leq 1$ we can choose a $k>\max(\frac{1}{2\alpha},\frac{1}{8\varepsilon})$ and $a_m=\begin{cases} 1 & \text{if } m \text{ has no prime factor } >X^{\alpha} ,\\ 0 & \text{otherwise} .\end{cases}$ Our lemma guarantees that the interval $(X-X^{1/2+\varepsilon},X]$ contains numbers n in the form $n=\ell m_1 m_2$ where m_i has no prime factor $>X^{\alpha}$ and $\ell<\frac{X}{M^2}=X^{1/2k}<X^{\alpha}$. This gives the theorem.

3. At first we reduce the proof of (4) to estimating a certain integral. Our basic tool is the Perron integral formula (Lemma 3.12 of [6]). We define

$$M(s)=\sum_{M<m\leq 2M} a_m m^{-s} , \quad L(s)=\sum_{L_1<\ell\leq L_2} \ell^{-s} ,$$

where $L_1=\frac{1}{5}X^{\frac{1}{2k}}$ and $L_2=2X^{\frac{1}{2k}}$. By Perron formula we can express the left hand side of (4) as an integral taken on a vertical line of the complex plane. We have

(5) $$\sum_{X-Y<n\leq X} d_n = \frac{1}{2\pi i}\int_{\frac{1}{2}-iT}^{\frac{1}{2}+iT} L(s)M^2(s)\frac{X^s-(X-Y)^s}{s}\, ds + O\Big(\frac{X\log^3 X}{T}\Big) .$$

We can provide a fairly small error term by choosing $T=\frac{X^{1+\delta/2}}{Y}= =X^{1/2-1/8k-\delta/2}$. The major part of the integral is that around $\frac{1}{2}+i0$. Choosing $T_o=X^{1/4k}$ and using the facts that

$$L(s) = \frac{L_2^{1-s}-L_1^{1-s}}{1-s} + O(L_2^{-1/2}) \quad \text{for } s=\tfrac{1}{2}+it ,\ |t|\leq T_o ,$$

$$\frac{X^s-(X-Y)^s}{s} = YX^{s-1} + O(|s-1|Y^2X^{-\frac{3}{2}}) \quad \text{for } s=\tfrac{1}{2}+it ,$$

$$|M(\tfrac{1}{2}+it)|^2 \ll M$$

we get again from the Perron formula that

$$\frac{1}{2\pi i}\int_{\frac{1}{2}-iT_o}^{\frac{1}{2}+iT_o} L(s)M^2(s)\frac{X^s-(X-Y)^s}{s}\, ds =$$

$$\frac{1}{2\pi i}\int_{\frac{1}{2}-iT_o}^{\frac{1}{2}+iT_o}\frac{L_2^{1-s}-L_1^{1-s}}{1-s}M^2(s)YX^{s-1}ds+O\Big((L_2^{-\frac{1}{2}}YX^{-\frac{1}{2}}+Y^2X^{-\frac{3}{2}}L_1^{\frac{1}{2}})T_oM\Big)=$$

(6) $$\frac{1}{2\pi i}\int_{\frac{1}{2}-iT_o}^{\frac{1}{2}+iT_o}\Big(\sum_{M<m\leqq 2M}\frac{a_m/m}{m^{s-1}}\Big)^2\frac{(X/L_1)^{s-1}-(X/L_2)^{s-1}}{s-1}ds+O\Big(\frac{Y}{\log^A X}\Big)=$$

$$=Y\Big(\sum_{M<m\leqq 2M}\frac{a_m}{m}\Big)^2+O\Big(\frac{Y}{\log^A X}\Big)$$

for all $A>0$. Combining (5) and (6) we arrive at

(7) $$\sum_{X-Y<n\leqq X}d_n=Y\Big(\sum_{M<m\leqq 2M}\frac{a_m}{m}\Big)^2+O\Big(\frac{Y}{\log^A X}+\frac{Y}{X^{1/2}}R\Big)$$

for all $A>0$, where

$$R=\int_{T_o}^{T}|L(\tfrac{1}{2}+it)M^2(\tfrac{1}{2}+it)|dt\ .$$

From (7) it is enough to prove that

(8) $$R\ll\frac{X^{1/2}}{\log^A X}\quad\text{for all}\quad A>0\ .$$

4. Next we prove (8). To bound R we use three important principles, the mean-value theorem of Dirichlet polynomials (Theorem 6.1 of [5]) which states

$$\int_{-T}^{T}|M(\tfrac{1}{2}+it)|^2dt\ll(M+T)\sum\frac{|a_m|^2}{m}\ ,$$

the Halász-Montgomery-Huxley large-value theorem [4] which states

$$|\{|t|\leqq T\ :\ V<|M(\tfrac{1}{2}+it)|\leqq 2V\}|\ll\Big(\frac{M}{V^2}+\frac{MT}{V^6}\Big)\log^2 X\ ,$$

and the fact that for $T_o<t\leqq T$ and for all $A>0$ we have

(9) $$L(\tfrac{1}{2}+it)\ll\frac{L_2^{1/2}}{\log^A X}\ ,$$

which follows for example from van der Corput's bound for trigonometrical sums (Theorem 5.13 of [6]). Note that the mean-value theorem when is applied to $L(\frac{1}{2}+it)^k$ gives

$$\int_{-T}^{T} |L(\tfrac{1}{2}+it)|^{2k} dt \ll (L_1^k+T) \sum \frac{1}{n} \Big(\sum_{\substack{n=n_1\dots n_k \\ L_1<n_i\le L_2}} 1 \Big)^2 \ll X^{1/2} \log^k X .$$

We divide the interval $[T_o,T]$ into parts and denote the integral over the set Ω_o on which $|M(\frac{1}{2}+it)| \le M^{1/4}$ by R_o and over the set $\Omega(V)$ on which $V<|M(\frac{1}{2}+it)| \le 2V$ by $R(V)$. As $|M(\frac{1}{2}+it)| \le M^{1/2}$ trivially, it is possible to cover the interval $[T_o,T]$ by using $\ll \log X$ sets $\Omega(V)$ together with Ω_o. From Hölder's inequality and the mean-value theorem

$$R_o \le \Big(\int_{\Omega_o} |L(\tfrac{1}{2}+it)|^{2k} dt \Big)^{\frac{1}{2k}} \Big(\int_{\Omega} |M(\tfrac{1}{2}+it)|^{\frac{4k}{2k-1}} dt \Big)^{1-\frac{1}{2k}} \le$$

$$\le \Big(\int_{T_o}^{T} |L|^{2k} \Big)^{\frac{1}{2k}} \Big(\int_{T_o}^{T} |M|^2 \Big)^{1-\frac{1}{2k}} \max_{t\in\Omega_o} |M(\tfrac{1}{2}+it)|^{\frac{1}{k}} \ll$$

$$\ll (X^{1/2}+T)^{\frac{1}{2k}} (M+T)^{1-\frac{1}{2k}} M^{1/4k} \log^{1/2} X \ll \tag{10}$$

$$\ll X^{\frac{1}{4k}+(\frac{1}{2}-\frac{1}{8k}-\frac{\delta}{2})(1-\frac{1}{2k})+(\frac{1}{2}-\frac{1}{4k})\frac{1}{4k}} \log^{1/2} X \ll$$

$$\ll X^{\frac{1}{2}-\delta(\frac{1}{2}-\frac{1}{4k})} \log^{1/2} X \ll \frac{X^{1/2}}{\log^A X}$$

for all $A>0$. From the large-value theorem for $M^{1/4}<V\le T^{1/4}$

$$\int_{\Omega(V)} 1 \, dt \ll \frac{MT}{V^6} \log^2 X$$

and

$$R(V) \le \Big(\int_{\Omega(V)} |L(\tfrac{1}{2}+it)|^{2k} dt \Big)^{\frac{1}{2k}} \Big(\int_{\Omega(V)} |M(\tfrac{1}{2}+it)|^{\frac{4k}{2k-1}} dt \Big)^{1-\frac{1}{2k}} \ll$$

$$\ll \Big(\int_{T_o}^{T} |L|^{2k} \Big)^{\frac{1}{2k}} \Big(V^{\frac{4k}{2k-1}} \int_{\Omega(V)} 1 \, dt \Big)^{1-\frac{1}{2k}} \ll$$

$$\ll \Big(X^{1/2}+T \Big)^{\frac{1}{2k}} \Big(MTV^{\frac{4k}{2k-1}-6} \Big)^{1-\frac{1}{2k}} \log^3 X \ll \tag{11}$$

$$\ll X^{\frac{1}{4k}} T^{1-\frac{1}{2k}} V^{\frac{3}{k}-4} M^{1-\frac{1}{2k}} \log^3 X \ll$$

$$\ll X^{\frac{1}{4k}} T^{1-\frac{1}{2k}} M^{\frac{1}{4k}} \log^3 X \ll \frac{X^{1/2}}{\log^A X}$$

for all $A>0$. Finally if $T^{1/4}<V$ then from the large-value theorem

$$\int_{\Omega(V)} 1\, dt \ll \frac{M}{V^2}\log^2 X$$

and from (9)

$$R(V) \ll \max_{t\in\Omega(V)} |L(\tfrac{1}{2}+it)|\, V^2 \int_{\Omega(V)} 1\, dt \ll \frac{X^{\frac{1}{4k}} M}{\log^A X} \ll \frac{X^{1/2}}{\log^A X} \quad . \tag{12}$$

Now (8) follows from (10) , (11) and (12). This completes the proof.

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