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# On Heegaard Diagrams of 3-manifolds

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The following text is intended to give a survey of (parts of) the theory of Heegaard decompositions for non-specialists in low dimensional topology. Therefore first the basic concepts are defined and illustrated by simple examples, and only thereafter do we discuss some recent results on Heegaard decompositons of Seifert fibre spaces, obtained in joint work with M. Boileau and M. Rost.

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In the following the symbol □ either denotes the end of a (sketch of a) proof or indicates that the assertion is given without further arguments.

## 1. Heegaard decompositions of 3-manifolds

In books on algebraic topology manifolds which are used to illustrate invariants are mostly surfaces, $S^n, P^n$ and perhaps a few further ones, e.g., lens spaces. One obstruction to the use of 3-manifolds as examples is that there is no general simple way to describe them, in contrast to the perceptual pictures of surfaces as a 2-sphere with handles. Of course, there have been models in 3-topology and, since all 3-manifolds are triangulable and the Hauptvermutung is true for dimension 3 (i.e. any two triangulations of the same space have isomorphic subdivisions), these models represent all 3$-$manifolds. But they lack intuition and do not help to find invariants of the space considered. Nevertheless, I will describe three of them.

Assume the 3−manifold $M^3$ is given by a complex, for instance, a simplicial one. By iteratively dropping 2−cells separating different 3−cells we obtain a cellular complex which contains only one 3−cell. Hence, $M^3$ may be obtained from a 3−ball $D^3$ by identifying points of the boundary. More precisely: $\partial D^3 = S^2$ carries some 2−complex and the 2−cells are pairwise identified by homeomorphisms which can be extended to the boundary of the 2−cells. The result is the manifold $M^3$. The other way round, let us start with a 3−ball $D^3$, a complex on $\partial D^3$ and identification rules for 2−cells of $\partial D^3$. Now we ask whether the space obtained is a 3−manifold. There is a simple answer with a simple proof (see [ST, pg. 208]):

**1.1 Theorem.** *Let $X$ be obtained from $D^3$ by pairwise identification of 2−cells of a complex of $\partial D^3 = S^2$. Then $X$ is a closed 3−manifold if and only if the Euler characteristic is 0.* □

As an application let us see why lens spaces are lens spaces. The favorite definition of a lens space is the following: Represent $S^3$ by $\{(z_1, z_2) \mid z_i \in \mathbb{C}, z_1\overline{z_1} + z_2\overline{z_2} = 1\}$ and define $g : S^3 \to S^3$ , $(z_1, z_2) \mapsto (\zeta z_1, \zeta^q z_2)$ where $\zeta$ is a $p$−th root of unity, $p > 1$ and $\gcd(p, q) = 1$. Then the group $\mathbf{G} = \{1, g, ..., g^{p-1}\} \cong \mathbb{Z}_p$ acts freely on $S^3$, and $L(p, q) = S^3/\mathbf{G}$ is a 3−manifold: *a lens space.* The projection $\pi : S^3 \to L(p, q)$ is the universal cover of $L(p, q)$, i.e. $\pi_1 L(p, q) \cong \mathbb{Z}_p$. The lens space L(2,1) is equal to $P^2$, and this is the only lens space with fundamental group $\mathbb{Z}_2$. Clearly: $L(p, q) \cong L(p, q + kp), k \in \mathbb{Z}$, and by changing orientation or the interchanging coordinates we obtain the if-statement of the following theorem. That these conditions are also necessary is due to K. Reidemeister [Rei 1].

**1.2 Theorem.** $L(p, q) \cong L(p', q')$ *if and only if* $p' = p$ and $q' \equiv \pm q \bmod p$ or $q'q \equiv \pm 1 \bmod p$. □

But why are the spaces $L(p, q)$ called lens spaces? Take a lens $L$, subdivide the boundary in $p$ equal sections and identify a triangle of the upper side with the triangle on the lower side lying q steps further in positive direction. Let us call the obtained complex already now $L(p, q)$. Since $\gcd(p, q) = 1$ the quotient space carries a complex consisting of two 0−cells (one coming from top-bottom, the other from the equator), $(p + 1)$ 1−cells (one from the equator and $p$ from the arcs going from equator to top-bottom), $p$ 2−cells and one 3−cell, that is $\chi(L(p, q)) = 2 - (p + 1) + p - 1 = 0$ ; hence, the quotient is a manifold. It is easy to see that it is the space $L(p, q)$ resulting from the action of $\mathbb{Z}_p$ described above.

Both ways of constructing 3−manifolds, that is by identifying points of the boundary of a 3−cell or as the quotient space of a free action of a group (of isometries), obtained new life during the last decade evoked by the work of Thurston. For details see [Ed], [Sco], [Thu 1,2].

We will now leave this extraordinary vivid area and come to another approach of describing 3−manifolds. Let us again start with a (simplicial) complex $K$ of an orientable closed 3−manifold $M^3$ consisting of $a_i$ $i$−cells $(i = 0, 1, 2, 3)$. Let $\Gamma$ denote the 1−skeleton and $N(\Gamma) = H_g$ a closed regular neighbourhood of $\Gamma$ with $g = a_1 - a_0 + 1$. The space $H_g$ can be obtained as follows: Take a maximal tree $T \subset \Gamma$. The regular neighbourhood of $T$ is a 3−ball which is part of $N(\Gamma)$, and $N(\Gamma)$ is obtained from the 3−ball $N(T)$ by attaching $g$ 1−handles $D^2 \times I$. Since $M^3$ is orientable the handles are glued to $N(T)$ such that $\partial N(\Gamma)$ is an orientable surface of genus $g$.

Moreover, $H_g = N(\Gamma)$ is a *handlebody of genus g*, that is, $H_g$ is homeomorphic to the closed regular neighbourhood of a bouquet of $g$ circles in $\mathbb{R}^3$. The complement of $H_g$ is an open regular neighbourhood of the 1-skeleton of the complex dual to $K$; hence, $\overline{M^3 - H_g} = H'_g$ is also a handlebody of genus $g$. Let us interrupt our discussion for a definition:

**1.3 Definition.** Let $M^3$ be a closed orientable 3-manifold.

(a) A pair $(H_g, H'_g)$ of handlebodies of genus $g$ is called a *Heegaard decomposition of genus $g$ of $M^3$* if $M^3 = H_g \cup H'_g$ and $H_g \cap H'_g = \partial H_g = \partial H'_g$ is a closed orientable surface of genus $g$. If the side is not specified, that is, when we consider the pair $(M^3, \partial H_g)$, we use the expression *Heegaard splitting.*

(b) *Two Heegaard decompositons* $(H_g, H'_g)$ and $(\hat{H}_g, \hat{H}'_g)$ are called *homeomorphic* if there is an (orientation preserving) homeomorphism $\Phi$: $H_g \cup H'_g \to \hat{H}_g \cup \hat{H}'_g$ with $\Phi(H_g) = \hat{H}_g$, $\Phi(H'_g) = \hat{H}'_g$. Similar for Heegaard splittings by looking at homeomorphisms of pairs $(M^3, \partial H_g)$.

(c) The minimal genus among the genera of all Heegaard decompositions of $M^3$ is called the *Heegaard genus* of $M^3$ and is denoted by $h(M^3)$.

Using this notation we can formulate the result from above as follows:

**1.4 Theorem.** *Every closed orientable 3-manifold admits a Heegaard decomposition.* □

Since a manifold carries many different complexes the genus of some Heegaard decomposition is not an invariant of $M^3$; in fact, one can add handles: Let $\gamma \subset H'_g$ be a simple arc and $\Delta \subset H'_g$ a disc such that $\partial\Delta = \gamma \cup (\Delta \cap \partial H_g)$, $\partial\gamma = \gamma \cap \partial H_g$. If $N(\gamma)$ is a closed regular neighbourhood of $\gamma$, then $H_{g+1} = H_g \cup N(\gamma)$ and $\overline{H'_g - N(\gamma)} = H'_{g+1}$ are handlebodies of genus $g+1$.

**1.5 Definition.** The step from the Heegaard decomposition $(H_g, H'_g)$ to $(H_{g+1}, H'_{g+1})$ is called an elementary stabilization and a sequence of such steps a *stabilization.* The inverse procedure is called a *reduction.* A Heegaard decomposition is called *minimal* or *irreducible* if it cannot be reduced, that is, does not result from a stabilization of a Heegaard decomposition of smaller genus.

Using the fact that two triangulations of a 3-manifold have isomorphic subdivisions the following result can be obtained (see [Rei 1], [Sin], [Cr]).

**1.6 Proposition.** *Any two Heegaard decompositions $(H_g, H'_g)$ and $(J_f, J'_f)$ of a 3-manifold $M^3$ lead to homeomorphic Heegaard decompositions $(\tilde{H}_k, \tilde{H}'_k)$ and $(\tilde{J}_k, \tilde{J}'_k)$ by stabilization.* □

**1.7 Remark.** Can one choose in Proposition 1.6 always $k \leq f + g$? This question is not yet decided. A positive answer to it would imply in particular that every Heegaard decomposition of $S^3$ of genus $g \geq 1$ is reducible. This is a known non-trivial result of F. Waldhausen [Wa 1], see Theorem 1.17, and this throws some light on the difficulty of this problem.

**1.8 Examples.** (a) A Heegaard decomposition of genus 0 gives a 3-manifold which is the union of two 3-balls glued together along there boundaries; hence it is the 3-sphere. Obviously, $S^3$ is the only manifold with Heegaard genus 0. Next take a circle $\gamma$ in a plane in $S^3$ and let $H_1$

be a regular neighbourhood of $\gamma$ and $H_1' = \overline{S^3 - H_1}$. Then $(H_1, H_1')$ is the "standard Heegaard decomposition of genus 1 of $S^3$".

(b) Let us again consider the lens space $L(p,q)$. Take the axis $\alpha$ of the lens and a regular neighbourhood $H_1$ of $\alpha$. Then $H_1$ is a solid torus. The closure of the complement of $H_1$ is also a solid torus. Hence, the lens spaces have Heegaard genus 1 and the described Heegaard decompositions are minimal.

(c) If we take the double of a solid torus $H_1$ we obtain a Heegaard decomposition $(H_1, H_1')$ of genus 1 where the boundary of a meridian disc of $H_1$ also bounds a meridian disc of $H_1'$. Therefore the manifold obtained is $S^2 \times S^1$. The sphere $S^3$, the lens spaces and $S^2 \times S^1$ are the only 3–manifolds having a Heegaard decompositions of genus 1.

(d) Consider in $S^3$ the standard solid torus $H_1$ and the standard Heegaard decomposition $(H_1, H_1')$. Fix on $\partial H_1$ a small disc $\Delta$ and define $F = \overline{\partial H_1 - \Delta}$. The regular neighbourhood of $F$ is homeomorphic to $F \times I$ and looks like the rubber of a car tyre. The complement, the air, forms a handlebody of genus 2; the rubber does the same: take two disjoint simple arcs $\alpha, \beta \subset F$ with $\partial\alpha \cup \partial\beta \subset \partial F$; then $\alpha \times I$, $\beta \times I$ are discs and $\overline{F \times I - N(\alpha \times I \cup \beta \times I)}$ - here $N$ indicates regular neighbourhood - is homeomorphic to a 2–disc cross $I$, that is a 3–ball. Hence, $(F \times I, \overline{S^3 - F \times I})$ is a Heegaard decomposition of genus 2 of $S^3$. (This construction has been used by K. Reidemeister [Rei 3].) In the same way we can construct Heegaard decompositons of genus 2 for lens spaces or, in general, a Heegaard decomposition of genus $2g$ when starting the same construction from a Heegaard decomposition of genus $g$.

These are the only simple examples. Next we come to more complicated examples: Heegaard decompositions of Seifert fibre spaces.

**1.9 Seifert manifolds.** Consider a 3–manifold $M^3$ with an action of the group $S^1$ where no point is fixed by all group elements. The orbits decompose $M^3$ into circles, and the orbifold $M^3/S^1$ is a closed surface $F$, called *basis of the Seifert fibration.* However, the projection $\pi : M^3 \to F$ does *not* define a fibre bundle in the usual sense, it is a foliation, but of a very special type. If we exclude a finite number of points $x_1, ..., x_m$ of $F$,

$$\pi\colon\ \pi^{-1}(F - \{x_1, ..., x_m\}) \to (F - \{x_1, ..., x_m\})$$

is a locally trivial fibration. Every point $x_i$ has a disc neighbourhood $D_i^2$ such that $\pi^{-1}(D_i^2)$ is a solid torus $D_i^2 \times S^1$, the core of which is mapped to $x_i$. If we use polar coordinates $(r, \varphi)$ for $D_i^2$ and $\psi$ for $S^1$ then the fibres are

$$\{(r, \varphi + \frac{b_i}{a_i} \cdot \psi, \psi) \mid 0 \le \psi \le 2\pi\}, \quad \text{where } a_i > 1, a_i, b_i \in \mathbb{Z}, \gcd(a_i, b_i) = 1.$$

When $\psi$ runs from 0 to $2\pi$ every fibre with $r \neq 0$ is passed exactly once, but the central fibre with $r = 0$ is passed $a_i$ times. It is called an *exceptional fibre of type* $(a_i, b_i)$ (or $b_i/a_i$). See Fig. 1.1. *For simplicity let us assume that M and F are orientable.* Then the Seifert manifold (= Seifert fibre space) is denoted by $S(g; e_0; b_1/a_1, \ldots, b_m/a_m)$ where $g$ is the genus of $F$, $e_0 \in \mathbb{Q}$ and $e = e_0 + \sum_{i=1}^m b_i/a_i \in \mathbb{Z}$ is the Euler class. Here the number $e$ represents an obstruction to the existence of a section of the fibration. It also appears in the presentation of the fundamental group, see 2.3. The rational number $e_0$ is defined by the above equation. It turns out that the fractions $b_i/a_i$ modulo 1 and $e_0$ are invariants of the manifold. (For details see [Or], [OVZ], [Sei], [Sie].)

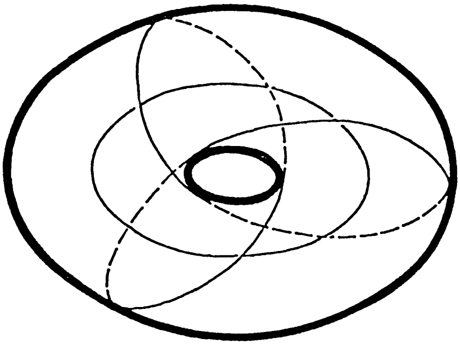


Fig. 1.1

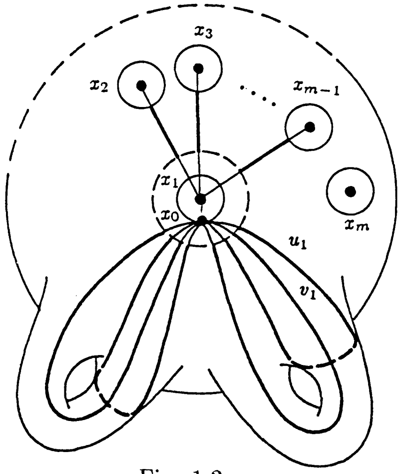



Fig. 1.2

**1.10 Proposition.** *Let $M = S(g; e_0; b_1/a_1, \ldots, b_m/a_m)$ with $1 < a_1 \leq \cdots \leq a_m$ and $\gcd(a_i, b_i) = 1, 1 \leq i \leq m$. If $m \geq 2$, $M$ admits a Heegaard decomposition of genus $2g + m - 1$; if $m \leq 1$, $M$ admits a Heegaard splitting of genus $2g + 1$.*

*Sketch of Proof* [BoiZ]. Choose a point $x_0$ on $\partial D_1$ and a system $(u_1, v_1, \ldots, u_g, v_g)$ of simple closed curves on $F - \{x_1, \ldots, x_m\}$ passing through $x_0$ such that the surface obtained from $F$ by cutting along $u_1, \ldots, v_g$ is a disc; see Fig. 1.2. The restriction of the Seifert fibration to $\overline{F - \bigcup D_i}$ is a trivial $S^1$-fibration. Therefore we may assume that $x_0$ and the curves $u_1, \ldots, v_g$ are in $M$. For $m \geq 2$ we consider the graph $\Gamma$ consisting of the curves $u_1, \ldots, v_g$, the exceptional fibres over $x_1, \ldots, x_{m-1}$ and $m - 1$ segments going from $x_1$ to $x_0$ and $x_i$, $2 \leq i \leq m - 1$, see Fig. 1.2. A regular neighbourhood $N(\Gamma)$ of $\Gamma$ is a handlebody of genus $2g + m - 1$. To see that $\overline{M - N(\Gamma)}$ is also a handlebody, consider the $2g+m-2$ segments indicated by heavysolid lines in Fig. 1.2. The preimage of each of them is an annulus in $M$; hence, a disc in $\overline{M - N(\Gamma)}$. Cutting $\overline{M - N(\Gamma)}$ along these $2g + m - 2$ discs we obtain a solid torus which is a regular neighbourhood of the $m$-th exceptional fibre. A similar argument will do it for $m \leq 1$. □

Clearly, in connection with the notion of Heegaard decompositions there arise questions of the following type:

**1.11 Problems.** (a) *Given a manifold $M^3$. Determine $h(M^3)$.*

(b) *Given two Heegaard decompositions of minimal genus of the same manifold. Are they homeomorphic? Does $M^3$ have only finitely many classes of homeomorphic Heegaard decompositions? Have irreducible Heegaard decompositions of a manifold $M^3$ equal genus?*

(c) *Given a Heegaard decomposition. Is it minimal? How can this question be decided?*

(d) *Get invariants of the manifold from a Heegaard decomposition.*

(e) *Give convenient descriptions of Heegaard decompositions.*

In general, only partial results are known in connection with these problems, and most of these are for Seifert manifolds. In the following we will mainly deal with the questions (a), (b) and (e) for Seifert manifolds and will only mention some results related to the other questions.

Before looking to these problems let us purely listen papers related to them, (more or less ordered according to the date of publication):

(a) [Ha 1], [Bu], [Oc 1, 3], [TO], [BoiZ], [Ko 1, 2], [Jo], [Mo 5, 6].

(b) [Wa 1], [Eng], [Sta], [Bi 2], [BGM], [Mo 2, 3], [Bon 2], [BonO], [HR], [MoW], [MoS], [BRZ 1, 2], [BoiO], [CG 2], [Mor 1,2], [BCZ].

(c) [Whi 1], [Zi 5], [Wa 1, 2], [Sta], [Ha 1], [VKF], [ViK], [BiM], [HOT], [Oc 2], [Mok], [Ka 1-3], [Os], [CG 1], [Mo 7].

(d) [BiC], [Jon], [Cas].

(e) [Poi], [Whi 1], [Rei 3], [Zi 1-5], [Wa 2], [Ka 2], [BRZ], [Mor 1,2].

Heegaard decompositions are often applied in 3-dimensional topology, e.g. [Sta], [Ro], [FL], [BuZ]. For instance, they can be used to show Stiefel's theorem that 3-manifolds are parallelizable, see [Lau 1].

Before going on with Heegaard diagrams we recall the concept of connected sums of 3−manifolds. Let $M^3$ be a closed 3−manifold, $\Sigma^2 \subset M^3$ a (tame) 2−sphere separating $M^3$: $M^3 = X_1 \cup X_2, X_1 \cap X_2 = \partial X_1 = \partial X_2 = \Sigma^2$. Attaching 3−balls $B_1, B_2$ to $X_1, X_2$ gives closed 3−manifolds $M_i^3 = X_i \cup B_i$ where $X_i \cap B_i = \partial B_i = \Sigma^2$ $(i = 1,2)$ and $M^3$ is called the *connected sum of* $M_1^3$ *and* $M_2^3$; notation: $M^3 = M_1^3 \# M_2^3$. The topological type of $M^3$ does not depend on the choices of the balls $B_i$ or the glueing mapping. A 3−manifold $M^3$ is called *prime* if an equation $M^3 = M_1^3 \# M_2^3$ implies that $M_1^3$ or $M_2^3$ is homeomorphic to $S^3$. The manifold $M^3$ is called *irreducible* if every 2−sphere in $M^3$ bounds a 3−ball. Irreducibility implies primeness. However, $S^2 \times S^1$ is prime, but not irreducible; this is the only closed orientable 3−manifold with this property (see [Hem, 3.13]). The following often used result is due to H. Kneser [Kn], see also the stronger version of J. Milnor [Mi] and the isotopy theorem of Laudenbach [Lau 2], [Lau 3, Ch. III, IV]:

**1.12 Theorem.** *Every* 3−*manifold is a connected sum of finitely many prime* 3−*manifolds. The prime factorization is uniquely determined up to a permutation.* □

From Heegaard decompositions $(H_{(1)}, H'_{(1)})$ and $(H_{(2)}, H'_{(2)})$ of the manifolds $M_1^3$ and $M_2^3$ one can easily construct a Heegaard decomposition of $M^3 = M_1^3 \# M_2^3$: choose the balls $B_1$ and $B_2$ in $H'_{(1)}$ and $H'_{(2)}$, respectively, such that they touch the boundaries of the corresponding handlebodies in a disc. Then $H_{(1)} \cup H_{(2)}$ and $\overline{H'_{(1)} - B_1} \cup \overline{H'_{(2)} - B_2}$ are handlebodies and form a Heegaard decomposition of $M^3$. For the Heegaard genera we obtain the following inequality:

$$h(M_1^3 \# M_2^3) \leq h(M_1^3) + h(M_2^3).$$

W. Haken proved the much more difficult result that in fact equality holds:

**1.13 Theorem** [Ha 1]. *If* $M^3 = M_1^3 \# M_2^3$ *then* $h(M^3) = h(M_1^3) + h(M_2^3)$. □

For the proof Haken showed: If $(H, H')$ is a Heegaard decomposition of $M^3 = M_1 \# M_2$ then there exists a curve $\gamma$ with the following properties:

$(*)$ $\gamma$ is a simple closed curve on $\partial H = \partial H'$ which is not contractible on $\partial H$ but bounds disks $D$ and $D'$ in $H$ and $H'$; here $D \cap \partial H = D' \cap \partial H' = \gamma$.

Obviously, the sphere $D \cup D'$ now defines either a decomposition of $M^3$ into a connected sum of two 3-manifolds different from $S^3$ or a reduction of the Heegaard decomposition. Of course, if we consider a Heegaard decomposition $(H, H')$ of an arbitrary manifold the existence of a curve with the property $(*)$ gives some simplification. For instance, assume that for arbitrary $M^3$ the condition $\pi_1(M^3) = 1$ yields the existence of a curve with the property $(*)$ for every Heegaard decomposition $(H, H')$ of $M^3$. By induction we decompose $(H, H')$ into Heegaard decompositions of genus 1 of manifolds which are also simply connected. As we have seen in 1.8 these summands are also 3-spheres and, hence, $M^3$ also. In other words, the above assumption implies the Poincaré conjecture; in fact, by the Theorem 1.17 of Waldhausen the assumption is even equivalent to it. One may formulate this in more algebraic versions to get some algebraic equivalences to the Poincaré conjecture, see [Sta].

Until now we have not given a general method for presenting Heegaard decompositions; we have only described some examples. Of course, a Heegaard decomposition $(H_g, H_g')$ of $M^3$ is determined, up to a homeomorphism, if the identification mapping of $\partial H_g$ and $\partial H_g'$ is known (up to isotopy and conjugation by homeomorphisms). One could try to use the mapping class groups of surfaces, as done by J. Birman and J. Powell [Bi 2, 3], [BiP], but this has not yet intensively been studied. Let us now describe the classical approach:

**1.14 Heegaard diagrams.** The common way is to cut the handlebodies $H_g, H_g'$ along discs into balls: There are $g$ disjoint discs $\Delta_1', \dots, \Delta_g' \subset H_g$ with $\delta_i' = \partial \Delta_i' \subset \partial H_g$ such that $H_g - \bigcup_{i=1}^{g} N(\Delta_i')$ is a 3−ball; similar for $H_g'$ where we use discs $\Delta_i$ with boundaries $\delta_i$. On $F_g = \partial H_g = \partial H_g'$ we now obtain two systems of pairwise disjoint simple closed curves: $(\delta_1', \dots, \delta_g')$ and $(\delta_1, \dots, \delta_g)$. This is called a *Heegaard diagram.* Clearly, a Heegaard diagram determines the Heegaard decomposition up to homeomorphisms, but there are many different Heegaard diagrams with the same decomposition. The elementary stabilization adds curves $\delta_{g+1}'$ and $\delta_{g+1}$ to the given systems where $\delta_{g+1}' \cap \delta_i$ (and $\delta_{g+1} \cap \delta_i'$) is empty for $i \neq g+1$ and one point if $i = g+1$.

Heegaard diagrams were introduced by Heegaard [Hee] to construct examples of 3-manifolds and compute their Betti numbers. Shortly thereafter Poincaré [Poi] used the same concept and gave the description of it which is more or less the modern one. In fact, Heegaard splittings were already considered by Dyck some ten to fifteen years earlier in [Dy] which contains a detailed description of the genus 1 Heegaard diagrams. The classification of 3-manifolds can be transformed into some equivalence of Heegaard diagrams as done by Reidemeister [Rei 1] and Singer [Sin]. In the study of Heegaard diagrams there arise non-trivial equivalences for the same genus in addition to stabilization, and it seems rather hopeless to get to a classification of 3-manifolds this way.

**1.15 An alternative form of Heegaard diagrams.** Consider only the curves $(\delta_1, \ldots, \delta_g)$, but now as curves on the boundary of the handlebody $H_g$ where the equivalence is defined by homeomorphisms of $H_g$. Then $H_g$ and the system $(\delta_1, \ldots, \delta_g)$ determines the manifold $M^3$ and we also call $(H_g; \delta_1, \ldots, \delta_g)$ a Heegaard diagram of $M^3$. The stabilization adds one handle to $H_g$ and a longitude $\delta_{g+1}$ of this handle to $(\delta_1, \ldots, \delta_g)$. The inverse procedure corresponds to a reduction. If this is possible we say that the Heegaard diagram can be *reduced.*

We come back to the general situation in Section 2, but consider next the case of genus 1. In this case we have a solid torus $H_1 = D^2 \times S^1$, a *meridian* $m = \partial D^2 \times 1$ and a *longitude* $\ell = 1 \times S^1$ on the torus $T = \partial H_1 = \partial D^2 \times S^1 = S^1 \times S^1$. (Here we put $D^2 = \{z \in \mathbb{C} \mid |z| \leq 1\}$.) The meridian is uniquely determined up to isotopy and reversing of the orientation. The longitude can be altered by isotopy, reversing of orientation and adding multiples of the meridian. If an orientation of $H_1$ (and so on $T$) has to be respected, the orientations of $m$ and $\ell$ can be altered simultanously only. The pair $(m, \ell)$ forms a basis for $\pi_1(\partial H_1) = H_1(\partial H_1)$.

Let us now in addition consider a positive meridian-longitude pair $(m', \ell')$ of the solid torus $H_1'$. Then (where $\sim$ denotes homologous)

$$m' \sim qm + p\ell \quad , \quad \ell' \sim \alpha m + \beta \ell \quad \text{on} \quad T = \partial H_1 \quad \text{where} \quad q\beta - p\alpha = 1,$$

and the numbers $p, q$ determine how $H_1'$ is glued to $H_1$ and thus the Heegaard decomposition. Let us denote Heegaard diagram and decomposition by $(H_1; (p, q))$. By simple considerations one obtains the following proposition.

**1.16 Proposition.** (a) *The Heegaard decomposition $(H_1; (p, q))$ represents $S^3$ if $p = 1$, $S^2 \times S^1$ if $p = 0$ and $L(p, q)$ for $p \geq 2$.*

(b) *The Heegaard decompositions $(H_1; (p, q))$ and $(H_1; (p_*, q_*))$ are homeomorphic by an orientation preserving homeomorphism if and only if $p = p_*$ and $q \equiv q_* \bmod p$. There is an orientation reversing homeomorphism mapping $(H_1; (p, q))$ to $(H_1; (p_*, q_*))$ if and only if $p = p_*$ and $q \equiv -q_* \bmod p$.*

(c) *The Heegaard diagrams $(H_1; (p, q))$ and $(H_1; (p, \bar{q}))$ with $q\bar{q} \equiv \pm 1 \bmod p$ belong to the same manifold, namely if they correspond to the Heegaard decompositions $(H_1, H_1'), (\bar{H}_1, \bar{H}_1')$ then there is a homeomorphism of $M^3$ onto itself sending $H_1$ to $\bar{H}_1'$ and $H_1'$ to $\bar{H}_1$. The condition $q\bar{q} \equiv \pm 1 \bmod p$ is also necessary for the existence of a homeomorphism changing the solid tori, that is, for the homeomorphy of the Heegaard splittings.* $\square$

Let us now consider the connected sum of two lens spaces $L(p, q)$ and $L(r, s)$. We obtain Heegaard decompositions of genus 2 of the sum by glueing two Heegaard decompositions of genus 1 of the factors together. They have the Heegaard diagrams $(H_2; (p, q), (r, s))$ and $(H_2^*; (p, q), (r, s_*))$ where $s_* s \equiv -1 \bmod r$. Here $(p, q)$ represents a curve on the first handle, $(r, s), (r, s_*)$ on the second. ABoth diagrams define the same 3-manifold $M^3$, a connected sum of two lens spaces. Assume that there is a homeomorphism $h$: $M^3 \to M^3$ mapping $H_2$ to $H_2^*$ or to $\overline{M^3 - H_2^*}$.

Using the methods of [Mi] one sees that there is a homeomorphism (even an isotopy [Lau 2, 3]) mapping one of the 2-spheres to the other, and we may assume that they coincide. Thus $M^3 = X_1 \cup X_2, X_1 \cap X_2 = S^2$ and $H_2 \cap X_i$ , $H_2^* \cap X_i$ , $(\overline{M^3 \setminus H_2}) \cap X_i$ , $(\overline{M^3 \setminus H_2^*}) \cap X_i$ $(i = 1, 2)$ are solid tori. In $X_1$ we obtain twice the Heegaard decomposition $(H_1; (p, q))$. If the Heegaard decomposition of genus 1 of $L(p, q)$ is not symmetric, i.e. $q^2 \not\equiv \pm 1 \bmod p$, it follows that $h$ maps $H_2 \cap X_1$ onto $H_2^* \cap X_1$; hence, $h(H_2) = H_2^*$. This implies that $h(H_2 \cap X_2) = H_2^* \cap X_2$ and that the Heegaard decompositions $(H_1; (r, s))$ and $(H_1; (r, s_*))$ are homeomorphic which is possible only if $s^2 \equiv \pm 1 \bmod r$, see 1.16. Hence we obtain the result of R. Engmann [Eng] that, for $q^2 \not\equiv \pm 1 \bmod p, s^2 \not\equiv \pm 1 \bmod r$, *$L(p, q) \# L(r, s)$ has Heegaard splittings of genus 2 which are not homeomorphic*. For a complete classification of genus 2 Heegaard splittings of connected sums of two lens spaces see [MoS]. This has been the first negative answer to the last question in 1.11(b). Shortly later Birman, González-Acuña and Montesinos constructed prime Seifert manifolds of Heegaard genus 2 with two non-homeomorphic Heegaard decompositions of genus 2, see [BGM] and [Mo 2, 3]. We will describe an example due to Moriah [Mor 1,2] explicitly in Theorem 4.11. For examples of hyperbolic manifolds see [MoW]. Recently Casson and Gordon constructed *manifolds having infinitely many irreducible Heegaard splittings of different genus*, see [CG 2]; this gives, in particular, a negative answer to the last question in 1.11 (b).

There are also known some 3-manifolds which have only one type of irreducible Heegaard decompositions for some genus. The best known example is $S^3$, as shown by Waldhausen [Wa 1]. Only recently Bonahon-Otal [BonO] showed a similar property for lens spaces.

**1.17 Theorem** [Wa 1]. *Let $(H_g, H_g')$ be a Heegaard decomposition of $S^3$ of genus $g \geq 1$. Then there is a pair of meridians $m$ and $m'$ of $H_g$ and $H_g'$, respectively, such that $m \cap m'$ consists of one point where proper intersection takes place. Hence, $(H_g, H_g')$ is obtained from a Heegaard decomposition of genus $g - 1$ by stabilization. Furthermore, any two Heegaard decompositions of $S^3$ of equal genus are homeomorphic and thus every Heegaard decomposition $(H_g, H_g')$ of $S^3$ is symmetric in the following sense: there is a homeomorphism of $S^3$ mapping $H_g$ to $H_g'$ .* □

**1.18 Theorem** [BonO]. *Every Heegaard decomposition of genus $g > 1$ of a lens space is obtained by stabilization from a Heegaard decomposition of genus $g - 1$. Moreover, if $(H_g, H_g')$ and $(\tilde{H}_g, \tilde{H}_g')$ are Heegaard decompositions of $L(p, q)$ and $g > 1$ then there is an orientation preserving homeomorphism $f$: $L(p, q) \to L(p, q)$ such that $f(H_g) = \tilde{H}_g$ and $f(H_g') = \tilde{H}_g'$.* □

As we have seen in 1.16 the last claim does not hold for $g = 1$. In the proof of Waldhausen in [Wa 1] the curves of two systems of meridians for the two handlebodies are ordered, and they are reduced to systems in which a curve of one of the systems is only intersected by curves of the other systems with numbers not larger as its own. This is proved by using arguments on curves on handlebodies. In the proof of [BonO] methods introduced by Schubert [Sch], see also [Bon 2], are used which have not been applied in the theory of Heegaard decompositions before, in particular they use Morse functions. J.-P. Otal [Ot] has used this method also to reprove Waldhausen's theorem for $S^3$.

**1.19 Heegaard decompositions and Morse theory.** Let us now give another interpretation of a Heegaard decomposition $H_g, H'_g$ of the closed 3-manifold $M$. First we take a point and its regular neighbourhood, a ball $B_0$, in the interior of $H_g$ such that $\overline{H_g - B_0}$ consists of $g$ cylinders $A_1 = \alpha_1 \times D^2, \ldots, A_g = \alpha_g \times D^2$ where the $\alpha_i$'s are arcs, the cores of the handles of $H_g$. The ball $B_0$ is called a $0-$*handle* and the cylinders are called $1-$*handles*:

$$H_g = B_0 \cup A_1 \ldots \cup A_g,\ A_i \cap A_j = \emptyset \text{ if } i \neq j,\ A_i \cap B_0 = \partial A_i \cap \partial B_0 \cong S^0 \times D^2.$$

Next let $\Delta_1, \ldots, \Delta_g$ be a system of discs of $H'_g$ and $D^3_1, \ldots, D^3_g$ be their regular neighbourhoods in $H'_g$ such that $\overline{H'_g - \cup_{j=1}^g D^3_j}$ is a ball, in particular, $D^3_j \cong \Delta_j \times I$, $D^3_j \cap H_g = \partial\Delta_j \times I$. Then the space $M_0 = H_g \cup D^3_1 \cup \ldots \cup D^3_g$ is obtained from $H_g$ by attaching $2-$*handles.* It is bounded by a 2-sphere and $M$ is obtained from $M_0$ by adding the 3-*handle* $\overline{H'_g - \cup_{j=1}^g D^3_j}$.

This interpretation of a Heegaard decomposition suggests some generalizations of the concept:

1) There is no need to postulate that the number of 2-handles is as big as the number of 1-handles and we may assume that the manifold $M$ has boundary and that $M = H_g \cup K_h$ where $K_h$ is obtained from the product of a closed orientable surface of genus $g$ and the interval by attaching $h$ 2-handles. We will consider this situation for the special case where $h = g - 1$ in Section 4.

2) This description is very close to the construction of manifolds using singularities of Morse functions: we choose such a function which has one singularity of index 0 on the lowest level, $g$ singularities of index 1 followed when going up by $g$ singularities of index 2 and, finally, 1 of index 3 — or in the more general concept as described in 1), one singularity of index 0, g singularities of index 1 and $h$ singularities of index 2. As already mentioned, the approach of this type has effectively been used by Bonahon-Otal [Bon 2], [BonO].

3) The description of Heegaard decompositions using handles can also been used to construct Heegaard splittings in higher dimensions.

## 2. On the rank of $\pi_1 M^3$ and the Heegaard genus of $M^3$

There have been few attemps to calculate invariants of 3-manifolds using Heegaard decompositions or diagrams. The following approach has often been used. Obviously, a Heegaard diagram $(H_g; \delta_1, \ldots, \delta_g)$ gives rise to a presentation of $\pi_1 M^3$ by taking as generators a system of free generators $s_1, \ldots, s_g$ of $\pi_1 H_g$ (corresponding possibly to the cores of the handles) and as relators the expressions of the curves $\delta_1, \ldots, \delta_g$ as words $R_1 = R_1(s_1, \ldots, s_g), \ldots,\ R_g = R_g(s_1, \ldots, s_g)$:

$$\pi_1 M^3 = \langle s_1, \ldots, s_g \mid R_1, \ldots, R_g \rangle.$$

Hence, the fundamental group $\pi_1 M^3$ of a closed (orientable) 3-manifold $M^3$ admits a presentation with as many relations as generators, that is a so-called *balanced presentation.* If $M^3$ has a Heegaard decomposition of genus $g$ then $\pi_1 M^3$ can be generated by $g$ elements. To formulate this in a more sophisticated way we make the following definition. This is the reason why not all groups can be fundamental groups of 3-manifolds, for instance, $\mathbb{Z}^4$ or $\mathbb{Z}_4 \otimes \mathbb{Z}_4$ do not have balanced presentations.

**2.1 Definition.** In the situation above the presentation $\langle s_1, \dots, s_g \mid R_1, \dots, R_g \rangle$ of $\pi_1 M^3$ is called *geometric.* The minimal number of elements needed to generate a group $\mathbf{G}$ is called the *rank of* $\mathbf{G}$, denoted by $r(\mathbf{G})$. For a 3−manifold $M^3$ we denote the rank of $\pi_1 M^3$ by $r(M^3)$.

**2.2 Proposition.** $r(M^3) \le h(M^3)$. $\square$

Clearly, $0 = h(M^3)$ implies $M^3 = S^3$, and $0 = r(M^3)$. The Poincaré conjecture can be formulated in the following way:

$$r(M^3) = 0 \iff h(M^3) = 0.$$

For the lens spaces and $S^2 \times S^1$ we have also the equation $r(M^3) = h(M^3)$, namely $= 1$, since these spaces have cyclic fundamental groups and Heegaard diagrams of genus 1. Waldhausen (see [Wa 2], [Ha 2]) has asked whether for every 3−manifold equality is true and suggested a study of this more general question instead of the Poincaré conjecture. In 2.3 we will show that for most of the Seifert fibre spaces rank and Heegaard genus coincide. However there are examples where this is not the case and others for which this question is not yet decided, see Theorem 2.6.

**2.3 The rank of fundamental groups of Seifert manifolds.** Using the Seifert-van Kampen theorem it is easy to prove that the Seifert manifold $S(g; e_0; b_1/a_1, \dots, b_g/a_g)$ has a fundamental group with presentation

$$\mathbf{G} = \langle s_1, \dots, s_m, u_1, v_1, \dots, u_g, v_g, f \mid s_1 \dots s_m \prod_{j=1}^{g} [u_j, v_j] f^e,$$
$$[s_i, f],\ [u_j, f],\ [v_j, f],\ s_i^{a_i} f^{b_i}\ (1 \le i \le m\ ,\ 1 \le j \le g) \rangle$$

where $e = e_0 + \sum_{i=1}^m b_i/a_i$. ($[a, b]$ denotes the commutator $aba^{-1}b^{-1}$.) If $2g + \sum_{i=1}^m (1 - b_i/a_i) \ge 2$, in particular, if $2g + m \ge 4$, then $f$ has infinite order. Moreover, f generates the centre of $\mathbf{G}$ if there is a strict inequality. (For details see [Or ], [OVZ].)

The group $\mathbf{F} = \mathbf{G}/\langle f \rangle$ has the presentation

$$\langle s_1, \dots, s_m, u_1, v_1, \dots, u_g, v_g\ [s_i^{a_i}, (\prod_{i=1}^{m} s_i \cdot \prod_{j=1}^{g} [u_j, v_j] \rangle,$$

and, thus, is isomorphic to a Fuchsian group (or a group acting on $\mathbb{R}^2$), see [ZVC, 4.6, 4.7]. Of course,

$$(1) \qquad r(\mathbf{G}) \ge r(\mathbf{F}).$$

For the rank of the Fuchsian group one would expect $r(\mathbf{F}) = m+2g-1$, if $m > 0$, and $r(\mathbf{F}) = 2g$ if $m = 0$; at least, these numbers are the minimal numbers of generators defined by fundamental domains of the Fuchsian groups, see the text following 2.4. However, this is not the case as shows.

**2.4 Theorem** ([PRZ], [Zi 6]).

$$r(\mathbf{F}) = \begin{cases} 2g & \textit{if } m = 0, \\ m-2 & \textit{if } g = 0,\ m \textit{ even, one } a_i \textit{ odd, all others being } 2, \\ 2g+m-1 & \textit{for the remaining cases.} \quad \square \end{cases}$$

The *geometric rank of the Fuchsian group* $\mathbf{F}$ is defined as follows: Let $D$ be a fundamental domain for $\mathbf{F}$. Let $r_D$ be the number of pairs $\{\varphi, \varphi^{-1}\}$ where $\varphi(D) \cap D$ contains an arc. By taking from each pair one element we obtain a system of generators for $\mathbf{F}$. The minimum of the $r_D$ over all fundamental domains is called the *geometric rank of* $\mathbf{F}$. In the first and the third case in 2.4 the geometric and the algebraic ranks coincide, however in the second case the geometric rank is $m-1$. This example has been discovered by Burns, Karrass, Pietrowski and Purzitzky. We present it in a slightly more complicated version after 2.5. For details see [PRZ]. From 2.1, 2.4, 1.10 and the inequality $r(\mathbf{F}) \leq r(\mathbf{G}) = r(M^3) \leq h(M^3)$ we obtain most of the the following theorem:

**2.5 Theorem** [BoiZ]. *Let* $M^3 = S(g; e_0; b_1/a_1, \ldots, b_m/a_m)$ *with* $1 < a_1 \leq \ldots \leq a_m$ *and* $\gcd(a_i, b_i) = 1$ *for* $1 \leq i \leq m$.

(i) *If* $M^3 = S(0; e_0; 1/2, \ldots, 1/2, b_m/2\ell+1)$ *with* $\ell > 0$, $m$ *even and* $m \geq 4$, *then* $m-2 = r(M^3) \leq h(M^3) \leq m-1$. *If, in addition,* $e_0 = \pm 1/2(2\ell+1)$, *then* $r(M^3) = h(M^3) = m-2$. *Moreover:*

(*) *If* $m = 4$ *and* $e_o \neq \pm 1/2(2\ell+1)$ *then* $2 = r(M^3) < h(M^3) = 3$.

(ii) *If* $g > 0$ *and* $m = 2$, *or if* $m \geq 3$ *and if* $M^3$ *does not belong to the case (i), then* $r(M^3) = h(M^3) = 2g+m-1$.

(iii) *If* $m = 1$, *then* $r(M^3) = h(M^3) = 2g$ *if* $e_0 = \pm 1/a_1$; *otherwise* $r(M^3) = h(M^3) = 2g+1$. *If* $m = 0$, *then* $r(M^3) = h(M^3) = 2g$ *if* $e_0 = \pm 1$; *otherwise* $r(M^3) = h(M^3) = 2g+1$.

The cases (ii) and (iii) are obtained by standard arguments using the existence of a section for the cases where the Heegaard genus becomes smaller than in the usual case. In the following we deal with the case (i). Let us first show for the case $m = 4$ that the unexpected diminishing of the rank of $\mathbf{F}$ lifts to a diminishing of the rank of $\mathbf{G} = \langle s_1, s_2, s_3, s_4, f \mid s_1^2 f,\ s_2^2 f,\ s_3^2 f,\ s_4^{2\ell+1} f^b,\ s_1 s_2 s_3 s_4 f^e \rangle$. A consequence of the relations $s_i^2 f$ is that $[s_i, f], 1 \leq i \leq 4$ is a relation. Define $x = s_1 s_2,\ y = s_1 s_3$. Then

$$w = xy^{-1}x^{-1}y = s_1 s_2 s_3 f^{-1} s_1 f^{-1} s_2 f^{-1} s_1^{-1} s_1 s_3 = (s_1 s_2 s_3)^2 f^{-3}$$

and $w \in \langle x, y \rangle$, the subgroup of $\mathbf{G}$ generated by $X$ and $Y$. Further,

$$\begin{aligned} s_4^{-2} &= (f^e s_1 s_2 s_3)^2 = w f^{2e+3}, \\ s_4 &= s_4^{-2\ell} f^{-b} = w^\ell f^{2e\ell+3\ell-b}, \\ s_3 &= (s_1 s_2)^{-1} s_1 s_2 s_3 = (s_1 s_2)^{-1} s_4^{-1} f^{-e} = x^{-1} w^{-\ell} f^{-2e\ell-3\ell+b-e} \\ &= x^{-1} w^{-\ell} f^\gamma \in \langle x, y, f \rangle \qquad \text{where } \gamma = -2e\ell - 3\ell + b - e, \\ s_1 &= s_1 s_3 s_3^{-1} = y w^\ell x f^{-\gamma} \in \langle x, y, f \rangle, \\ s_2 &= s_1^{-1} s_1 s_2 = x^{-1} w^{-\ell} y^{-1} x f^\gamma \in \langle x, y, f \rangle. \end{aligned}$$

This implies that

$$\begin{aligned} 1 &= s_1^2 f = (y w^\ell x)^2 f^{-2\gamma+1} \qquad \Rightarrow \qquad f^{-2\gamma+1} \in \langle x, y \rangle, \\ 1 &= s_2^2 f = (x^{-1} w^{-\ell} y^{-1} x)^2 f^{2\gamma+1}; \end{aligned}$$

hence, $f^2 \in \langle x, y \rangle$ and $f \in \langle x, y \rangle$. This shows that $x, y$ generate $\mathbf{G}$. (The example of Burns, Karrass, Pietrowski, Purzitzky is obtained when putting $f = 1$.)

In fact, by a more careful calculation one obtains that *there is a presentation for* $\mathbf{G}$ *with the generators* $x$, $y$ *and two defining relations,* a so-called *balanced presentation.* J.M. Montesinos [Mo 5, 6] has shown that one can pass from a presentation defined by a Heegaard diagram of genus 3 to a balanced presentation with 2 generators by using only extended Nielsen processes; this gives a geometric interpretation of the presentation mentioned.

It remains to show that $h(M^3) = 3$ for the exceptional cases 2.6 (*). We will only sketch the proof; for details see [BoiZ]. The proof is based on two well known theorems. The first is a result of Birman-Hilden and Viro which is valid only for handlebodies of genus 2 and does not have an analogue for higher genus.

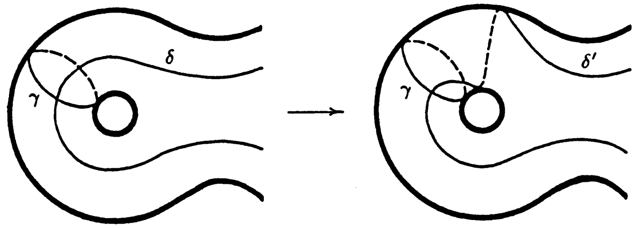
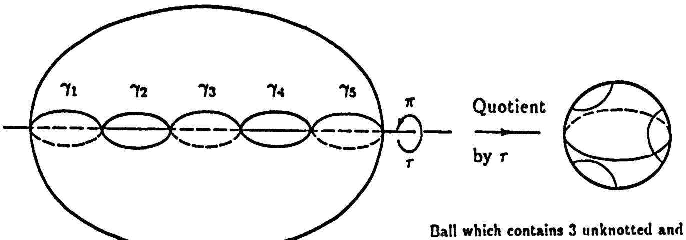



Fig. 2.1

A *Dehn twist* on a surface $S$ along a curve $\gamma$ is a homeomorphism which is the identity outside a regular neighbourhood $N(\gamma)$ of $\gamma$ and gives a twist inside $N(\gamma)$, see Fig. 2.1. It is well known, see [De], [Li], [Bi, 4.3], that for every surface the Dehn twists along a finite system of curves (which is nearly the usual canonical system) generate the group of orientation preserving homeomorphisms up to isotopies; more precisely, the isotopy classes of such twists generate the group of isotopy classes of orientation preserving homeomorphisms. For the particular case of a closed orientable surface $S_2$ of genus 2 every orientation preserving homeomorphism is a product of the Dehn twists along the curves $\gamma_1, \dots, \gamma_5$ which are drawn in Fig. 2.2. If $\tau$ denotes the hyperelliptic involution, i.e. the rotation by $\pi$ described in Fig. 2.2, then each of the Dehn twists along $\gamma_1, \dots, \gamma_5$ can be chosen in a $\tau$-equivariant form, that is, such that every pair of points one of which is the image of the other under $\tau$ is mapped to a pair with the same property. Thus every orientation preserving homeomorphism of $S_2$ is isotopic to a $\tau$-equivariant one. Now consider $S_2$ as the boundary of a handlebody $H_2$ of genus 2 and extend $\tau$ to an involution of $H_2$ in the way suggested by Fig. 2.2. The factor space $H_2/\tau$ is a 3-ball $D^3$ and $H_2 \to D^3$ is a branched covering where the branching occurs along three unlinked arcs, see Fig. 2.2.




Handlebody of genus 2 with the standard rotation of angle $\pi$ about an axis meeting the boundary in 6 points.

Ball which contains 3 unknotted and unlinked arcs corresponding to the image of the image of the fixed point set of $\pi$.

Fig. 2.2

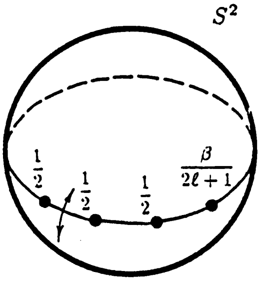
Next consider a 3-manifold $M^3$ which has a Heegaard decomposition $(H_2, H_2')$ of genus 2. The hyperelliptic involution on the surface can be extended to both handlebodies and, thus, to $M^3$, see Fig. 2.2. On each handlebody the involution defines a branched covering over a 3-ball and, hence, a branched covering $M^3 \to S^3$ of order 2 where the branch points are on the link formed by the six arcs from the two 3-balls. In every 3-ball the three arcs are unlinked, that is, there are three pairwise disjoint discs each of which is bounded by one of the arcs considered and an arc on the boundary of the 3-ball. A presentation of a link of this type is called a 3-*bridge presentation;* in each ball the system of arcs is trivial and all complications of the link are due to the glueing mapping used to identify the boundaries of the balls. — Hence, we have shown the following.

**2.6 Proposition** ([BiH],[Vi]). (a) *Let $S_2$ be a closed orientable surface of genus 2. There is an involution $\tau$: $S_2 \to S_2$ such that every orientation preserving homeomorphism $f$: $S_2 \to S_2$ is isotopic to a homeomorphism $f_*$ such that $f_*^{-1} \circ \tau \circ f_* = \tau$.*

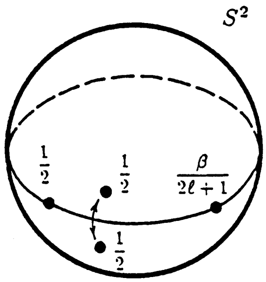
(b) *A closed orientable 3-manifold $M^3$ admits a Heegaard decomposition of genus 2 if and only if $M^3$ is a 2-fold covering of $S^3$ branched along a link with a 3-bridge presentation.* □

For other relations between Heegaard decompositions, bridge presentations and branched coverings see [Hee], [Bu], [Mo 4]. The next important result we need is due to Tollefson [To] and Bonahon [Bon 1], namely that every involution of a Seifert fibre space $M^3$ is conjugate to a fibre preserving one. Let us assume that the involution $\tau$ from above is already fibre-preserving. One has to discuss the different cases according to the effect of $\tau$ to the fibres and, in particular, to the exceptional fibres of $M^3$. We do this here for the case when $\tau$ reverses the orientation of the fibres.

For $M^3 = S(0; e_0; 1/2, 1/2, 1/2, b/2\ell + 1)$ the bases of the Seifert fibration is $S^2$ with four exceptional points, corresponding to the exceptional fibres. On $S^2$, $\tau$ induces an involution $\bar{\tau}$ which reverses the orientation of $S^2$, since $\tau$ preserves the orientation of $M^3$ and reverses the orientation of the fibres. Since $\tau$ has a non-empty fixed point set, so does $\bar{\tau}$ too. Therefore $\bar{\tau}$ looks like a reflection through the equation of $S^2$. Since $\tau$ respects the set of exceptional fibres, $\bar{\tau}$ permutes the images of the exceptional fibres in $S^2$, and there are the two possibilities, shown in Fig. 2.3.



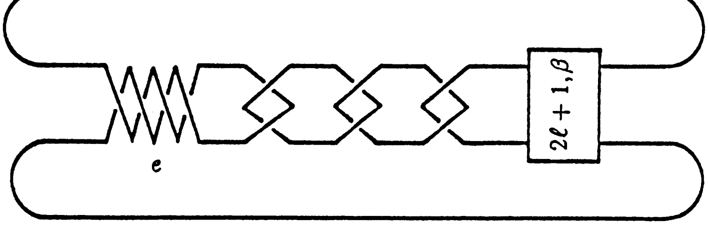

a) $\tau$ respects each exceptional fibre




b) $\tau$ exchanges two exceptional fibres of type $1/2$

Fig. 2.3

a) In this case $M^3/\tau$ is $S^3$ and the branch set is the link from Fig. 2.4, called the *Montesinos link* $\mathbf{m}(0; e_0; 1/2, 1/2, 1/2, b/(2\ell + 1))$. (For details see [Mo 1,3], [BuZ, Section 12], [Zi 8].) It is a link consisting of 3 components one of which is a non-tivial 2–bridge knot. This component needs at least 2 bridges in every bridge presentation, so there are necessarily at least 4 bridges for the branch set.




Montesinos link $\mathbf{m}(0; e_0; 1/2, 1/2, 1/2, \beta/2\ell + 1), e = e_0 + 3/2 + \beta/2\ell + 1 \in \mathbb{Z}$

Fig. 2.4

b) In the second case the quotient $M^3/\tau$ is not $S^3$, contrasting with the result in 2.6 (b). In fact, according to Montesinos' construction ([Mo 1, 3], [BuZ, Section 12]), $M/\tau$ is obtained by Dehn surgery with coefficient 1/2 along a trivial solid torus in $S^3$, see Fig. 2.5, that is, the solid torus drawn is replaced by another one having as meridian a curve going twice along the longitude and once along the meridian. After the surgery this solid torus correponds to the image in $M^3/\tau$ of a regular neighbourhood of the two exceptional fibres of type 1/2 which are exchanged by $\tau$. Therefore $\pi_1(M/\tau) \cong \mathbb{Z}_2$ and $\tau$ defines a 2–fold branched covering of $M^3$ over the lens space $L(2,1)$, not over $S^3$.

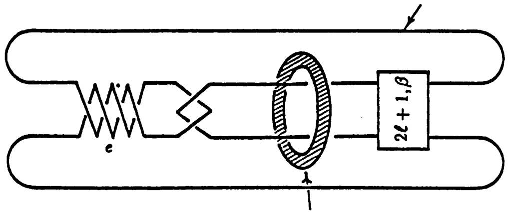



Fig. 2.5

Hence, if $\tau$ reverses the orientation of the fibres, $M^3$ is never a 2–fold covering of $S^3$ branched along a link admitting a 3–bridge presentation. In a similar way, using other special results on Seifert fibre spaces, one deals with the case when $\tau$ preserves the orientation of the fibres. $\square$

**2.7 Remarks.** (a) The proof of Theorem 2.5 for $m = 4$ described above essentially depends on Proposition 2.6. For the cases in 2.5 (ii) with $m \geq 6$ the Heegaard genus is not determined and it seems much more difficult to deal with this case than for the case $m = 4$.

(b) Using connected sums of the manifolds from above one can find 3–manifolds $K^3$ where $h(K^3) - r(K^3)$ is arbitrarily large, since Heegaard genus and rank behave additively for connected sums of 3–manifolds, see 1.13 and [ Zi 6], [ZVC, 2.8.2]. Up to now, no prime manifold is known where the difference of Heegaard genus and rank of the fundamental group is greater than 1. In fact, no other examples of prime manifolds, except those given in 2.5(*), are known to have a positive difference, although it seems rather likely that there must be more (see [Mo 6]).

## 3. Simple closed curves on handlebodies

In this section we will collect some results on curves on handlebodies which can be used in the theory of Heegaard decompositions as we will do in the following section. Most of the results are from [Zi 1,2,3]. In the following we consider an oriented handlebody $H = H_g$ of genus $g$; the orientation induces an orientation on the boundary $S = \partial H$. Therefore we can define intersection numbers between curves in $S$ or between curves on $S$ and discs $\Delta_i \subset H$ with $\Delta_i \cap S = \partial\Delta$ and use an expression like "positive intersection".

A system $\Delta = (\Delta_1, \dots, \Delta_g)$ of disjoint discs in the handlebody $H$ is called a *system of cuts of* $H$ if $\Delta_i \cap \partial H = \partial\Delta_i$ and the complement of a regular neighbourhood of these discs is ball. Let $\delta_*$ be a simple closed curve on $S = \partial H$ such that $\delta_* \cap \Delta = \emptyset$. Since $\overline{H - N(\Delta)}$ is a 3−ball and $\delta_*$ lies in its boundary there is a disc $\Delta_* \subset H$ with $\delta_* = \partial\Delta_*$ and $\Delta_* \cap \bigcup_{i=1}^{g} \Delta_i = \emptyset$. Assume that there is a curve $\kappa \subset S$ such that $\kappa \cap \delta_*$ and $\kappa \cap \partial\Delta_i$, for some $i \in \{1, \dots, g\}$, consist of one point each and that the intersection is proper in both cases. Then $\Delta' = (\Delta_1, \dots, \Delta_{i-1}, \Delta_*, \Delta_{i+1}, \dots, \Delta_g)$ is also a system of cuts of $H$. The step from $\Delta$ to $\Delta'$ is called a *bifurcation*. See Fig. 3.1. It is easily proved that one can go from one system of cuts of $H$ to an arbitrary other one by finitely many bifurcations.

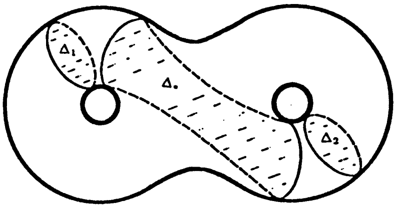



Fig. 3.1

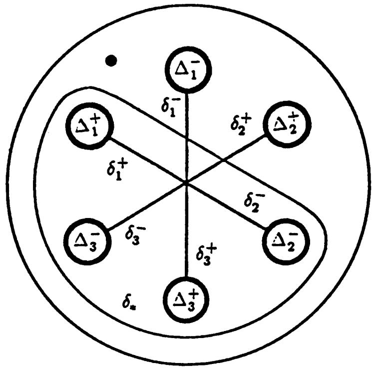



Fig. 3.2

Let us now determine the changes of free generators of $\pi_1 H$ induced by a bifurcation. Take the basepoint $P_\# \in S - \partial\Delta$ and complete $\partial\Delta$ by curves $(\vartheta_1, \dots, \vartheta_g)$ such that the set $S - (\partial\Delta \cup \bigcup_{j=1}^{g} \vartheta_j)$ is a disc. The curves $\vartheta_1, \dots, \vartheta_g$ define free generators of $\pi_1 H$. By dividing $S$ along the trace $(\delta_1, \dots, \delta_g)$ of $\Delta$ we obtain the 2−sphere $S^2$ with $2g$ holes $(\delta_1^+, \delta_1^-, \dots, \delta_g^+, \delta_g^-)$ which are connected with $P_\#$ by arcs $(\vartheta_1^+, \vartheta_1^-, \dots, \vartheta_g^+, \vartheta_g^-)$ obtained from the $\vartheta_i$. See Fig. 3.2. If we cut along all these arcs the sphere with $2g$ holes becomes a disc. Let $\delta_*$ be a simple closed curve on the sphere with holes which intersects any $\vartheta_j^\pm$ at most once. The curve $\delta_*$ divides the sphere into two domains and the curves $\delta_i^+$ and $\delta_i^-$ have to be in different domains, if we want to replace $\Delta_i$ by a disc $\Delta_*$ with boundary $\delta_*$; this condition is also sufficient. Now $\Delta' = (\Delta_1, \dots, \Delta_{i-1}, \Delta_*, \Delta_{i+1}, \dots, \Delta_g)$ defines a new system $(S_1', \dots, S_g')$ of free generators of $\pi_1 H$. By calculating the words for the curves $\vartheta_j$ with respect to the new system of cuts we see how the old generators are expressed by the new ones: there is a permutation $\begin{pmatrix} 1 & \cdots & g \\ n(1) & \cdots & n(g) \end{pmatrix}$

such that

$$S_{n(j)}=\begin{cases} S'_{n(j)} & \text{for } 1\le j<g_1,\\ S'_{n(j)}S_i'^{\varepsilon} & \text{for } g_1\le j<g_2,\\ S_i'^{-\varepsilon}S'_{n(j)} & \text{for } g_2\le j<g_3,\\ S_i'^{-\varepsilon}S'_{n(j)}S_i'^{\varepsilon} & \text{for } g_3\le j\le g,\end{cases}$$

where $\varepsilon=\pm1$ and $n(i)=1$. In the first case both discs $\Delta^+_{n(j)},\Delta^-_{n(j)}$ are in the same domain as $P_{\#}$, in the last case they both belong to the other domain, in the second and third, one of the discs is in the same domain as $P_{\#}$, the other is not. By taking a homeomorphism of $H$ which sends the new system $\Delta'$ of discs to the old preserving subscripts and orientations, we obtain the automorphism of $\pi_1 H$ which has the following effect on the generators:

$$S_{n(j)}\mapsto\begin{cases} S_{n(j)} & \text{for } 1\le j<g_1,\\ S_{n(j)}S_i^{\varepsilon} & \text{for } g_1\le j<g_2,\\ S_i^{-\varepsilon}S_{n(j)} & \text{for } g_2\le j<g_3,\\ S_i^{-\varepsilon}S_{n(j)}S_i^{\varepsilon} & \text{for } g_3\le j\le g.\end{cases}$$

Automorphisms of this form are called *Whitehead automorphisms.* They have the following remarkable properties.

**3.1 Whitehead automorphisms.** In the free group $\langle S_1,\dots,S_g \mid -\rangle$ let $|W|$ denote the usual length with respect to the generators $S_1,\dots,S_g$. Two words $W$, $W'$ are called *Nielsen equivalent* if there is an automorphism $\alpha$ of the free group with $\alpha(W)=W'$. The main property now is: *If $W$ does not have minimal length among all words which are Nielsen equivalent to $W$ then there exists a Whitehead automorphism $\alpha$ such that $|\alpha(W)|<|W|$. Therefore, if the words $W$ and $W'$ are Nielsen equivalent and their lenghts cannot be reduced by applying Whitehead automorphisms, then they have the same length. Moreover, if two words $W$, $W'$ have minimal length and are Nielsen equivalent then there is a sequence $\alpha_1,\dots,\alpha_k$ of Whitehead automorphisms and automorphisms induced by permutations of the set $\{S_1,S_1^{-1},\dots,S_g,S_g^{-1}\}$ such that*

$$\alpha_k\circ\cdots\circ\alpha_1(W)=W' \quad \textit{and}\ |\alpha_j\circ\cdots\circ\alpha_1(W)|=|W| \quad \textit{for} \quad 1\le j\le k;$$

*in other words, the application of the $\alpha_i$ does not increase the length of the word.*

The results quoted above can be generalized to the case of systems of words. Another generalization, important for topological problems, deals with system of conjugacy classes of words and uses the *cyclic length*, that is the length obtained after free and cyclic reductions; for example, $S_1S_2S_1^{-1}$ has the cyclic length 1. *Two systems $(W_1,\dots,W_k)$ and $(W_1',\dots,W_k')$ of words* of the free group are called *equivalent* if there is an automorphism $\alpha$ of the free group and elements $L_1,\dots,L_k$ such that $\alpha(W_i)=L_i^{-1}W_i'L_i$ for $1\le i\le ka$. Of course, the length used for this case is the cyclic length. Clearly, this type of equivalence is the right one for dealing with Heegaard diagrams. There are only finitely many Whitehead and permutation automorphisms; hence, *it can be decided in a finite number of steps whether two words or systems of words are equivalent or not.*

The results described in 3.1 have been found by J.H.C. Whitehead in [Whi 1, 2]; new proofs are given by Rapaport [Ra] and Higgins-Lyndon [HiL]. The theorem of Whitehead allows one to calculate generators for the stabilizer of an element of a free group, that is for the group of automorphisms of the free group which fix the element. McCool [Mc] discovered how to calculate defining relations for the stabilizer, which leads to a better understanding of the proof of Higgins-Lyndon.

Let us consider some examples to become more familiar with the Whitehead method.

**3.2 Examples.** (a) Consider the word $S^pT^{-q}$ with $p, q > 2$ in the free group $\langle S, T \mid - \rangle$. Since inner automorphism do not change the length one can decrease length only by applying the following Whitehead automorphisms (or their products with inner automorphisms):

$$S \mapsto S, \quad T \mapsto TS^\varepsilon \text{ or } S \mapsto ST^\varepsilon, \quad T \mapsto T$$

where $\varepsilon \in \{1, -1\}$. The image of the above word is $S^p(TS^\varepsilon)^{-q}$ in the first case and it is easy to check that the length increases. Similar for the other case; so the word $S^pT^{-q}$ is minimal.

(b) Consider the words $S^5V^{-3}S^5V^{-4}$ and $S^5V^{-2}S^5V^{-2}S^5V^{-3}$ of $\langle S, V \vert - \rangle$. It is easy to check that are also minimal. Further consider the word $S^5T^{-7} \in \langle S, T \mid - \rangle$. Since the lengths of these elements are 17, 22 and 12 they are not Nielsen equivalent.

What is interesting about these words is that they give isomorphic groups if we introduce them as single defining relations:

$$\begin{aligned}\langle S, T \mid S^5 = T^7\rangle &= \langle S, T, V \mid S^5 = T^7, V = T^2\rangle = \langle S, T, V \mid S^5V^{-3} = T, V = T^2\rangle \\ &= \langle S, T, V \mid S^5V^{-3} = T, V = (S^5V^{-3})^2\rangle = \langle S, V \mid S^5V^{-3}S^5V^{-4}\rangle.\end{aligned}$$

When we define $V = T^3$ we obtain in the same way the relator $S^5V^{-2}S^5V^{-2}S^5V^{-3}$, and so all these three presentations with a single defining relation are from the 'same' group. Since the relations are not Nielsen equivalent they give a counterexample to a conjecture of Magnus concerning one-relator groups, see [MKS, p. 401]; in fact, this has been the first one found by McCool-Pietrowski [McP] and in [Zi 6]. We will return to it later.

To study simple closed curves on $H_g$ ($g \geq 2$) it is convenient to determine *handles* of $H_g$ by a system $\beta = \{\beta_1, ..., \beta_g\}$ of $g$ disjoint simple closed curves on $\partial H_g$, the *belt curves*, bounding $g$ disjoint discs $B_j$ in $H_g$ which cut $H_g$ into solid tori $Q_1, ..., Q_g$ and a ball $D^3$. If $g = 2$, it suffices to take only one curve $\beta$ and forget the ball. Each solid torus $Q_j$ determines uniquely, up to isotopy and reversal of orientation, a meridian $\delta_j$ of $H_g$ such that $\delta_j \cap \beta_j = \emptyset$. Any system of pairwise disjoint simple closed curves on $\partial H_g$ can be deformed by an isotopy of $H_g$ into a system $\kappa$ transverse to $\beta$. Assume that $\kappa \cap Q_j$, for some $Q_j$, contains an arc $\tau$ which is homotopic on $\partial Q_j$ to one of the arcs on $\beta_j$ determined by the endpoints of $\tau$. Then $\tau$ can be pushed by an isotopy of $H_g$ to the belt $\beta_j$ and further to the other side, so we may assume that arcs of this trivial type do not occur.

**3.3 Connections.** The remaining arcs on one handle are called *connections*. They are arcs on the boundary of a handle $Q = Q_j$ which start and end on the belt curve $\beta = \beta_j$ and which cannot be continuously deformed on $\partial Q$ to $\beta$. (For a moment we will drop the subscript $j$, since all arguments are valid for every handle $Q_j$.) Let $\tau$ be a system of pairwise disjoint connections on $\partial Q$. *Two disjoint connections on* $Q$ are called *parallel* if they, together with two arcs on $\beta$, bound a rectangle on $\partial Q$. If we ignore the orientations of the members of $\tau$, at most three classes of parallel connections occur in $\tau$ according to Fig. 3.3.

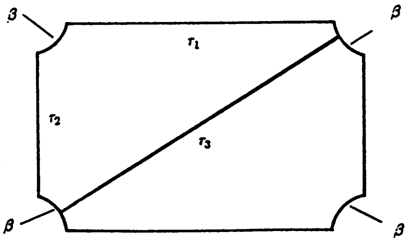



Fig. 3.3

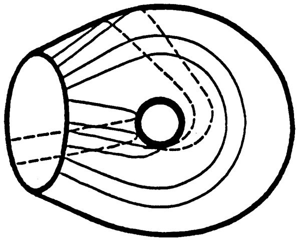


Fig. 3.4

We complete $\delta$ by a longitude $\vartheta$, $\vartheta \cap B = 0$, to a canonical system of curves on $\partial Q$ and we assume that we cannot reduce the number of intersections of $\tau$ with $\delta \cup \vartheta$; that is, we forbid an arc in $\tau$ which together with an arc on $\delta$ or $\vartheta$ bounds a disc on $\partial Q$. For a fixed oriented connection the intersections with $\delta$ or $\vartheta$ are all positive or all negative. If a connection has intersection numbers $a$ and $a'$ with $\delta$ and $\vartheta$, respectively, then we call $(a, a')$ the *type of the connection.* If two disjoint connections are not parallel and have types $(a, a'), (b, b')$, respectively, then $\begin{vmatrix} a & b \\ a' & b' \end{vmatrix} = \pm 1$.

A homeomorphism $h$ of $Q$ maps $\delta$ to a curve isotopic to $e'\delta$ and $\vartheta$ to $e\vartheta + m\delta$ with $e', e \in \{1, -1\}, m \in \mathbb{Z}$; $h$ preserves the orientation of $Q$ when $e' = e$. Connections of type $(a, a')$ are mapped to connections of type $(e'a, ea' + ma)$. This can be done for all numbers $e', e \in \{1, -1\}$, $m \in \mathbb{Z}$.

Let $\tau$ consist of two or three subsystems of parallel connections of types $(a, a'), (b, b'), (c, c')$ where members of different systems are not parallel. Let $\bar{\tau}$ be another system of connections of types $(a, a''), (b, b''), (c, c'')$. Since $\begin{vmatrix} a & b \\ a'' & b'' \end{vmatrix} = \pm 1, \ldots$ it follows that $a'' = \pm a' + \mu a$, $b'' = \pm b' + \mu b$ and $c'' = \pm c' + \mu c$; hence, when the numbers of the connections of the corresponding types are the same there is a homeomorphism $h : Q \to Q$ mapping $\tau$ to $\bar{\tau}$. Thus, if the three first numbers are different then already the first numbers determine the types up to homeomorphisms of $Q$. Of course, in the discussion above we did not forbid that two of the numbers $a, b, c$ coincide or one to be the negative of the other. Assume that $b = a + c$ if there are in fact three types of connections. Assume that $a = c$. Since $ac' - a'c = \pm 1$ it follows that $a = c = 1$, $b = 2$. If $a = b$ then $a = b = 1$, $c = 0$. For all other cases the three numbers $a, b, c$ are distinct. If a connection of type $(0, a')$ occurs then $a' = \pm 1$ and $|b| = |c| = 1$ for possibly other connections. In general already the intersection numbers with the meridian $\delta$ determine the distribution of the connections to the classes of parallels. We record this result in a weak form which we will use in the following. The last assertion of it becomes evident from Fig. 3.4.

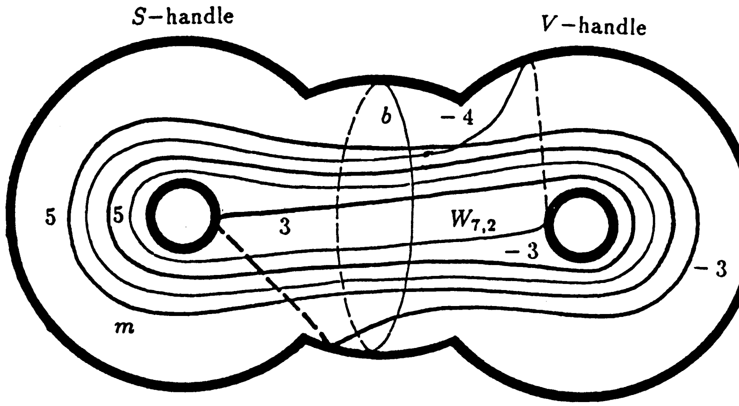
**3.4 Proposition.** *Let $\tau$ be a system of pairwise disjoint connections on a handle $Q$ such that there are at least two non-parallel ones and for one connection the (algebraic) intersection* *number with a meridian of $Q$ is bigger than 2 in absolute value. Then the intersection numbers with a meridian of $Q$ determine the connections up to a homeomorphism of $Q$. Moreover, $\tau$ is determined up to a homeomorphism of $Q$ if the quantities of the different collections of connections is known.* □

Because there are only finitely many possibilities to join finite systems of connections on the different handles, these results allow to decide whether a given system of conjugacy classes of words in a free group of rank $g$ can be realized by a system of disjoint simple closed curves on the boundary of a handlebody of genus $g$. Fortunately, the case when one of the intersection numbers with a meridian is 0 can be avoided, but, nevertheless, if all numbers are ones or twos the ones do not decide to which system of parallels the connection in question belongs. For Details see [Zi 1-3].

**3.5 Heegaard diagrams of genus** 2. Now the situation is much simpler. First we need only one belt curve $\beta$ on $\partial H_2$. Let $W$ be a system of conjugacy classes of elements of $\pi_1 H_2$, each conjugacy class being represented by a reduced cyclic word with powers of the generators as syllables such that consecutive syllables belong to different generators. If $W$ is obtained from a system $\kappa'$ of disjoint simple closed curves on $\partial H_2$ then there exists a homeomorphism of $H_2$ mapping $\kappa'$ to a system $\kappa$ such that $W$ is obtained by reading off the words belonging to $\kappa$ with respect to $(\delta_1, \delta_2\}$. Clearly, if connections of type $(0, a')$ do not occur, the number of syllables in $W$ equals the number of points in $\kappa \cap \beta$. The connections of type $(0, a')$ can be avoided, see [Zi 2]. This makes it easy to decide whether a given system of words in $\pi_1 H_2$ can be represented by disjoint simple closed curves on $\partial H_2$. To illustrate this we present here two examples coming from the group of the torus knot $\mathbf{t}(5,7)$, which has the presentation $\mathbf{G}(5,7) = \langle s, t \mid s^5 t^{-7}\rangle$; the meridian of the knot is $\mu = s^{-2}t^3$, see [BuZ, 3.28].

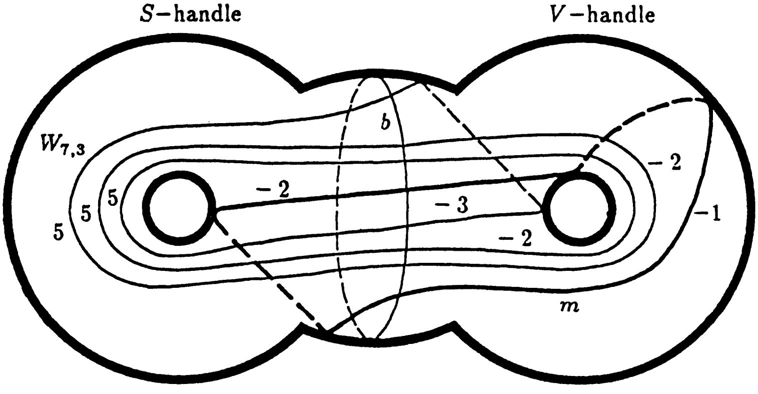
**3.6 Examples.** First we consider the one-relator presentation corresponding to the pair of generators $(s, v = t^2)$, see Example 3.2 (b): $\mathbf{G}(5,7) = \langle s,\ v = t^2 \mid s^5 v^{-3} s^5 v^{-4}\rangle$. In Fig. 3.5 one can see that the relator $s^5 v^{-3} s^5 v^{-4}$ and the meridian $\mu = s^3 v^{-3} s^5 v^{-3} s^5 v^{-3}$ cannot be realized by disjoint simple closed curves on the boundary of a genus 2 handlebody. We say, see 2.1, that *this one-relator presentation of the group* $\mathbf{G}(5,7)$ *of the torus knot* $\mathbf{t}(5,7)$ *is* not *geometric.* (Of course, the connections on the handlebody $H_2$ look more complicated than those in Fig. 3.5, but the original curves and their pictures can be mapped one to the other by a homeomorphism of the surface $\partial H_2$ which in general does not extend to $H_2$.)

For our second example we consider the one-relator presentation corresponding to the pair of generators $(s, v = t^3)$, see Example 3.2 (b): $\langle s, v \mid s^5 v^{-2} s^5 v^{-2} s^5 v^{-3}\rangle$. In Fig. 3.6 one can see that the relator $s^5 v^{-2} s^5 v^{-2} s^5 v^{-3}$ and the meridian $\mu = s^{-2} v$ can be realized by disjoint simple closed curves on the boundary of a genus 2 handlebody. Thus *this one-relator presentation of the group* $\mathbf{G}(5,7)$ *is geometric.* For final results in this direction see Section 4.




$W_{7,2}(V^{-1},S^p) \sim S^5V^{-3}S^5V^{-4} \qquad m = S^3V^{-3}S^5V^{-3}S^5V^{-3}$

Fig. 3.5




$W_{7,3}(V^{-1},S^p) \sim S^5V^{-2}S^5V^{-2}S^5V^{-3} \qquad m = S^{-2}V$

Fig. 3.6

## 4. Heegaard decompositions of knot exteriors.

The title of this section suggests that the concept of Heegaard decompositions has to be extended to manifolds with boundary as described in 1.19.

**4.1 Torus knots.** Consider the "standard" Heegaard decomposition $(H_1, H_1')$ of $S^3$, see 1.8 (a). Let $\delta$ and $\delta'$ be meridians of the solid tori $H_1$ and $H_1'$, respectively, such that $\delta \cap \delta'$ consists of one point. These curves generate the homology group of the torus $T = \partial H_1$ which is isomorphic to $\mathbb{Z}^2$. Let us recall some well known results on curves on a torus. *If $\gamma$ is a simple closed curve on $T$ then $\gamma \sim q\delta + p\delta'$ where* $\gcd(p,q) = 1$. *If two simple closed curves are homologous on $T$ then they are isotopic. To any pair $(p,q)$ of relatively prime integers there exists a simple closed curve $\gamma$ on $T$ with $\gamma = q\delta + p\delta'$.* Proofs can be found in many places. By $\mathbf{t}(p,q)$ we denote a simple closed curve on $T$ of the homology class $q\delta + p\delta'$ and call it *the $(p,q)$ torus knot.*

**4.2 The standard Heegaard decomposition of $E(p,q)$.** Consider a regular neighbourhood $N = T \times [-1,1]$ of the torus $T = T \times 0$, a regular neighbourhood $A_0 \cong S^1 \times [-1,1]$ of the knot $\mathbf{t}(p,q) = S^1 \times 0$ on $T$, and $A = \overline{T - A_0}$. Then $U = A_0 \times [-1,1] \subset N$ and $V = \overline{N - U} = A \times [-1,1]$ are annuli $\times$ interval and $U$ is a regular neighbourhood of $\mathbf{t}(p,q)$. Define $Q_1 = \overline{H_1 - N}$ and $Q_2 = \overline{H_1' - N}$. Then $E(p,q) = \overline{S^3 - V} = Q_1 \cup Q_2$ is the *exterior of the knot* $\mathbf{t}(p,q)$. Next we divide the annulus $A$ into two closed discs $\Delta, B$ the intersection of which consists of two disjoint segments. Then $H_2 = Q_1 \cup (B \times [-1,1]) \cup Q_2$ is a handlebody of genus 2 and $\overline{E(p,q) - H_2} = \Delta \times [-1,1]$ is a disc $\times$ interval. Hence, $(H_2, \overline{E(p,q) - H_2})$ is a Heegaard decomposition of genus 2 of $E(p,q)$; we call it the *standard Heegaard decomposition of genus* 2 *of* $E(p,q)$ and denote it by $HD_0$. A diagram of it, also denoted by $HD_0$, is given by $H_2$ and the curve $\delta = \partial\Delta$. (This construction mentioned already in 1.8 (d) is from [Rei 3].) See Fig. 4.1.

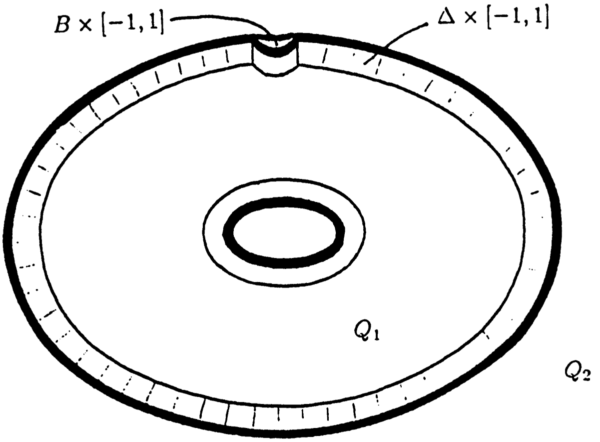
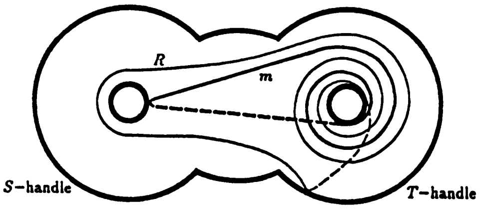



Fig. 4.1

**4.3 Different Heegaard diagrams for $\mathbf{t}(p,q)$.** Let us now consider the torus knot $\mathbf{t}(p,q)$. Let $a, b \in \mathbb{Z}$ such that $pb + qa = 1$. We illustrate the construction for the torus knot $\mathbf{t}(5,7)$; here $a = -2$, $b = 3$. Starting with the Heegaard diagram $(H_2; s^5t^{-7}, s^{-2}t^3)$ of $S^3$, see Fig. 4.2, we add one handle and obtain the Heegaard diagram $(H_3; s^5t^{-7}, s^{-2}t^3, v^{-1}t^3)$ of $S^3$, see Fig. 4.3. Next we move the knot meridian $\mu$, comp. 3.5, from $s^{-2}t^3$ to $s^{-2}v$, see Fig. 4.4, and then replace $s^5t^{-7}$ by $s^5v^{-2}t^{-1}$ and $v^{-1}t^3$ by $v^{-1}(s^5v^{-2})^3$, see Fig. 4.5-7, and finally we cut off the $T$-handle, see Fig. 4.8. This gives the following sequence of Heegaard diagrams of $S^3$:

**4.4**

$$\begin{aligned}
&(H_2;\ s^5t^{-7}, \mu = s^{-2}t^3),\\
&(H_3;\ s^5t^{-7}\ ,\ \mu = s^{-2}t^3\ ,\ v^{-1}t^3),\\
&(H_3;\ s^5t^{-7}\ ,\ \mu' = s^{-2}v\ ,\ v^{-1}t^3),\\
&(H_3;\ s^5v^{-2}t^{-1}\ ,\ \mu' = s^{-2}v,\ v^{-1}t^3),\\
&(H_3;\ s^5v^{-2}t^{-1},\ \mu' = s^{-2}v,\ s^5v^{-2}s^5v^{-2}s^5v^{-3}),\\
&(H_2;\ \mu' = s^{-2}v\ ,\ s^5v^{-2}s^5v^{-2}s^5v^{-3}).
\end{aligned}$$




$\langle S,T \mid R = S^pT^{-q}\ ,\ m = S^aT^b\rangle$

Fig. 4.2

If we drop the meridian we obtain a sequence of Heegaard diagrams of $E(p,q)$; thus $(H_2; s^5t^{-7})$ and $(H_2; s^5v^{-2}s^5v^{-2}s^5v^{-3})$ are Heegaard diagrams of genus 2 of $E(5,7)$. Notice that for the first diagram the words do not determine the connections on both handles, but for the second diagram only those on the $S$−handle are not characterized by the words. We must add the second number of the connections which are 2 for the $S$−handle and 3 on the $T$−handle. On the $V$−handle we do not need this additional information, see 3.10.

The third and the fourth step in 4.4 correspond to the steps in the euclidean algorithm to find gcd(7, 3):

$$7 = \underline{2} \cdot 3 + 1$$
$$3 = 3 \cdot 1 + 0$$

the $\underline{2}$ and $\underline{2} + 1$ are the exponents of $V^{-1}$ in the following expressions.

We want to show that these Heegaard decompositions of $E(5,7)$ are not homeomorphic. Otherwise there would be an isomorphism

$$\Phi : \langle s, t \mid - \rangle - \langle , v \mid - \rangle$$

inducing an isomorphism

$$\varphi : \langle s, t \mid s^5t^{-7} \rangle \to \langle s, v \mid s^5v^{-2}s^5v^{-2}s^5v^{-3} \rangle.$$

By a theorem closely related to the Freiheitssatz, see [MKS, Theorem 4.11], it follows that $\Phi(s^5t^{-7})$ is conjugate to $(s^5v^{-2}s^5v^{-2}s^5v^{-3})^{\pm 1}$. But this is impossible as we have seen in 3.2 (b).

If we replace $s$ by $u = s^2$ and do the same construction as above, we obtain the Heegaard diagram $(H_2; t^7u^{-2}t^7u^{-3}, u^{-1}t^3)$ of $S^3$ and $(H_2; t^7u^{-2}t^7u^{-3})$ of $E(5,7)$. It is not homeomorphic to the other two. Hence, *$E(p,q)$ admits three non-homeomorphic Heegaard decompositions of genus* 2. In fact, these are all the types, see 4.8.

**4.5 Heegaard diagrams of the exteriors of torus knots.** To attack the problem for arbitrary torus knots one has

(a) to find the non-Nielsen-equivalent 1−relator presentations of the fundamental group

$$\pi_1 E(p,q) = \mathbf{G}(p,q) = \langle s, t \mid s^pt^{-q} \rangle \quad \text{for} \quad p, q \geq 2, \quad \gcd(p,q) = 1;$$

(b) to generalize the steps 4.4;

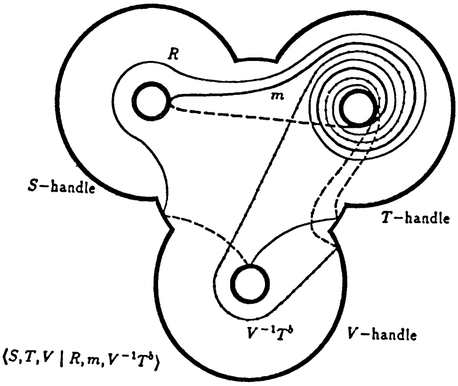



Fig. 4.3

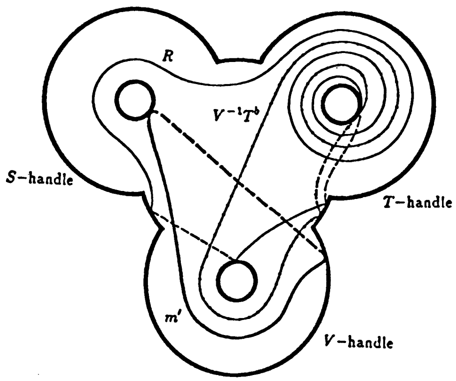



Fig. 4.4

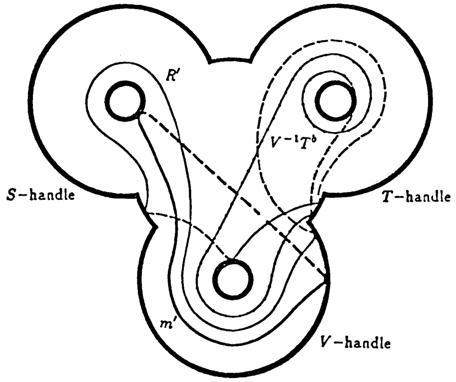



Fig. 4.5

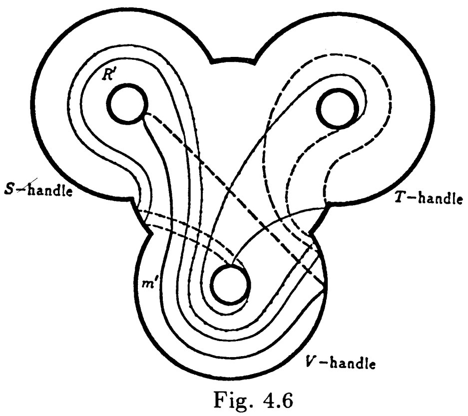



Fig. 4.6

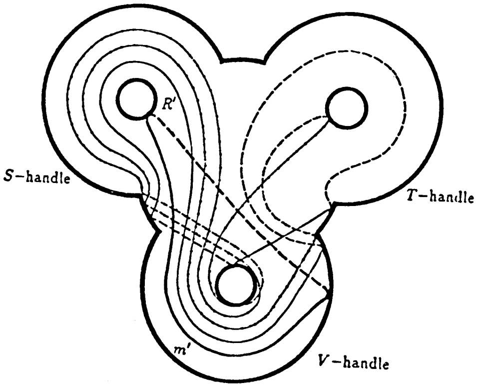



Fig. 4.7

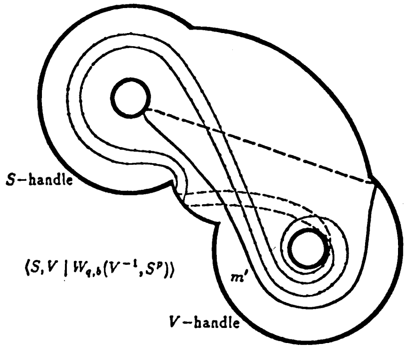



Fig. 4.8

(c) but to do so one has to relate the euclidean algorithm for finding $\gcd(q,b)$ with changes of connections on one handle and curves on $H_3$;

(d) to show that any two simple closed curves that realize the appearing words are uniquely determined up to a homeomorphism of $H_2$.

The problem 4.5(a) has been solved by D. Collins: By [Zi 7, Theorem 5.1], each pair of generators of $\mathbf{G}(p,q)$ is Nielsen equivalent to a pair $(s^a, t^b)$ such that

$$\gcd(a,p) = \gcd(b,q) = \gcd(a,b) = 1,\ 0 < 2a \leq pb,\ 0 < 2b \leq qa$$

(equality may hold only if $pb = 2$ or $qa = 2$). Moreover two such pairs $(s^a, T^b)$ and $(s^{a'}, T^{b'})$ of generators of $\mathbf{G}(p,q)$ are Nielsen equivalent iff $a = a'$ and $b = b'$ or, for the case $p = q$, $a = b'$ and $b = a'$. One-relator presentations for $\mathbf{G}(p,q)$ can be obtained if $a = 1$ or $b = 1$ (see [Zi 7, section 6]). Moreover these are the only possibilities by [Col, Theorem 2.6].

Therefore we have the following:

**4.6 Theorem** ([Zi 7] and [Col]). *The group $\mathbf{G}(p,q) = < s,t \mid s^p = t^q >$, with $p,q \geq 2$, has only finitely many one-relator presentations which are not Nielsen equivalent. They correspond to the pairs of generators $(s^a, t)$ or $(s, t^b)$ with $\gcd(a,p) = 1$ and $0 < 2a \leq p$ or $\gcd(b,q) = 1$ and $0 < 2b \leq q$, respectively.* □

The possible relators can be determined as follows. Define a function $f_{m,n} : \mathbb{Z} \to \{1,2\}$ by

$$f_{m,n}(k) = f_{m,n}(k') \quad \text{if} \quad k \equiv k' \bmod (m+n) \quad \text{and}$$

$$f_{m,n}(k) = \begin{cases} 1 & \text{if} \quad 0 < k \leq m, \\ 2 & \text{if} \quad m < k \leq n+m\,. \end{cases}$$

Now define a word $W_{m,n}(x_1,x_2)$ by $W_{m,n}(x_1,x_2) = \prod_{i=0}^{m+n-1} x_{f_{m,n}(1+im)}$. If $\gcd(m,n) = 1$ this word corresponds to a free generator of the free group $\langle x_1, x_2 \mid - \rangle$ which gives in the abelianized group $\mathbb{Z}^2$ the primitive element $(m,n)$. $W_{m,n}$ is determined by these properties up to conjugation. See [Nie 1] and [OZ]. In the following we will use the expression $W_{m,n}$, although, in fact, we should take a conjugate of it.

**4.7 Corollary.** *The relation corresponding to the generators $(u = s^a, t)$ and $(s, v = t^b)$ from 4.6 are $W_{p,a}(u^{-1}, t^q)$ and $W_{q,b}(v^{-1}, s^p)$, respectively.* □

If we apply the euclidean algorithm to $q, b$ with $0 < 2b < q$ then we start with the equation $q = n_1 \cdot b + r_1$, $\quad 0 \leq r_1 < b$. Here $n_1$ plays the role of the $\underline{2}$ in 4.3 and is $\geq 2$ because $2b < q$. It follows that $v^{-1}$ appears in $W_{q,b}(v^{-1}, s^p)$ with exponents $n_1$ and $n_1 + 1$ and the last one occurs at least once. Thus we may apply 3.4 to see that the word $W_{q,b}(v^{-1}, s^p)$ defines the connections on the $V$–handle up to a homeomorphism. To determine the Heegaard diagram we have only to add the second number for the $S$-handle and we write $(W_{q,b}(v^{-1}, s^p); q, -)$; similar for the other Heegaard diagrams: $(s^p t^{-q}; q, p)$, $\quad (W_{p,a}(u^{-1}, t^q); -, p)$, respectively. After proving some additional properties of connections one sees that all steps of 4.4 can be done in the general case and this gives the existence of three, in general, non-homeomorphic Heegaard decompositions of $E(p,q)$ (see [BRZ 1, 2], [Mor 1]). This is part of the following theorem:

**4.8 Theorem** [BRZ 1, 2]. *Any Heegaard decomposition of genus 2 of the exterior $E(p,q)$ of the torus knot $\mathbf{t}(p,q)$ is homeomorphic to one of the following three, which are described by Heegaard diagrams:*

$$HD_0 \leftrightarrow (s^p t^{-q}; q, p),\ HD_S \leftrightarrow (W_{p,a}(u^{-1}, t^q); -, p),\ HD_T \leftrightarrow (W_{q,b}(v^{-1}, s^p); q, -),$$

*where $aq + bp = 1$. They are pairwise not homeomorphic, except in the following cases:* i) *for $|p-q| = 1$ they are all homeomorphic; if* ii) $|p-q| \neq 1$ *the Heegaard diagrams $HD_s$ and $HD_T$ are not homeomorphic; now $HD_0$ is homeomorphic to $HD_s$ (or $HD_T$, respectively) if and only if $q \equiv \pm 1 \bmod p$ (or $p \equiv \pm 1 \bmod q$, respectively).*

We have sketched the proof of the existence of three, in general, non-homeomorphic Heegaard decompositions of $E(p,q)$. To see that those are all, thus to solve Problem 4.5 (d), is more difficult. One can do it by looking carefully to intersection numbers of suitable curves on $\partial H_2$. Another approach uses alternating products in free groups. The most efficient way is the following: The parallels of the knot $\mathbf{t}(p,q)$ on the torus $T$ and its parallels in the solid tori $H_1, H_1'$ together with the cores of the solid tori form a Seifert fibration of $E(p,q)$ with a disc as base space and the cores as exceptional fibres of orders $p$ and $q$. More general, one can consider all Seifert fibre spaces with two exceptional fibres of types $\alpha/p, \beta/q$ and base a disc. Let us denote such a space by $S(\alpha/p, \beta/q)$. They all have Heegaard decompositions of genus 2 and fundamental group $\langle s,t \mid s^p t^{-q}\rangle$. The steps described in 4.4 can be performed for these spaces and it follows, using the known classification theorem for Seifert fibre spaces, the following theorem:

**4.9 Theorem** [BRZ 2]. *The manifold $S(\alpha/p, \beta/q)$ admits three Heegaard decompositions $HD_0$, $HD_s$, $HD_T$, represented by the following Heegaard diagrams:*

$$HD_0 \leftrightarrow (s^p t^{-q}; \lambda, \mu),\ HD_s \leftrightarrow (W_{p,\alpha}(u^{-1}, t^q); -, \mu), HD_T \leftrightarrow (W_{q,\beta}(v^{-1}, s^p); \lambda, -).$$

*Here $\alpha\lambda \equiv 1 \bmod p$ and $\beta\mu \equiv 1 \bmod q$. Any Heegaard decomposition of genus 2 of $S(\alpha/p, \beta/q)$ is homeomorphic to one of these. Moreover:*

(a) *$HD_0$ is homeomorphic to $HD_T$ (or $HD_S$) if and only if $\beta \equiv \pm 1 \bmod q$ or $\alpha \equiv \pm 1 \bmod p$, respectively).*

(b) *If $\beta \equiv \pm 1 \bmod q$ and $\alpha \equiv \pm 1 \bmod p$ then $HD_0$, $HD_S$, $HD_T$ are homeomorphic.*

(c) *$HD_S$ and $HD_T$ are homeomorphic if and only if either case (b) occurs or $\alpha/p \equiv \pm\beta/q \bmod 1$ (that is, $p = q$ and $\alpha \equiv \beta \bmod p$).*

*In general there are exactly three non-homeomorphic Heegaard decompositions of $S(\alpha/p, \beta/q)$.*

□

M. Boileau and J.P. Otal [BoiO] used Theorem 4.9 to classify the Heegaard decompositions of genus 2 of some closed Seifert fibre spaces; moreover, they obtained the following slightly stronger result:

**4.10 Theorem.** *Let $M^3 = S(0; e_0; \pm 1/p, \pm 1/q, \gamma/r)$ be a Seifert fibre space over $S^2$ with three exceptional fibres. Then any two Heegaard decompositions of genus 2 of $M^3$ are isotopic, except in the case of a Brieskorn manifold $\Sigma(2,3,a)$ with $\gcd(a,3) = 1$ and $a \geq 7$ or a manifold $S(0;\ 1/4b;\ 1/2,\ (-b)/4,\ (4^{-1})/b)$ with $b \geq 5$ odd; here $(n)$ and $(n^{-1})$ denote the number $n$ or its inverse modulo the denominator. In these exceptional cases there are exactly two classes with respect to homeomorphisms or isotopies; these are also the examples of* [BGM]. □

Theorem 4.9 can also be used to find non-homeomorphic Heegaard decompositions of closed Seifert fibre space with three exceptional fibres and orbitspace a 2−sphere.

**4.11 Theorem** [BCZ], [Mor 2]. *Let $M^3 = S(0; e_0; \alpha_1/p, \beta/q, \gamma/r)$ be a Seifert fibre space over $S^2$ with three exceptional fibres.*

(a) *If $\alpha \not\equiv \pm 1 \bmod p$, $\beta \not\equiv \pm 1 \bmod q$ and $\gamma \not\equiv \pm 1 \bmod r$ then $M$ admits exactly three pairwise non isotopic Heegaard decompositions of genus 2, namely $HD(1,2), HD(2,3), HD(3,1)$. (These Heegaard decompositions are described below.)*

(b) *If $\alpha \not\equiv \pm 1 \bmod p$, $\beta \not\equiv \pm 1 \bmod q$ and $\gamma \equiv \pm 1 \bmod r$ then $M$ admits exactly two non isotopic Heegaard decompositions of genus 2, namely $HD(1,2)$ and $HD(3,1)$ (the last one is isotopic to $HD(2,3)$).*

(c) *Any two Heegaard decompositions of genus 2 of $M^3 = S(0; e_0; \alpha_1/p, \pm 1/q, \pm 1/r)$ are isotopic (namely to $HD(1,2)$) except in the case when*

(i) *$M$ is the Brieskorn manifold $V(2,3,a) = \{z \in \mathbb{C}^3 \mid z_1^2 + z_2^3 + z_3^a = 0, \|z\| = 1\}$ with $\gcd(3,a) = 1$ and $a \geq 7$;*

(ii) *$M$ is the algebraic manifold $W(2,4,b) = \{z \in \mathbb{C}^3 \mid z_1^2 + (z_2^3 + z_3^b)z_2 = 0, \|z\| = 1\}$ with $\gcd(2,b) = 1$ and $b \geq 5$.*

*In each exceptional case $M$ admits, up to isotopy, an additional Heegaard splitting which is not isotopic to $HD(1,2)$. (They are also the examples of* [BGM].)

*If two Heegaard decompositions of $M$ are homeomorphic then they are isotopic, except in the case when there is a homeomorphism of $M$ mapping one exceptional fibre to another one, that is, for example, when $\alpha/p \equiv \beta/q \bmod 1$.* □

This theorem has been obtained by Moriah [Mor 2] using some calculations of number theory and by Boileau-Collins-Zieschang [BCZ] applying Dehn's solution of the conjugacy problem for Fuchsian groups.

There is an easy way to construct Heegaard decompositions of a Seifert manifold by looking at some particular arcs on the orbit space of the Seifert fibration, see 1.10 and [BoiZ]. By definition, the orbit space of $S(\alpha/p, \beta/q)$ is a disc $D$ with two exceptional points corresponding to the exceptional fibres. We denote the projection by $\pi : S(\alpha/p, \beta/q) \to D$. Let us consider three circles $c_0, c_S, c_T$ on $D$ with the properties that $\pi^{-1}(c_0)$ is a torus parallel to the boundary of $S(\alpha/p, \beta/q)$ and that $\pi^{-1}(c_S)$ (resp. $\pi^{-1}(c_T)$) bounds a regular neighbourhood of the exceptional

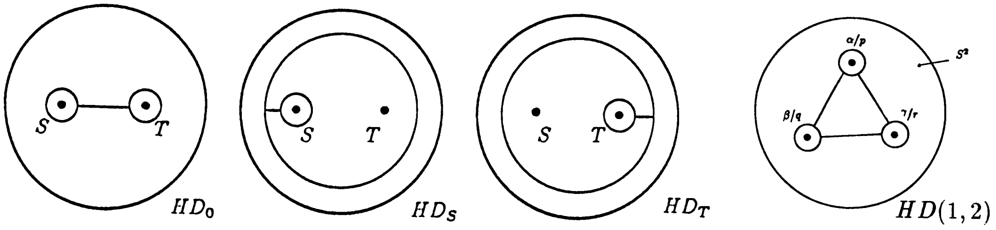



Fig. 4.9

fibre of order $p$ (resp. $q$). Furthermore, let $a_0, a_S, a_T$ be three pairwise disjoint arcs on $D$ joining the three circles: $a_0$ joins $c_S$ to $c_T$, $a_S$ (resp. $a_T$) joins $c_S$ (resp. $c_T$) to $c_0$. See Fig. 4.9.

**4.12 The standard Heegaard decompositon $HD_0$ of genus 2 of** $S(\alpha/p, \beta/q)$ is obtained as follows (see [BoiZ, 1.3]): Let $H_2$ be the handlebody in $S(\alpha/p, \beta/q)$ formed by the regular neighbourhoods of the exceptional fibres (bounded by $\pi^{-1}(c_S)$ and $\pi^{-1}(c_T)$) joined by a regular neighbourhood $Z$ of an essential arc of the annulus $\pi^{-1}(a_0)$. The complement of $H_2$ in $S(\alpha/p, \beta/q)$ is the union of $(S^1 \times S^1) \times [0,1]$ and a 2–handle $D^2 \times [0,1]$ corresponding to a regular neighbourhood of $\pi^{-1}(a_0) - Z$, that is this 2–handle is a regular neighbourhood of the fibred annulus $\pi^{-1}(a_0)$ cut along a regular neighbourhood of the essential arc joining $\pi^{-1}(c_T)$ to $\pi^{-1}(c_S)$. This gives the standard presentation of the group $\mathbf{G}(p,q)$.

**4.13 The Heegaard decomposition $HD_T$ of genus 2** corresponding to the Heegaard diagram $(W_{q,b}(v^{-1}, s^p; \lambda, -)$, where $\lambda\alpha \equiv 1 \bmod p$ is obtained as follows: Let $Y_2$ be the surface of genus 2 obtained by glueing the torus $\pi^{-1}(c_T)$ to the torus $\pi^{-1}(c_0)$ by the boundary of a regular neighbourhood $U$ of an essential arc of the annulus $\pi^{-1}(a_T)$. This surface $Y_2$ bounds a handlebody in $S(\alpha/p, \beta/q)$: As first meridian disc take $\overline{\pi^{-1}(a_T) - U}$; the second is a meridian disc of the regular neighbourhood of the exceptional fibre of order $p$ bounded by the torus obtained from $\pi^{-1}(c_0)$ and $\pi^{-1}(c_T)$ after cutting along the annulus $\pi^{-1}(a_T)$. Therefore the two cores of the two handles of $H_2$ obtained in this way are 1) the exceptional fibre of order $p$ and 2) a section $w$ of a regular neighbourhood of the exceptional fibre of order $q$ (which is a section of the fibration induced by $\pi^{-1}(c_T)$; it intersects the boundary of $\pi^{-1}(a_T)$ in one point). Moreover, the complement of $H_2$ has the form $(S^1 \times S^1) \times [0,1]$ union a 2–handle $D^2 \times [0,1]$ which corresponds to a regular neighbourhood of a meridian disc of $\pi^{-1}(c_T)$. It is not difficult to show that this Heegaard decomposition has the diagram $(W_{q,b}(v^{-1}, s^p); \lambda, -)$. The third Heegaard decompostion is obtained by joining $\pi^{-1}(c_S)$ and $\pi^{-1}(c_0)$. For more details see [BRZ 2].

The Heegaard decomposition $HD(1,2)$ is obtained from $HD_0$ by filling in the neighbourhood of the third exceptional fibre of type $\gamma/r$, that is, one of the handlebodies consists of the regular neighbourhoods of the graph consisting of the two exceptional fibres of type $\alpha/p$ and $\beta/q$ and an "unknotted" arc joining them and the other is the closure of its complement. The decompositions $HD(2,3)$ and $HD(3,1)$ correspond to $HD_T$ and $HD_S$, respectively.

# References


[BCZ] Boileau, M.; D.J. Collins; H. Zieschang: *Scindements de Heegaard des petites variétés de Seifert.* C.R. Acad. Sci. Paris **I-305**, 557-560 (1986)

[BGM] Birman, J.S.; F. González-Acuña; J.M. Montesinos: *Heegaard splittings of prime 3−manifolds are not unique.* Mich. Math. J. **23**, 97-103 (1976)

[Bi 1] Birman, J.S.: *Braids, links, and mapping class groups.* Ann. Math. Studies **82**. Princeton, N.J.: Princeton University Press 1974

[Bi 2] Birman, J.S.: *On the equivalences of Heegaard splittings of closed orientable 3-manifolds.* Ann. Math. Studies **84**, 137-164 (1975)

[Bi 3] Birman, J.S.: *Special Heegaard splittings for closed oriented 3-manifolds.* Topology **17**, 157-166 (1978)

[BiC] Birman, J.S.; R. Craggs: *The $\mu$-invariant of 3-manifolds and certain structural properties of the group of homeomorphisms of a closed orientable surface.* Trans. Amer. Math. Soc. **237**, 283-309 (1978)

[BiH] Birman, J.S.; H. Hilden: *Heegaard splittings and branched coverings of $S^3$.* Trans. Amer. Math. Soc. **213**, 315-352 (1975)

[BiM] Birman, J.S.; J.M.Montesinos: *On minimal Heegaard splittings.* Michigan Math. J. **27**, 47-57 (1980)

[BiP] Birman, J.S.; J. Powell: *Special representations for 3-manifolds.* In: *Geometric Topology* (ed. J.C. Cantrell) pp. 23-51. New York- San Francisco-London: Academic Press 1979

[BoiO] Boileau, M.; J.-P. Otal: *Groupes des difféotopies de certaines varietés de Seifert.* C.R. Acad. Sci. Paris **303-I**, 19-22 (1986)

[BoiZ] Boileau, M.; H. Zieschang: *Heegaard genus of closed orientable Seifert 3-manifolds.* Invent. math. **76**, 455-468 (1984)

[Bon 1] Bonahon, F.: *Involutions et fibrés de Seifert dans les variétés de dimension 3.* Thèse de $3^{eme}$ cycle, Orsay 1979

[Bon 2] Bonahon, F.: *Difféotopies des espaces lenticulaires.* Toplogy **22**, 305-314 (1983)

[BonO] Bonahon, F.; J.-P. Otal: *Scindements de Heegaard des espaces lenticulaires.* Ann. scient. Éc. Norm. Sup. **16**, 451-466 (1983)

[BRZ 1] Boileau, M.; Rost, M.; H. Zieschang: *Décompositions de Heegaard des extérieurs des noeuds toriques et des variétés de Seifert associées.* C.R. Acad. Sci. Paris **302-I**, 661-664 (1986)

[BRZ 2] Boileau, M.; M. Rost; H. Zieschang: *On Heegaard decompositions of torus knot exteriors and related Seifert fibre spaces.* Math. Ann. **279**, 553-581 (1988)

[Bu] Burde, G.: *On branched coverings of $S^3$.* Canad. J. Math. **23**, 84-89 (1971)

[BuZ] Burde, G.; H. Zieschang: *Knots.* Berlin: de Gruyter 1985

[Cas] Akbulut, S.; McCarthy, J.: *Casson's invariant for oriented homology 3-spheres.* (preprint)

[CG 1] Casson, A.; C.Mc. Gordon: *Reducing Heegaard splittings.* (to appear)

[CG 2] Casson, A.; C.Mc. Gordon: *Manifolds with irreducible Heegaard splittings of arbitrarily high genus.* (to appear)

[Col] Collins, D.J.: *Presentations of the amalgamated free product of two infinite cycles.* Math. Ann. **237**, 233-241 (9178)

[Cr] Craggs, R.: *A new proof of the Reidemeister-Singer theorem on stable eqivalence of Heegaard splittings.* Proc. Amer. Math. Sco. **57**, 143-147 (1976)

[Dy] Dyck, W.: *On the "Analysis situs" of three dimensional spaces.* Brit. Assoc. Adv. Sci. Report 1884, 648

[Ed] Edmonds, A.L.: *Transformation groups and low-dimensional manifolds.* Contemp. Math. **36**, 339-366 (1985)

[Eng] Engmann, R.: *Nicht-homöomorphe Heegaard-Zerlegungen vom Geschlecht 2 der zusammenhängenden Summe zweier Linsenräume.* Abh. Math. Sem. Univ. Hamburg **35**, 33-38 (1970)

[FL] Fathi, A.; F. Laudenbach: *Difféomorphismes pseudo-Anosov et décomposition de Heegaard.* C.R. Acad. Sci. Paris **291**, Sér. A, 423-425 (1980)

[Ha 1] Haken, W.: *Various aspects of the 3-dimensional Poincaré problem.* In: Topology of manifolds, pp. 140-152. Chicago: Markham 1970

[Ha 2] Haken, W.: *Some results on surfaces in 3-manifolds.* Studies in Modern Topology, M.A.A. Studies in Math. **5**, 39-98 (1968)

[Hee] Heegaard, P.: *Forstudier til en topologisk teori för der algebraiske sammenhäng.* Köbenhavn: Det Nordiske Forlag Ernst Bojesen 1898. French transl.: *Sur l'Analysis situs.* Bull. Soc. Math. France **44**, 161-242 (1916)

[Hem] Hempel, J.: *3-Manifolds.* Annals Math. Studies **86**. Princeton, N.J.: Princeton Univ. Press 1976

[HiL] Higgins, P.L.; R.C. Lyndon: *Equivalence of elements under automorphisms of a free group.* J. London Math. Soc. **8**, 254-258 (1975)

[HR] Hodgson, C.; J.H. Rubinstein: *Involutions and isotopies of lens spaces.* Lect. Notes in Math. **1144**, 60-96 (1985)

[HOT] Homma, T.; M. Ochia; M. Takahashi: *An algorithm for recognizing $S^3$ in 3-manifolds with Heegaard splittings of genus two.* Osaka J. Math. **17**, 625-648 (1980)

[Jo] Johannson, K.: *On surfaces and Heegaard surfaces.* (to appear)

[Jon] Johnson, D.: *Quadratic forms and the Birman-Craggs homomorphisms.* Trans. Amer. Math. Soc. **261**, 235-254 (1980)

[Ka 1] Kaneto, T.: *A note on an example of Birman-Montesinos.* Proc. Amer. Math. Soc. **83**, 425/426 (1981)

[Ka 2] Kaneto, T.: *On simple loops on a solid torus of general genus.* Proc. Amer. Math. Soc. **86**, 551-552 (1982)

[Ka 3] Kaneto, T.: *On genus 2 Heegaard diagrams for the 3-sphere.* Trans. Amer. Math. Soc. **276**, 583-597 (1983)

[Kn] Kneser, H.: *Geschlossene Flächen in dreidimensionalen Mannigfaltigkeiten.* Jahresber. Deutsch. Math. Ver. **38**, 248-260 (1929)

[Ko 1] Kobayashi, T.: *Structures of full Haken manifolds.* (to appear)

[Ko 2] Kobayashi, T.: *Torus decomposition with incompressible tori intersecting a Heegaard surface in essential loops.* MSRI Preprint 13712-85

[Lau 1] Laudenbach, F.: *Topologie des variétés de dimension 3.* Exposé au Sém. de Topologie d'Orsay 1971

[Lau 2] Laudenbach, F.: *Sur les 2-sphères d'une variété de dimension 3.* Ann. of Math. **97**, 57-81 (1973)

[Lau 3] Laudenbach, F.: *Topologie de la dimension trois: homotopie et isotopie.* Astérisque **12**. Paris: Soc. Math. France 1974

[Li] Lickorish, W.B.R.: *A finite set of generators for the homeotopy group of a 2-manifold.* Proc. Cambridge Phil. Soc. **60**, 769-778 (1964); Errata: Proc. Cambridge Phil. Soc. **62**, 679-681 (1966)

[LS] Lyndon, R.C.; Schupp, P.E.: *Combinatorial Group Theory.* Berlin-Heidelberg-New York: Springer 1977

[Mc] McCool, J.: *A presentation for the automorphism group of a free group of finite rank.* J. London Math. Soc. **8**, 259-266 (1974)

[McP] McCool, J.; A. Pietrowski: *On free products with amalgamation of two infinite cyclic groups.* J. Algebra **18**, 377-383 (1971)

[MKS] Magnus, W.; Karrass, A.; Solitar,D.: *Combinatorial Group Theory: Presentations of Groups in Terms of Generators and Relations.* New York: Wiley 1966

[Mi] Milnor, J.: *A unique decomposition theorem for 3−manifolds.* Amer. J. Math **84**, 1-7 (1962)

[Mo 1] Montesinos, J.M.: *Variedades de Seifert que son recubridores ciclicos ramificados de dos hojas.* Bol. Mat. Mexicana **18**, 1-32 (1973)

[Mo 2] Montesinos, J.M.: *Minimal plat presentations of prime knots and links are not unique.* Canad. J. Math. **28**, 161-167 (1976)

[Mo 3] Montesinos, J.M.: *Revetements ramifies de noeuds, espaces fibres de Seifert et scindement de Heegaard.* Prepublication Orsay 1978

[Mo 4] Montesinos, J.M.: *Lectures on branched coverings.* In: *Atti del Conveg-no-Studio: Sulla Geometria delle Variet Differnziabili* (ed. I. C. Gasparini), pp. 127-167. Bologna: Pitagora Editrice Bologna 1985

[Mo 5] Montesinos, J.M.: *Note on a result of Boileau-Zieschang.* Durham Symposium 1984 (to appear)

[Mo 6] Montesinos, J.M.: *Genero de Heegaard y rango de una 3-variedad.* (to appear)

[Mo 7] Montesinos, J.M.: *Two questions on Heegaard diagrams of $S^3$.* (to appear)

[Mok] Morikawa, O.: *A counterexample to a conjecture of Whitehead.* Math. Sem. Notes Kobe Univ. **8**, 295-299 (1980)

[Mor 1] Moriah, Y.: *Heegaard splittings and group presentations.* Ph.D. thesis, University of Texas, Austin 1986 and Preprint

[Mor 2] Moriah, Y.: *Heegaard splittings of Seifert fibered spaces.* Invent. math. (to appear)

[MoS] Montesinos, J.M.; C. Safont: *On the Birman invariants of Heegaard splittings.* MSRI Preprint 14012-85

[MoW] Montesinos, J.M.; W. Whitten: *Construction of two-fold branched covering spaces.* (to appear)

[Nie 1] Nielsen, J.: *Die Isomorphismen der allgemeinen, unendlichen Gruppe mit zwei Erzeugenden.* Math. Ann. **78**, 385-397 (1918)

[Nie 2] Nielsen, J.: *Über die Isomorphismen unendlicher Gruppen ohne Relation.* Math. Ann. **79**, 269-272 (1919)

[Nie 3] Nielsen, J.: *Die Isomorphismengruppen der freien Gruppen.* Math. Ann. **91**, 169-209 (1924)

[Oc 1] Ochiai, M.: *Heegaard splittings of $F \times S^1$.* Yokohama Math. J. **25**, 109-112 (1977)

[Oc 2] Ochiai, M.: *A counterexample to a conjecture of Whitehead and Volodin-Kuznetsov-Fomenko.* J. Math. Soc. Japan **31**, 687-691 (1979)

[Oc 3] Ochiai, M.: *On Haken's theorem and its extension.* Osaka J. Math. **28**, 461-468 (1983)

[Or] Orlik, P.: *Seifert manifolds.* Lecture Notes in Math. **291**. Berlin-Heidelberg-New York: Springer-Verlag 1972

[Os] Osborne, R.P.: *Heegaard diagrams of lens spaces.* Proc. Amer. Math. Soc. **84**, 412-414 (1982)

[Ot] Otal, J.-P.: *Sur les scindements de Heegaard de la sphere $S^3$.* (Preprint Orsay)

[OVZ] Orlik, P.; E. Vogt; H. Zieschang: *Zur Topologie gefaserter dreidimensionaler Mannigfaltigkeiten.* Topology **6**, 49-64 (1967)

[OZ] Osborne, R.P.; H. Zieschang: *Primitives in the free group on two generators.* Invent. math. **63**, 17-24 (1981)

[Poi] Poincaré, H.: *Cinquième complément à l'analysis situs.* Rend. Circ. Mat. Palermo **18**, 45-110 (1904)

[PRZ] Peczynski, N.; G. Rosenberger; H. Zieschang: *Über Erzeugende ebener diskontinuierlicher Gruppen.* Invent. math. **63**, 161-180 (1975)

[Rei 1] Reidemeister, K.: *Zur dreidimensionalen Topologie.* Abh. Math.Sem. Univ. Hamburg **9**, 189-194 (1933)

[Rei 2] Reidemeister, K.: *Homotopieringe und Linsenräume.* Abh. Math. Sem. Univ. Hamburg **11**, 102-109 (1935)

[Rei 3] Reidemeister, K.: *Über Heegaard Diagramme.* Abh. Math. Sem. Univ. Hamburg **25**, 140-145 (1961/62)

[Ro] Rolfsen, D.: *Knots and Links.* Math. Lect. Series. Berkeley, Cal.: Publish or Perish, Inc. 1976

[Sch] Schubert, H.: *Knoten mit zwei Brücken.* Math. Z. **65**, 133-170 (1956)

[Sco] Scott, P.: *The geometries of 3-manifolds.* Bull. London Math. Soc. **15**, 401-487 (1983)

[Se] Seifert, H.: *Topologie dreidimensionaler gefaserter Räume.* Acta Math. **60**, 147-238 (1933)

[Sie] Siebenmann, L.: *Introduction aux espaces fibres de Seifert.* Notes multicopis, Orsay 1977-1978

[Sin] Singer, J.: *Three-dimensional manifolds and their Heegaard diagrams.* Trans. Amer. Math. Soc. **35**, 88-111 (1933)

[ST] Seifert, H.; W. Threlfall: *Lehrbuch der Topologie.* Leipzig: Teubner 1934

[Sta] Stallings, J.: *How not to prove the Poincaré conjecture.* Ann. Math. Studies **60**, 83-88. Princeton, N.J.: Princeton University Press 1966

[TO] Takahashi, M.; M. Ochiai: *Heegaard diagrams of torus bundles over $S^1$.* Comment. Math. Univ. Sancti Pauli **31**, 63-69 (1982)

[To] Tollefson, J.L.: *Involutions of Seifert fiber spaces.* Pacific J. Math. **74**, 519-529 (1978)

[Thu 1] Thurston, W.P.C.: *Three dimensional manifolds, Kleinian groups and hyperbolic geometry.* Bull. Amer. Math. Soc. **6**, 357-381 (1982)

[Thu 2] Thurston, W.P.C.: *The geometry and topology of 3−manifolds.* (in preparation, Princeton University)

[Vi] Viro, O.Ya.: *Linkings, two-sheeted branched coverings and braids.* Math. USSR-Sbornik **16**, 223-236 (1972). Engl transl. of: Mat. Sbornik **87**, 216-228 (1972)

[ViK] Viro, O.Ya; V.L. Kobel'skii: *The Volodin-Kuznetsov-Fomenko conjecture on Heegaard diagrams is false.* Uspechi Mat. Nauk **32**:5, 175-176 (1977)

[VKF] Volodin, I.A.; V.E. Kuznetsov; A.T. Fomenko: *The problem of discriminating algorithmically the standard three-dimensional sphere.* Russian Math. Surveys **29**:5, 71-172 (1974) (Engl. transl. of: Uspechi Mat. Nauk **29**:5, 71-168(1974))

[Whi 1] Whitehead, J.H.C.: *On certain sets of elements in a free group.* Proc. London Math. Soc. **41**, 48-56 (1936)

[Whi 2] Whitehead, J.H.C.: *On equivalent sets of elements in a free group.* Ann. of Math. **37**, 782-800 (1936)

[Wa 1] Waldhausen, F.: *Heegaard-Zerlegungen der 3-Sphäre.* Topology **7**, 195-203 (1968)

[Wa 2] Waldhausen, F.: *Some problems on 3-manifolds.* Proc. Symposia in Pure Math. **32** (part 2), 313-322 (1978)

[Zi 1] Zieschang, H.: *Über einfache Kurven auf Vollbrezeln.* Abh. Math. Sem. Univ. Hamburg **25**, 231-250 (1962)

[Zi 2] Zieschang, H.: *Über einfache Kurvensysteme auf einer Vollbrezel vom Geschlecht 2.* Abh. Math. Sem. Univ. Hamburg **26**, 237-247 (1963)

[Zi 3] Zieschang, H.: *Classification of simple systems of paths on a solid pretzel of genus 2.* Soviet Math. **4**, 1460-1463 (1963). Transl. of Doklady Acad. Sci. USSR **152**, 841-844 (1963)

[Zi 4] Zieschang, H.: *Alternierende Produkte in freien Gruppen.* Abh. Math. Sem. Univ. Hamburg **27**, 13-31 (1964)

[Zi 5] Zieschang, H.: *On simple systems of paths on complete pretzels.* Amer. Math. Soc. Transl. (2) **92**, 127-137 (1970). Transl. of Mat. Sbornik **66**, 230-239 (1965)

[Zi 6] Zieschang, H.: *Über die Nielsensche Kürzungsmethode in freien Produkten mit Amalgam.* Invent. math. **10**, 4-37 (1970)

[Zi 7] Zieschang, H.: *Generators of the free product with amalgamation of two infinite cyclic groups.* Math. Ann. **277**, 195- 221 (1977)

[Zi 8] Zieschang, H.: *Classification of Montesinos knots.* In: Topology Conference Leningrad, 1982 (ed. L.D. Faddeev + A.A. Mal'cev). Lecture Notes in Math. **1060**, 378-389 (1984)

[ZVC] Zieschang, H.; E. Vogt; H.-D. Coldewey: *Surfaces and Planar Discontinuous Groups.* Lecture Notes in Math. **835**. Berlin-Heidelberg- New York: Springer 1980

[ZZ] Zieschang, H.; B. Zimmermann: *Endliche Gruppen von Abbildungsklassen.* Math. Ann. **240**, 41-62 (1979)