Astérisque

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Astérisque, tome 168 (1988), p. 157-166

<http://www.numdam.org/item?id=AST_1988__168__157_0>

Société Mathématique de France
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On the representations of reductive groups with disconnected centre

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1. We consider a connected reductive algebraic group defined over a finite field $\mathbb{F}_q$ with Frobenius map $F : G \to G$.
Let $\hat{G}^F$ denote the set of irreducible representations up to isomorphism of the finite group G^F over $\overline{\mathbb{Q}}_\ell$ (ℓ is a prime not dividing q). In the case where the centre Z_G of G is connected, a parametrization for $\hat{G}^F$ was given in [4] ; this is extended here to the general case (i.e. we allow Z_G to be disconnected). The proof will be by a reduction to the case where Z_G is connected using a method in [4, 14.1]. The results of this paper were obtained during the summer of 1983 and were announced in [5].

2. We denote by $G^\star$ a connected reductive group defined over $\mathbb{F}_q$, dual to G, as in [2]. We again denote by F the corresponding Frobenius map. (The same notation will be used for the Frobenius map of any algebraic variety defined over $\mathbb{F}_q$). As in [2] we have a natural bijection

$$\{(T',\theta)\} \bmod G^F\text{-conjugacy} \leftrightarrow \{(T,s)\} \bmod G^{\star F}\text{-conjugacy}$$

where T' (resp. T) runs over the F-stable maximal torus of G (resp. $G^\star$), $\theta : T'^F \to \overline{\mathbb{Q}}_\ell^\star$ is a character and s is an element of T^F. If (T',θ), (T,s) correspond in this way we consider the virtual representation $R_{T'}^G(\theta)$ defined in [2]; we shall also write $R_T^G(s)$ instead of $R_{T'}^{G'}(\theta)$.

For a semisimple element $s \in G^{\star F}$, let $(\hat{G}^F)_s$ be the set of all $\rho \in \hat{G}^F$ appearing with non-zero multiplicity in $R_T^G(s)$ for some F-stable maximal torus $T \subset G$.

The subset $(\hat{G}^F)_s$ of $\hat{G}^F$ depends only on the $G^{\star F}$-conjugacy class of s. We

★ Supported in part by the National Science Foundation.

have a partition $\hat{G}^F = \bigsqcup_s (\hat{G}^F)_s$ where s runs over the semisimple elements of $G^{\star F}$ up to $G^{\star F}$-conjugacy. (See [2], [3]).

3. Let $\pi : G \to G_{ad}$ be the adjoint quotient of G. We have a natural isomorphism

$$G_{ad}^F/\pi(G^F) \cong (Z_G/Z_G^o)_F$$

(the subscript F denotes F-coinvariants, i.e. largest quotient on which F acts trivially) ; it is defined by the correspondence $G_{ad}^F \ni g \to \dot{g}^{-1}F(\dot{g}) \in Z_G$, where $\dot{g} \in G$ satisfies $\pi(\dot{g}) = g$.

The group G_{ad}^F acts naturally on G^F by automorphisms $g : g_1 \to \dot{g}g_1\dot{g}^{-1}$ ($g,\dot{g}$ as above). Hence G_{ad}^F acts naturally on $\hat{G}^F$. Clearly this action is trivial on the subgroup $\pi(G^F)$ hence it induces an action of $G_{ad}^F/\pi(G^F)$ on $\hat{G}^F$. It is easy to check that this action (extended by linearity to virtual representations) leaves fixed each $R_T^G(s)$; hence it leaves stable each subset $(\hat{G}^F)_s$ of $\hat{G}^F$. We have thus defined an action of $(Z_G/Z_G^o)_F$ on $(\hat{G}^F)_s$.

4. We fix a semisimple element $s \in G^{\star F}$ and we denote $H = Z_{G^\star}(s)$. If $x \in H^F$, then conjugation by x is an automorphism of H^o (over $\mathbb{F}_q$) ; hence it defines an automorphism of $\hat{H}^{oF}$ which leaves stable the set $(\hat{H}^{oF})_1$ of unipotent representations. If $x \in H^{oF}$, the corresponding automorphism of $(\hat{H}^{oF})_1$ is trivial, so we have a natural action of H^F/H^{oF} on $(\hat{H}^{oF})_1$.

5. With these notations, we can now state our main result.

Proposition 5.1. There exists a surjective map $\psi : (\hat{G}^F)_s \to (\hat{H}^{oF})_1$ mod action of H^F/H^{oF} with the following properties.

The fibres of ψ are precisely the orbits of the action of $(Z_G/Z_G^o)_F$ on $(\hat{G}^F)_s$, (see Sec.3). If Θ is a H^F/H^{oF}-orbit on $(\hat{H}^{oF})_1$ and Γ is the stabilizer in H^F/H^{oF} of an element in Θ, then the fibre $\psi^{-1}(\Theta)$ has precisely $|\Gamma|$ elements. If $\rho \in \psi^{-1}(\Theta)$ and T is an F-stable maximal torus of $G^\star$ containing s, then

$$(\rho : R_T^G(s))_{G^F} = \varepsilon_G \varepsilon_H \sum_{\bar\rho \in \Theta} (\bar\rho : R_T^{H^o}(1))_{H^{oF}} .$$

Here (:) denotes the standard inner product of virtual representations and $\varepsilon_G = (-1)^{\sigma(G)}$, $\sigma(G) = \mathbb{F}_q$-rank of G; in $R_T^{H^o}(1)$, 1 stands for the trivial character of T^F.

Remarks. The sets $(\hat{H}^{oF})_1$ are described explicitly in [4]; they are insensitive to the centre of H^o. The action of outer automorphisms on the set of unipotent representations of a connected reductive group is easy to describe; for example when that group is simple modulo its centre, of type $\neq D_{2n}$, this action is trivial. The multiplicities $(\bar\rho : R_T^H(1))_{H^F}$ are also described explicitly in [4]. Hence the proposition gives an explicit parametrization of $(\hat{G}^F)_s$ and explicit formulas for the multiplicities $(\rho : R_T^G(s))_{G^F}$.

6. Now let G' be a connected reductive group over $\mathbb{F}_q$ with connected centre. Let $s' \in G'^{\star F}$ be semisimple and let $H' = Z_{G'^\star}(s')$. Then both groups $(Z_{G'}/Z^o_{G'})_F$ and H'^F/H'^{oF} are trivial and from 5.1 we obtain the following known result.

Corollary 6.1. [4]. If $Z_{G'}$ is connected, then there exists a bijection $\psi : (\hat{G}'^F)_{s'} \longrightarrow (\hat{H}'^{oF})_1$ such that $(\rho' : R_{T'}^{G'}(s'))_{G'^F} = \varepsilon_{G'}\varepsilon_{H'}(\bar\rho' : R_{T'}^{H'^o}(1))_{H'^{oF}}$ for any $\rho' \in (\hat{G}'^F)_{s'}$ and any F-stable maximal torus T' of G' containing s'; here $\bar\rho = \psi(\rho)$.

7. If G is as in Sec. 1, we say that $i : G \to G'$ is a regular imbedding if G' is a connected reductive group over $\mathbb{F}_q$ with connected centre, i is an isomorphism of G with a closed subgroup of G' and $i(G)$, G' have the same derived subgroup.

We shall need the following simple result.

Lemma 7.1. [1, 2.3.2]. If G is semisimple and $i : G \to G'$, $\bar{i} : G \to \bar{G}$ are regular imbeddings, then there exists a connected reductive group G'' over $\mathbb{F}_q$ and regular imbeddings $j : G' \to G''$, $\bar{j} : \bar{G} \to G''$ such that $j \circ i = \bar{j} \circ \bar{i}$.

8. With G as in Sec. 1, we fix a regular imbedding $G \to G'$. To this corresponds by duality a surjective homomorphism $\delta : G'^{\star} \to G^{\star}$ (over $\mathbb{F}_q$) whose kernel K is a central torus in $G'^{\star}$. We have a natural isomorphism $K^F \xrightarrow{\sim} \mathrm{Hom}(G'^F/G^F, \bar{\mathbb{Q}}_{\ell}^{\star})$, $k \mapsto \theta_k$. We consider the action of K^F on $\hat{G}'^F$ given by $k : \rho' \to \rho' \otimes \theta_k$. The action of $k \in K^F$ on $\hat{G}'^F$ defines a bijection $(\hat{G}'^F)_{s_1} \xrightarrow{\sim} (\hat{G}'^F)_{ks_1}$ for any semisimple $s_1 \in G'^{\star F}$.

Now let $s' \in G'^{\star F}$ be semisimple, $H' = Z_{G'^{\star}}(s')$. Let $K^F_{s'}$ be the set of all $k \in K^F$ which map $(\hat{G}'^F)_{s'}$ into itself or, equivalently,

$$K^F_{s'} = \{k \in K^F | ks' \text{ is conjugate to } s' \text{ under } G'^{\star F}\}.$$

If $s = \delta(s') \in G^{\star F}$, and $H = Z_{G^{\star}}(s)$ we have a natural isomorphism $H^F/H^{OF} \simeq K^F_{s'}$ defined by the correspondence $H^F \ni x \to s'^{-1}\dot{x}s'\dot{x}^{-1} \in K^F$ where $\dot{x} \in G'^{\star F}$ satisfies $\delta(\dot{x})=x$. (Note that $\delta : G'^{\star F} \to G^{\star F}$ is surjective). Using this isomorphism the action of $K^F_{s'}$ on $(\hat{G}'^F)_{s'}$ becomes an action of H^F/H^{OF} on $(\hat{G}'^F)_{s'}$. Now δ defines a surjective homomorphism $H'^O \to H^O$ with kernel K, hence a bijection $(\hat{H}^{OF})_1 \xrightarrow{\sim} (\hat{H}'^{OF})_1$. Using this, the action of H^F/H^{OF} on $(\hat{H}^{OF})_1$ in Sec 4 becomes an action on $(\hat{H}'^{OF})_1$. We shall need the following strengthening of 6.1.

<u>Proposition 8.1.</u> <u>The isomorphism ψ in 6.1 can be chosen to be compatible with the action of H^F/H^{OF} on $(\hat{G}'^F)_{s'}$ and $(\hat{H}'^{OF})_1$ defined above</u>.

<u>Proof</u>. (a) Assume first that G is almost simple, simply connected. If G is a classical group, then ψ in 6.1 is uniquely determined ; in the remaining cases (with one exception) either H is connected (and there is nothing to prove) or ψ is uniquely determined. In these cases the result follows easily. The exception is : G of type E_7, $s \in G^{\star F}$ is such that $H = Z_{G^{\star}}(s)$ has two components and H^O modulo its centre is of type E_6. There are two representations in $(\hat{G}'^F)_{s'}$ which are not distinguished by their multiplicities in the $R^{G'}_{T'}(s')$. We must show that they are in the same orbit of $H^F/H^{OF} \cong \mathbb{Z}/2$. If they are not, they would remain irreducible on restriction to G'^F. But their restrictions to G'^F are reducible by an argument in [4, p.353] .

(b) Assume next that $G = G_1 \times G_2 \times \ldots \times G_n$ with almost simple, simply connected factors G_i permuted by F and that $G' = G'_1 \times G'_2 \times \dots \times G'_n$ where $G_i \to G'_i$ are regular imbeddings over an extension of $\mathbb{F}_q$ and the G_i are again permuted by F. In this case we group together the factors in the various orbits of F and we are reduced to the case where F permutes cyclically the indices. In that case we have $G^F = G_1^{F^n}$, $G'^F = G'^{F^n}$ and the result follows by applying (a) to G_1 and G'_1 instead of G and G'.

(c) Assume that G is simply connected. We decompose G in a product $G_1 \times G_2 \times \dots \times G_n$ as in (b) ; we imbed it in $\overline{G} = \overline{G}_1 \times \dots \times \overline{G}_n$ where $G \to \overline{G}$ is like $G \to G'$ in (b). Let $G' \to G''$, $\overline{G} \to G''$ be as in 7.1. We can find $s'' \in G''^{\star F}$ which maps to $s' \in G'^{\star F}$ and to some element $\overline{s} \in \overline{G}^{\star F}$ under $G'^{\star} \leftarrow G''^{\star} \to \overline{G}^{\star}$. Since G', G'', $\overline{G}$ have connected centre, we get by restriction bijections $(\hat{G}'^F)_{s'} \xleftarrow{\approx} (\hat{G}''^F)_{s''} \xrightarrow{\approx} (\hat{\overline{G}}^F)_{\overline{s}}$. There are compatible with the actions of H^F/H^{oF}. Since the case $(G \to \overline{G}, \overline{s})$ is handled by (b), the cases $(G \to G'', s'')$ and $(G \to G', s')$ follow.

(d) Assume that G is the derived group of G'. We can find a connected reductive group $\tilde{G}'$ over $\mathbb{F}_q$ with simply connected derived group $\tilde{G}$ and a surjective homomoprhism $\tilde{G}' \to G'$ (over $\mathbb{F}_q$) whose kernel is a central torus in $\tilde{G}'$. Then $\tilde{G}'$ has connected centre and $\tilde{G}' \to G'$ restricts to a finite covering $\tilde{G} \to G$. We have $G'^{\star} \subset \tilde{G}'^{\star}$ hence s' can be considered as an element of $\tilde{G}'^{\star F}$. Let $\tilde{s}$ be the image of $s \in G^{\star F}$ under the finite covering $G^{\star} \to \tilde{G}^{\star}$. Let $\tilde{H} = Z_{\tilde{G}^{\star}}(\tilde{s})$. We have a natural imbedding $H^F/H^{oF} \to \tilde{H}^F/\tilde{H}^{oF}$, induced by $G^{\star} \to \tilde{G}^{\star}$. Composition with $\tilde{G}'^F \to G'^F$ defines a bijection $(\hat{G}'^F)_{s'} \xrightarrow{\sim} (\hat{\tilde{G}}'^F)_{s'}$. Applying (c) to $(\tilde{G} \to \tilde{G}', s')$ we deduce the desired result for $(G \to G', s')$.

(e) We now consider the general case. Let G'' be the derived group of G. Let s'' be the image of $s \in G^{\star F}$ under $G^{\star} \to G''^{\star}$ and let $H'' = Z_{G''^{\star}}(s'')$. We have a natural imbedding $H^F/H^{oF} \to H''^F/H''^{oF}$. Applying (d) to $(G'' \to G', s')$ we deduce the desired result for $(G \to G', s')$. This completes the proof.

9. Let $A \subset B$ be finite groups such that A is normal in B and B/A is abelian. Then the abelian group B/A acts naturally on $\hat{A}$ (this is induced by the action of B on A by conjugation) and the abelian group $\widehat{B/A}$ acts naturally

on $\hat{B}$ by tensor product. The proofs of the results in this section are standard, and will be ommitted.

(a) Assume that any $\rho \in \hat{B}$ restricts to a multiplicity free representation of A. Then there is a unique bijection

$$\hat{A} \text{ mod action of } B/A \leftrightarrow \hat{B} \text{ mod action of } \widehat{B/A}$$

with the following properties. Let Θ be a B/A-orbit on $\hat{A}$ and let Θ' be the corresponding $\widehat{B/A}$-orbit on $\hat{B}$. Then if $\rho_0 \in \Theta'$, we have $\rho_0|A = \sum_{\tau\in\Theta} \tau$; if $\tau_0 \in \Theta$, we have $\mathrm{ind}_A^B\ \tau_0 = \sum_{\rho\in\Theta'} \rho$. Moreover, the stabilizer of ρ_0 in $\widehat{B/A}$ and the stabilizer of τ_0 in B/A are orthogonal to each other under the natural duality $B/A \times \widehat{B/A} \to \bar{\mathbb{Q}}_\ell^\star$.

We now want to find conditions which should imply that the assumptions of (a) holds.

(b) If B/A is cyclic then the assumption of (a) is automatically satisfied.

(c) Assume now that any $\rho \in \hat{B}$ has stabilizer I_ρ in $\widehat{B/A}$ isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ or $\mathbb{Z}/2\mathbb{Z}$ or $\{e\}$. Let $\hat{B}' = \{\rho \in \hat{B} | \rho|A$ is multiplicity free$\}$, $\hat{B}'' = \hat{B} - \hat{B}'$, $\hat{A}' = \{\tau \in \hat{A} | \tau$ appears in $\rho|A$ some $\rho \in \hat{B}\}$, $\hat{A}'' = \hat{A} - \hat{A}'$. Then the conclusions of (a) hold if $\hat{A},\hat{B}$ are replaced by $\hat{A}',\hat{B}'$. If $\rho \in \hat{B}''$ then $I_\rho = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ and $\rho|A = 2\tau$, $\tau \in \hat{A}''$; moreover $\mathrm{Ind}_A^B\ \tau = 2\rho$ and $\rho \mapsto \tau$ is a bijection $\hat{B}'' \xrightarrow{\sim} \hat{A}''$.

Let $x_i = \#\{\rho \in \hat{B} | |I\rho| = i\}$, $(i = 1,2,4)$ and $y = \#\hat{B}''$. Then $|\hat{B}| = x_1+x_2+x_4$, $|\hat{A}| = \frac{x_1}{p} + 4\frac{x_2+y}{p} + 16\frac{x_4-y}{p}$, $(p = |B/A|)$. Hence if we assume also that $|\hat{A}| = \frac{x_1}{p} + 4\frac{x_2}{p} + 16\frac{x_4}{p}$, then $y = 0$, so that the assumption of (a) is again satisfied.

10. <u>Proposition.</u> <u>Let</u> $G \subset G'$ <u>be a regular imbedding</u> (Sec. 7). <u>For any</u> $\rho' \in \hat{G}'^F$, <u>the restriction</u> $\rho'|G^F$ <u>is multiplicity free.</u>

<u>Proof</u>. a) Assume first that G is almost simple, simply connected and that $\dim Z_{G'} \leqslant 1$ except that $\dim Z_{G'} = 2$ when $G = \mathrm{Spin}_{4n}$ and char $\mathbb{F}_q \neq 2$. If $\dim Z_{G'} \leqslant 1$, or if $G = \mathrm{Spin}_{4n}$ is non-split over $\mathbb{F}_q$ (of odd characteristic) then G'^F/G^F is cyclic

and we may use 9(b). If $G = \mathrm{Spin}_{4n}$ is split over F_q of odd characteristic we use 9(c) as follows. First note that in this case the set $\hat{G}'^F$ and the action of $(G'^F/G^F)^\wedge$ on it are determined explicity by Proposition 8.1. From this we can compute explicitly the numbers x_1, x_2, x_4 in 9(c) for $A = G^F$, $B = G'^F$. On the other hand we can count directly the number of conjugacy classes in the split $\mathrm{Spin}_{4n}(F_q)$. This is the same as $|\hat{A}|$. We then compare $|\hat{A}|$ and $\frac{x_1}{p} + 4\frac{x_2}{p} + 16\frac{x_4}{p}$ $(p = |B/A|)$ and find that they are equal. We can apply 9(c) and we see that the proposition holds.

(b) Assume next that G is almost simple, simply connected but there is now no restriction on $\dim Z_{G'}$. We can find a regular imbedding $G \to \bar{G}$ which is like $G \to G'$ in (a). Let $G' \to G''$, $\bar{G} \to G''$ be as in 7.1. We have natural surjective maps $\hat{G}'^F \leftarrow \hat{G}''^F \to \hat{\bar{G}}^F$ defined by restriction. Let $\rho' \in \hat{G}'^F$ and let $\rho'' \in \hat{G}''^F$ be such that $\rho''|G'^F = \rho'$. Let $\bar{\rho} = \rho''|\bar{G}^F$. By (a), $\bar{\rho}|G^F$ is multiplicity free ; the restrictions $\rho'|G^F$, $\bar{\rho}|G^F$ coincide, hence $\rho'|G^F$ is multiplicity free.

(c) Assume that G is simply connected. Let $G \to \bar{G} \to G''$, $G' \to G''$ be as in 8(c). Arguing as in (b) we see that we can replace G' by $\bar{G}$ in which case we can use the case (b).

(d) Assume that G is semisimple. Let $\begin{matrix} \tilde{G} & \to & \tilde{G}' \\ \downarrow & & \downarrow \\ G & \to & G' \end{matrix}$ be as in 8(d). Then $\tilde{\rho}' = \rho'|\tilde{G}'^F$ is irreducible since $\tilde{G}'^F \to G'^F$ is surjective. By (c), $\tilde{\rho}'|\tilde{G}^F$ is multiplicity free. But $(\rho'|G^F)|\tilde{G}^F = \tilde{\rho}'|\tilde{G}^F$. Hence $\rho'|G^F$ is multiplicity free.

(e) We now consider the general case. Let G'' be the derived group of G. By (d), $\rho'|G''^F$ is multiplicity free. But $(\rho'|G^F)|G''^F = \rho'|G''^F$ hence $\rho'|G^F$ is multiplicity free. This completes the proof.

11. <u>Proof of Proposition 5.1.</u> Let $G \to G'$ be a regular imbedding (Sec.7), and let $s' \in G'^{\star F}$ be an element which maps to $s \in G^{\star F}$ under the corresponding homomorphism $G'^{\star} \to G^{\star}$. Let $H' = Z^0_{G'^{\star}}(s')$.

The action of G'^F/G^F on $\hat{G}^F$ (see Sec.9) factors through the action of

$G_{ad}^F/\pi(G^F)$ in Sec.3. (We have a natural surjective homomorphism $G'^F/G^F \to G_{ad}^F/\pi(G^F)$). Hence in the statement of 5.1. we can replace "orbits of $(Z_G/Z_G^{\circ})_F$ "by "orbits of G'^F/G^F ". The map ψ in the proposition is defined as the composition

$$\begin{array}{ll}
(\hat{G}^F)_s & \\
\downarrow & \\
(\hat{G}^F)_s \text{ mod action of } G'^F/G^F & \\
\downarrow & \text{, see Sec. 10 and 9(a)} \\
\bigcup_{k\in K^F} (\hat{G}'^F)_{s'k} \text{ mod action of } K^F \cong (G'^F/G^F)\hat{} & \\
\downarrow & \text{, see Sec. 8} \\
(\hat{G}'^F)_{s'} \text{ mod action of } K_{s'}^F = H^F/H^{\circ F} & \\
\downarrow & \text{, see Sec. 8.1} \\
(\hat{H}'^{\circ F})_1 \text{ mod action of } H^F/H^{\circ F} & \\
\downarrow & \text{, see Sec. 8} \\
(\hat{H}^{\circ F})_1 \text{ mod action of } H^F/H^{\circ F}. &
\end{array}$$

The properties of ψ follow easily from 9(a) ; for the multiplicity formula we use that :

$$R_T^G(s) = R_{T'}^{G'}(s')|G^F \qquad (T' = \text{inverse image of } T \subset G'^* \to G^*)$$

$$\mathrm{ind}_{G^F}^{G'^F}(\rho) = \sum_{\rho'\in\Theta'} \rho' \ + \text{representation of } G'^F \text{ outside } (\hat{G}'^F)_{s'} ,$$

where $\Theta' \subset (\hat{G}'^F)_{s'}$ is the $H^F/H^{\circ F}$-orbit determined by ρ. Θ' corresponds to a $H^F/H^{\circ F}$ orbit Θ_1' on $(\hat{H}'^{\circ F})_1$ and to a $H^F/H^{\circ F}$ orbit Θ on $\hat{H}^{\circ F})_1$. We have

$$\begin{aligned}
(\rho : R_T^G(s))_{G^F} &= (\rho : R_{T'}^{G'}(s')|G^F)_{G^F} \\
&= (\mathrm{ind}_{G^F}^{G'^F}(\rho) : R_{T'}^{G'}(s'))_{G'^F} \\
&= (\sum_{\rho'\in\Theta'} \rho' : R_{T'}^{G'}(s'))_{G'^F}
\end{aligned}$$

$$= \varepsilon_{G'}\varepsilon_{H'} \sum_{\rho_1' \in \Theta_1'} (\rho_1' : R_{T'}^{H'^{\circ}}(1))_{H'^{\circ F}}$$

by 6.1, 8.1,

$$= \varepsilon_G \varepsilon_H \sum_{\bar{\rho} \in \Theta} (\bar{\rho} : R_T^{H^{\circ}}(1))_{H^{\circ F}} .$$

This completes the proof.

12. In the setup of Sec.4, we say that an irreducible representation of H^F is unipotent if its restriction to $H^{\circ F}$ is a sum of unipotent representations of $H^{\circ F}$. Let $(\hat{H}^F)_1$ be the set of unipotent representations of H^F. It is easy to see that 9(a) (for $H^{\circ F} \subset H^F$) provides a surjective map

$$\psi' : (\hat{H}^F)_1 \to (\hat{H}^{\circ F})_1 \text{ mod action of } H^F/H^{\circ F}$$

with the following property : the fibres of ψ' and ψ over the same point have the same cardinal. Hence there exists a bijection

$$\psi'' : (\hat{G}^F)_s \xrightarrow{\sim} (\hat{H}^F)_1$$

such that $\psi = \psi' \circ \psi''$.

13. The parametrizations of $\hat{G}^F$ considered here and in [4] are to some extent non-canonical. It is likely that these will be canonical when they will be related to character sheaves. Note also that the crucial part of our proof (the multiplicity 1 statement in Sec. 10 for $Spin_{4n}(\mathbb{F}_q)$) involves some very long and unpleasant computations of the number of conjugacy classes and unpleasant computations of the number of conjugacy classes and irreducible representations of $Spin_{4n}(\mathbb{F}_q)$. Although these computations give the desired results, they don't show why the result holds. One can give a somewhat more satisfactory proof, using character sheaves on $Spin_{4n}$.

14. The parametrization of $\hat{G}^F$ given in [5] is in terms of a dual group over $\mathbb{C}$ rather than over $\bar{\mathbb{F}}_q$; however that parametrization is equivalent to the one given here.

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