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Introduction

These notes on character sheaves are an outgrowth of a seminar at the University of Utrecht (in 1985-1987) and lectures by one of us in Paris (in June 1987). The aim of the notes is to give an exposition of some of the main ideas of Lusztig's theory of character sheaves (contained in the five papers cited as [CS] in the references). His aim is to give a geometric theory of "characters" of a connected reductive algebraic group G over an arbitrary algebraically closed field k.
The "characters" in question are the character sheaves. They are certain perverse sheaves on G. We have to assume familiarity with these objects (treated at length in [BBD]). In no.1 some basic facts and auxiliary results about perverse sheaves are discussed very briefly.
In no.2 we introduce certain local systems on an algebraic torus T (up to a dimension shift these are the character sheaves on T). If X is the character group of T, the isomorphism classes of such local systems can be identified with the elements of $\hat{X} = X \otimes \mathbb{Q}/X$ with denominator prime to the characteristic exponent of the underlying field. If T is a maximal torus in a reductive group G, the Weyl group W of (G,T) operates on $\hat{X}$. If $\xi \in \hat{X}$ denote by W'_ξ the isotropy group of ξ in W.
No.3 contains a number of auxiliary algebraic results, for example about the groups W'_ξ. We have found it useful to introduce a generalized Hecke algebra $\mathcal{K} = \mathcal{K}_{\mathcal{O}}$, associated to a W-orbit $\mathcal{O}$ in $\hat{X}$ and a set of generators S of the Coxeter group W. It is an algebra over the ring of Laurent polynomials $\mathbb{Z}[t, t^{-1}]$, with a basis $(e_{\xi,w})_{\xi\in\mathcal{O},w\in W}$.
The multiplication rules generalize those for the usual Hecke algebra of (W, S), which one recovers if $\xi = \{0\}$ (see 3.3.1 for these formulas). The $e_{\xi,w}$ with fixed ξ and $w\xi = \xi$ span a subalgebra $\mathcal{H}'_\xi$ of $\mathcal{K}$ which is isomorphic to the Hecke algebra of groups like W'_ξ introduced in [CS, no.6]. We establish in no.3 some basic properties of the algebras $\mathcal{K}$ and introduce their Kazhdan-Lusztig elements $c_{\xi,w}$.
In no.4 we deal with material contained in Lusztig's book [L1], which is needed in the theory of character sheaves. Let B be a Borel subgroup of G containing the maximal torus T. The double coset $BwB (w \in W)$ is isomorphic as a variety to the product of T and an affine space. The local system on T parametrized by $\xi \in \hat{X}$ gives, by an appropriate pull-back, a local system $\mathcal{L}_{\xi,w}$ on G_w. This defines an irreducible perverse sheaf $A_{\xi,w}$ on G (the "perverse extension" of $\mathcal{L}_{\xi,w}$, or the intersection cohomology complex defined by $\mathcal{L}_{\xi,w}$), whose support is contained in the closure $\bar{G}_w$. There is a connection between $A_{\xi,w}$ and the Kazhdan-Lusztig elements $c_{\xi,w}$ of an algebra $\mathcal{K}$. We also describe, after [L1, Ch.1], the cohomology sheaves of the perverse sheaves $A_{\xi,w}$. We have found it convenient to work with G (and not with G/B, as in [loc.cit]). For example, we can then exploit equivariance properties of left- and right translations by B.
The character sheaves are introduced in no.5. If $A_{\xi,w}$ is as before with $w\xi = \xi$ one constructs a complex of sheaves $C_{\xi,w}$ on G, obtained by "making $A_{\xi,w}$ equivariant for conjugation". It turns out that $C_{\xi,w}$ is a direct sum of shifted irreducible perverse sheaves. The irreducible perverse sheaves thus obtained (for varying w and ξ) are the character sheaves on G. (The definition given in no.5 of character sheaves is not the one of [CS, no.2], but is equivalent to it by [loc.cit., 12.7]). We discuss in no.5 some basic properties of character sheaves. Important for the sequel is the construction (implicit in [CS, no.6]) of a general-

ized trace function τ on an algebra $\mathcal{K}$ (i.e. a linear map satisfying $\tau(uv) = \tau(vu), u, v \in \mathcal{K}$), with values in the Grothendieck group of character sheaves with Laurent polynomial coefficients (see 5.1.7). In no.5 we also discuss briefly the example $G = SL_n$, with $C_{\xi,w}$ such that w is a Coxeter element.
There is a machinery of parabolic induction and restriction for character sheaves. Much of the later sections is taken up with the discussion of these operations and their properties. In no.6 parabolic restriction is introduced, for conjugation equivariant perverse sheaves, as well as the important notion of a cuspidal perverse sheaf. This notion was introduced by Lusztig in [L2]. We review the results of that paper which we need. This included the precise description of irreducible cuspidal perverse sheaves (6.3.1). We also introduce, as in [L2], a stratification of G. Parabolic induction is introduced in no.7. Induction of an irreducible cuspidal perverse sheaf on a Levi group produces a direct sum of irreducible perverse sheaves on G. These are called admissible perverse sheaves. They are discussed in no.8. As a matter of fact, Lusztig has shown (under some mild assumptions) that the admissible perverse sheaves coincide with the character sheaves, as a consequence of results which we do not discuss here (see [CS, 17.8.5]). It is shown that the restrictions of the cohomology sheaves of an admissible perverse sheaf to the strata of the stratification mentioned before are locally constant. This is a consequence of results about the restriction of induced perverse sheaves to the centralizer of a semi-simple element (8.2).
Restriction and induction of character sheaves are taken up in no.9 and no.10. The basic result for restriction, discussed in 9.2, is the description of the restrictions of the perverse sheaves $C_{\xi,w}$. The proof given in [CS, nos.3,6] is not easy to follow. We hope that our approach, using the algebra $\mathcal{K}$, is more accessible. As a consequence of this result it follows that parabolic restriction carries a character sheaf to a direct sum of character sheaves. A similar result holds for induction. One can then develop formal properties of character sheaves which are quite similar to properties familiar in the character theory of finite groups of Lie type.
No.11 discusses the further analysis of character sheaves. Here material about Hecke algebras is needed (reviewed in 11.1 and 11.5, see also [Cu]). The function τ mentioned before induces a generalized trace on the Hecke algebras $\mathcal{H}'_\xi$, which is a linear combination with coefficients in the field of rational functions $E(t)$ (E being the coefficient field for cohomology), of trace functions defined by the irreducible representations of $\mathcal{H}'_\xi$. One of the main results (11.2.1) is that under an extra assumption on G, these coefficients are constants (this is equivalent to [CS, 14.9]). We review the proof given in [loc.cit., nos. 15,16] that one can attach to a character sheaf a two-sided Kazhdan-Lusztig cell in a suitable $\mathcal{H}'_\xi$.
Finally, in no.12 we state the main results of[CS] (one of which is that the extra assumption on G is almost always fulfilled). These results are proved in [CS] via a case by case analysis which we did no go into in these notes. Perhaps we can come back to this analysis in a sequel.

It will be clear from the preceeding review of the contents of these notes that we have rearranged considerably the presentation of [CS] and that we also have included relevant material from [L1] and [L2]. The proofs given here of the main results are fairly complete (of course, modulo results from Lie theory and the theory of perverse sheaves), although occasionally we have referred to the literature, if there was a straightforward reference.
As already mentioned, we did not include proofs of the results of [CS] which involve a case

by case analysis. Nor did we discuss in detail the results of [CS, nos. 9,10] about orthogonality relations, for example of generalized Green functions. However, the technical results needed for these are established. The orthogonality relations in question are discussed in [Sh]. There one also finds a review of the results about generalized Green functions of [CS, no.24].

If k is an algebraic closure of a finite field F_q and if G is defined over F_q there is a Frobenius morphism F on G, which can act on various objects. In the proofs one often uses, after Lusztig, a reduction to this situation, whose special features (like purity) can then be exploited.

We did not discuss the use of character sheaves in the theory of character of finite groups of Lie type. There is a description, as yet conjuctural, of the irreducible characters of such groups in terms of character sheaves (see [L1, p.348]). We want to mention in passing that our approach to character sheaves, starting with the perverse sheaves $A_{\xi,w}$ has a counterpart in the situation treated in [L1]. We hope to come back to this at another occasion.

If $k = \mathbf{C}$ one can attack to any perverse sheaf on G a regular holonomic $\mathcal{D}$-module. The $\mathcal{D}$-modules defined by character sheaves can be characterised by properties of their characteristic varieties (see [G] or [MV]). Ginzburg [G] also characterizes character sheaves by an admissibility property of their $\mathcal{D}$-modules. This can be used to give new proofs of several results of [CS] (over C). We did not go into these matters in our notes.

Contents

1 Perverse sheaves

In this section we review some basic facts about perverse sheaves, and we discuss some material to be needed later. For the theory of perverse sheaves see [BBD].
Let k be an algebraically closed field. We consider algebraic varieties over k.

1.1 Derived category

1.1.1. Let X be an algebraic variety over k. We deal with sheaves of vector spaces on X over a coefficient field E, and complexes of such sheaves. There are two cases:
(a) ℓ-adic sheaves, where ℓ is a prime number different from the characteristic of k. Then E is an extension of $\mathbf{Q}_\ell$ (for example an algebraic closure $\bar{\mathbf{Q}}_\ell$),
(b) $k = \mathbf{C}$, we consider sheaves on X for the classical topology. Now E can be any field (for example $\mathbf{Q}$).
We denote by $\mathcal{D}X$ the bounded derived category of constructible sheaves (of E-vector spaces) on X. For the definition in case (a) (which requires some care) see [D, p.148-149].
If K is a complex in $\mathcal{D}X$ we denote by H^iK or $H^i(K)$ its i^{th} cohomology sheaf, which is a constructible sheaf on X, by $H^i(X,K)$ the i^{th} hypercohomology group and by DK the Verdier dual.
1.1.2. If $f: X \to Y$ is a morphism of algebraic varieties we have functors between the derived categories $\mathcal{D}X$ and $\mathcal{D}Y$. As in [BBD, p.17] we denote them by $f_*, f^*, f_!, f^!$ (and not by $Rf_*, \ldots$).
Let U be an open subset of the algebraic variety X with complement F and denote by $j: U \hookrightarrow X$, $i: F \hookrightarrow X$ the inclusion maps. If $K \in \mathcal{D}X$ there is a canonical distinguished triangle $(j_!j^*K, K, i_*i^*K)$ in $\mathcal{D}X$.

1.2 Perverse sheaves

1.2.1. Let $\mathcal{D}X^{\leq 0}$ be the full subcategory of $\mathcal{D}X$ whose objects K satisfy dim supp $H^i(K) \leq -i$ for all integers i and put $\mathcal{D}X^{\geq 0} = D(\mathcal{D}X^{\leq 0})$.
Let $\mathcal{M}X$ be the full subcategory of $\mathcal{D}X$ whose objects are in $\mathcal{D}X^{\leq 0} \cap \mathcal{D}X^{\geq 0}$. The objects of $\mathcal{M}X$ are the *perverse sheaves* on X. $\mathcal{M}X$ is an abelian category in which all objects have finite length [BBD, p.112].
1.2.2. The inclusion of $\mathcal{D}X^{\leq 0}$ (resp. $\mathcal{D}X^{\geq 0}$) in $\mathcal{D}X$ has a right adjoint $\tau_{\leq 0}$ (resp a left adjoint $\tau_{\geq 0}$) and the functors $\tau_{\geq 0}\tau_{\leq 0}$, $\tau_{\leq 0}\tau_{\geq 0}$ are canonically isomorphic. If $K \in \mathcal{D}X$, the complex $\tau_{\geq 0}\tau_{\leq 0}\, K$ is a perverse sheaf ${}^pH^\circ K$.
We define a functor ${}^pH^i : \mathcal{D}X \to \mathcal{M}X$ by ${}^pH^iK = {}^pH^\circ(K[i])$, the square brackets denoting dimension shift. If (K, L, M) is a distinguished triangle in $\mathcal{D}X$ we have a long exact sequence in $\mathcal{M}X$

$$\ldots \to {}^pH^iK \to {}^pH^iL \to {}^pH^iM \to {}^pH^{i+1}K \to \ldots$$

Also, if $K \in \mathcal{D}X$ then ${}^pH^iK = 0$ if $|i|$ is large. See [BBD, no.I] for these facts.
A complex $K \in \mathcal{D}X$ is said to be *split* if K is isomorphic in $\mathcal{D}X$ to the direct sum $\oplus_i\, {}^pH^iK[-i]$ and to be *semi-simple* if it is split and all ${}^pH^iK$ are semi-simple objects in $\mathcal{M}X$. A *constituent* of the semi-simple complex K is an irreducible constituent of some ${}^pH^iK$, in the abelian category $\mathcal{M}X$.
1.2.3. **Irreducible perverse sheaves.** Let U be a locally closed, smooth, irreducible

subvariety of X of dimension d and let $\mathcal{L}$ be an irreducible local system on U. There exists a unique irreducible perverse sheaf $I = I(\bar{U}, \mathcal{L})$ on X whose support is the closure $\bar{U}$ and whose restriction to U is $\mathcal{L}[d]$ (see [BBD, p.112]). Any irreducible perverse sheaf on X can be obtained in this manner.

We call I the ***perverse extension*** to X of the local system $\mathcal{L}$. The intersection cohomology complex of Deligne-Goresky-MacPherson is $I[-d]$. We shall need only local systems $\mathcal{L}$ with finite monodromy, i.e. such that there is an étale covering $U' \to U$ such that the inverse image of $\mathcal{L}$ on U' is trivial.

We notice that $DI(\bar{U}, \mathcal{L}) = I(\bar{U}, \mathcal{L}^\vee)$, where $\mathcal{L}^\vee$ is the dual local system.

Perverse extensions $I(\bar{U}, \mathcal{L})$ exist for any local system $\mathcal{L}$ on U, not necessarily irreducible. Such a complex I is characterized by the following support conditions

$$\begin{aligned} \dim \operatorname{supp} H^i(A) &< -i \\ \dim \operatorname{supp} H^i(DA) &< -i, \end{aligned}$$

if $i > -\dim U$. Moreover, $A\mid_U = \mathcal{L}[\dim U]$ and

$$H^i(A) = 0 \text{ for } i < -\dim U.$$

We say that a perverse sheaf K is ***even*** if $H^i(K) = 0$ if $i \not\equiv \dim \operatorname{supp} K \pmod{2}$.

1.2.4. Examples.

(a) Let X be a flag variety associated to a semi-simple linear algebraic group over k. If U is a Bruhat cell in X and $\mathcal{L}$ the constant sheaf E, the corresponding perverse extension I is even. We shall discuss a more general result below (4.1).

(b) Let X be a smooth irreducible variety of dimension d and let $D_1, \ldots, D_r$ be smooth divisors with normal crossings in X. Let $\mathcal{L}$ be a one dimensional local system on $X - \bigcup_{i=1}^{r} D_i$, coming from a representation of the fundamental group of $X - \bigcup_{i=1}^{r} D_i$ which factors through a finite quotient of order prime to char k. Then $H^i I(X, \mathcal{L}) = 0$ if $i \neq -\dim X$.

In fact, If J is the set of $i \in [1, r]$ such that the local monodromy of $\mathcal{L}$ around D_i is non-trivial and $U = X - \bigcup_{i \in J} D_i$ then $\mathcal{L}$ can be extended to a local system $\bar{\mathcal{L}}$ on U. The restriction of $I(X, \mathcal{L})$ to U is $\bar{\mathcal{L}}[d]$ and the restriction of $I(X, \mathcal{L})$ to $X - U$ is zero.

These facts can be proved using the explicit construction of $I(X, \mathcal{L})$ given in [BBD, p.112]. It reduces, essentially, the proof to the case that $\dim X = 1$.

We next establish a lemma which is needed later.

1.2.5. Lemma. *Let K and L be two perverse sheaves on X.*

(i) $H_c^i(X, K \otimes L) = 0$ for $i > 0$;

(ii) *If K and L are irreducible then $H_c^0(X, K \otimes L) = 0$ if and only if K is not isomorphic to DL.*

Here H_c^i denotes hypercohomology with proper support. We sketch the proof given in [CS, 7.4].

There is a spectral sequence converging to $H_c^{\cdot}(X, K \otimes L)$ with E_2- term given by

$$E_2^{ij} = H_c^i(X, H^j(K \otimes L)).$$

From the definition of perverse sheaves it follows that

$$\dim \operatorname{supp} H^j(K \otimes L) \leq -\tfrac{1}{2} j,$$

whence $E_2^{ij} = 0$ if $i + j > 0$, and (i) follows.
Now let K and L be irreducible and put $d = \dim \operatorname{supp} K$. Then we see that $E_2^{i,-i} = 0$ unless $\operatorname{supp} K = \operatorname{supp} L$, $i = 2d$. We then have

$$E_2^{2d,-2d} = H_c^{2d}(X, H^{-d}(K) \otimes H^{-d}(L)).$$

Let U be a smooth open subset of supp$K$ such that the restrictions of $H^{-d}K$ resp. $H^{-d}L$ to U are local systems $\mathcal{L}$ resp. $\mathcal{M}$. Then

$$E_2^{2d,-2d} = H_c^{2d}(U, \mathcal{L} \otimes \mathcal{M}).$$

By Poincaré duality this is zero if and only if $H^\circ(U, (\mathcal{L} \otimes \mathcal{M})^\vee) = 0$, i.e. if and only if $\mathcal{L} \otimes \mathcal{M}$ is non-constant. The assertion (ii) follows.
1.2.6. Let $f : X \to Y$ be a morphism of algebraic varieties. We then have the four functors between $\mathcal{D}X$ and $\mathcal{D}Y$ of 1.1.2. As in [BBD, p.36] we say that such a functor is t-exact if it carries $\mathcal{D}X^{\leq 0}$ to $\mathcal{D}Y^{\leq 0}$ and $\mathcal{D}X^{\geq 0}$ to $\mathcal{D}Y^{\geq 0}$ or similarly with X and Y interchanged, as the case may be. We note the following facts:
(a) If f is finite then $f_! = f_*$ is t-exact and if f is étale then $f^! = f^*$ is t-exact ([BBD, p.69]);
(b) If f is smooth with connected fibers of dimension d then $f^! = f^*[2d]$ and $f^*[d]$ induces a fully faithful functor of $\mathcal{M}Y$ onto a subcategory of $\mathcal{M}X$ which is stable under taking subquotients ([BBD, p.108- 110]).

1.3 Finite ground fields

Assume that k is an algebraic closure of the finite field F_q of characteristic p.
1.3.1. Let X be an algebraic variety which is defined over F_q. We use the convention of [BBD, p.122], so we view X as being obtained by extension of scalars from a scheme X_0 over F_q. We have a Frobenius morphism $F : X \to X$ (raising coordinates to the q^{th} power). The fixed points set of F^n is the set $X(\mathsf{F}_{q^n})$ of points of X rational over F_{q^n}.
If S_0 is a sheaf of E-vector spaces on X_0 and S the sheaf on X which it defines we have a canonical isomorphism $\varphi : F^*S \tilde{\to} S$.
Recall that S_0 (or S) is said to be *punctually pure* of weight w if for each $x \in X(\mathsf{F}_{q^n})$ all complex absolute values of all eigenvalues of φ^n in the stalk S_x are $q^{\frac{1}{2}wn}$ and that S_0 (or S) is *mixed* of weight $\leq w$ if S_0 has a finite filtration whose successive quotients are pure of weight $\leq w$. Moreover, a complex K_0 in $\mathcal{D}X_0$ (or the corresponding complex K in $\mathcal{D}X$) is mixed if the cohomology sheaves H^iK_0 are mixed, it is (mixed) of weight $\leq w$ if the H^iK_0 have weight $\leq w + i$. Such a complex K_0 is of weight $\geq w$ if DK_0 is of weight $\leq -w$. The complex K_0 is *pure* of weight w if it is of weight $\leq w$ and $\geq w$.
1.3.2. With the previous notations, we have the following results about purity (see [BBD, pp.136, 138, 142]):
(a) *An irreducible mixed perverse sheaf is pure*;
(b) *A mixed complex K_0 is pure of weight w if and only if each perverse sheaf ${}^pH^iK_0$ is pure of weight $w + i$*;
(c) *If K_0 is a pure complex then K is semi- simple.*
From these purity results one deduces the following theorem, which holds for algebraic varieties over an arbitrary algebraically closed field k (see [BBD, p.163]).

(d) (*Decomposition theorem*) *Let $f : X \to Y$ be a proper morphism of algebraic varieties. If $K \in \mathcal{D}X$ is a semi-simple complex of geometric origin then so is f_*K.*
For the notion of perverse sheaves of geometric origin see [BBD, p. 162- 163]. We shall only encounter perverse sheaves of this kind.
The proof of the decomposition theorem uses a "reduction of characteristic 0 to characteristic p", explained in [BBD, no.6]. We shall have to use that reduction procedure. We shall then need the following lemma.
1.3.3. Let $f : X \to Y$ be a morphism of varieties over $\mathbf{F}_q$. Assume that X is a disjoint union of finitely many locally closed subvarieties $X_a (a \in A)$ and that there is a function $c : A \to \mathbf{N}$ such that the closure $\bar{X}_a$ is contained in $X_a \cup \bigcup_{c(b)<c(a)} X_b$, for any $a \in A$. Denote by f_a the restriction of f to X_a.
Lemma. *Let K be a mixed complex in $\mathcal{D}X$. Let K_a be its restriction to X_a. Assume that all complexes $(f_a)_! K_a$ are pure of weight 0. Then $f_! K$ is isomorphic in $\mathcal{D}Y$ to $\bigoplus_{a \in A} (f_a)_! K_a$.*
Put $Z_h = \bigcup_{c(a) \le h} X_a$, then Z_h is closed in X. Let i_h (resp. j_h) denote the inclusion $Z_h \to X$ (resp. $Z_h - Z_{h-1} \to X$). We have a distinguished triangle

$$((j_h)_! j_h^* K, (i_h)_* i_h^* K, (i_{h-1})_* i_{h-1}^* K)$$

in $\mathcal{D}X$, whence a distinguished triangle

$$(f_!(j_h)_! j_h^* K, f_!(i_h)_! i_h^* K, f_!(i_{h-1})_! i_{h-1}^* K)$$

in $\mathcal{D}Y$. As $Z_h - Z_{h-1}$ is the disjoint union of the X_a with $c(a) = h$, we have

$$f_!(j_h)_! j_h^* K = \bigoplus_{c(a)=h} (f_a)_! K_a.$$

We have a long exact sequence in $\mathcal{M}Y$ (see 1.2.2) :

$$\dots \xrightarrow{\delta} \bigoplus_{c(a)=h} {}^pH^j((f_a)_! K_a) \to {}^pH^j(f_!(i_h)_! i_h^* K) \to {}^pH^j(f_!(i_{h-1})_! i_{h-1}^* K)$$

$$\xrightarrow{\delta} \bigoplus_{c(a)=h} {}^pH^{j+1}((f_a)_! K_a) \to \dots.$$

Notice that $f_!(i_h)_! i_h^* K = f_! K$ if h is large and $= 0$ if $-h$ is large. Under the assumptions of the lemma $\bigoplus_{c(a)=h} {}^pH^j((f_a)_! K_a)$ is pure of weight j. By induction on h one deduces, using the exact sequence, that ${}^pH^j(f_!(i_h)_! i_h^* K)$ is pure of weight j. In particular, $f_! K$ is pure of weight 0 (see property (b) in 1.3.2). It also follows that the maps δ of the exact sequence are zero so we get a family of short exact sequences. Since $f_! K$ is semi-simple (property (c) of 1.3.2) we conclude from the short exact sequences that $f_! K$ has the asserted property.
1.3.4. **Characteristic functions.**
Let X be as in 1.3.1, let $K \in \mathcal{D}X$ and assume given an isomorphism $\varphi : F^* K \tilde{\to} K$. We define the *characteristic function*

$$\chi_{K,\varphi} : X(\mathbf{F}_q) \to E$$

by

$$\chi_{K,\varphi}(x) = \sum_i (-1)^i Tr(\varphi_x, H^i(K)_x).$$

More generally, we can define $\chi_{K,\varphi^n} : X(\mathbb{F}_{q^n}) \to E$. One can show that if K is a semi-simple perverse sheaf the family of functions $(\chi_{K,\varphi^n})_{n\geq 1}$ determines K up to isomorphism.
If necessary we write $\chi_{K,\varphi} = \chi^X_{K,\varphi}$.
1.3.5. In the situation of this section one has to take into account Tate twists, in constructions with perverse sheaves where weights are involved.

1.4 Equivariance

In this section X denotes an algebraic variety and G a *connected* algebraic group which acts on X. Let $a : G \times X \to X$ be the action morphism. It is a morphism whose fibers are all isomorphic to G.
1.4.1. **Definition.** *A perverse sheaf $K \in \mathcal{D}X$ is G-equivariant if the perverse sheaves $a^*K[\dim G]$ and $E \boxtimes K[\dim G]$ are isomorphic in $\mathcal{M}_{G\times X}$.*
Notice that by 1.2.6 (b), $a^*[\dim G]$ sends perverse sheaves to perverse sheaves. We denote by $\boxtimes$ the exterior tensor product of complexes. We say that a split complex $K \in \mathcal{D}X$ (1.2.2) is G- equivariant if all ${}^pH^iK$ are so.
Remark. One can define a notion of G-equivariance for complexes $K \in \mathcal{D}X$, see [MV, appendix]. The definition is more complicated. We shall not use it here.
We list a few properties, which are easy to prove. $K \in \mathcal{M}X$ is a G-equivariant perverse sheaf.
(a) All sheaves H^iK are G-equivariant in the sense of sheaf theory. If G acts trivially on X then G acts trivially on the H^iK;
(b) Any subquotient of K is G-equivariant;
(c) Let $f : X \to Y$ be a G-equivariant morphism, Y being a second G-variety. Then all ${}^pH^i(f_!K)$ are G-equivariant. If $L \in \mathcal{M}Y$ is G-equivariant then so are the ${}^pH^i(f^*L)$.
1.4.2. **Lemma.** *Let $f : X \to Y$ be a locally trivial principal fibre space with group G. Then $K \in \mathcal{M}X$ is G-equivariant if and only if there exists $L \in \mathcal{M}Y$ such that $K = f^*L[\dim G]$.*
First assume that $X = G \times Y$ is a trivial principal fibre space, f being the projection. Let $iy = (e, y)$. If $K \in \mathcal{M}X$ is equivariant it follows that K is isomorphic to f^*i^*K, so we may take $L = i^*K[-\dim G]$. Then L is perverse by 1.2.6 (b). If X is arbitrary one constructs L locally and uses a gluing argument (see [BBD, p.65], in [CS, 1.9.3] a somewhat different argument is given).

2 Kummer local systems on tori

2.1. Let T be an algebraic torus over k. Its character group is denoted by X. If p is the characteristic exponent of k we denote by $\mathbb{Z}_{(p)}$ the ring of rational numbers with denominator prime to p (so $\mathbb{Z}_{(p)} = \mathbb{Q}$ if char $k = 0$). Put

$$\hat{X} = \hat{X}(T) = \mathbb{Z}_{(p)} \otimes_{\mathbb{Z}} X / 1 \otimes_{\mathbb{Z}} X,$$

this is an abelian torsion group without p-torsion.
We consider local systems on T with coefficients in our fixed field E, which we assume now to be sufficiently large, say $E = \bar{\mathbb{Q}}_\ell$ or $\mathbb{C}$. Fix an isomorphism ψ of the group μ of roots of unity in k onto the group of roots of unity of order prime to p in E.
2.1.1. **Definition.** *A local system $\mathcal{L}$ of rank one on T is Kummer if there is an integer $n \neq 0$ prime to p such that $\mathcal{L}^{\otimes n} \simeq E$ (the constant sheaf).*
The set of isomorphism classes of Kummer local systems is an abelian group $\mathcal{K}T$.
2.1.2. **A construction.**
Let $m > 0$ be an integer prime to p. Consider the m^{th} power isogeny $m : T \to T$. It is a Galois covering whose group is the group ${}_mT$ of elements of T whose order divides m. If $x \in X$ define a character $\chi_{x,m}$ of ${}_mT$ by

$$\chi_{x,m}(t) = \psi(x(t)) \ (t \in_m T).$$

The character lifts to a one dimensional representation of the fundamental group $\pi_1(T, e)$, which defines a local system $\mathcal{L}_{x,m}$ of rank one on T. The following properties are immediate from the definition:
(i) $\mathcal{L}_{nx,mn} = \mathcal{L}_{x,m}$;
(ii) $\mathcal{L}_{x,m} = \mathcal{L}_{y,n}$ if and only if $my - nx \in mnX$;
(iii) $\mathcal{L}_{x,m}$ is trivial if and only if $x \in mX$.
It follows that $\mathcal{L}_{x,m}$ depends only on the class $\xi \in \hat{X}$ of $m^{-1} \otimes x$ (which we write $m^{-1}x$) modulo X. We write $\mathcal{L}_\xi = \mathcal{L}_{x,m}$, if necessary we write $\mathcal{L}_{\xi,T}$.
It is clear that $\mathcal{L}_\xi \otimes \mathcal{L}_\eta = \mathcal{L}_{\xi+\eta}$ $(\xi, \eta \in \hat{X})$, from which we see that $\mathcal{L}_\xi$ is a Kummer local system.
If $m : T \to T$ is as before we have

$$(1) \qquad m_*E = \bigoplus_{\xi \in \hat{X}, m\xi = 0} \mathcal{L}_\xi$$

Let $\varphi : T \to T'$ be a homomorphism of tori. The induced map of character groups defines a homomorphism $\hat{\varphi} : \hat{X}(T') \to \hat{X}(T)$.
2.1.3. **Lemma.** *For $\xi \in \hat{X}(T')$ we have $\varphi^*\mathcal{L}_{\xi,T'} = \mathcal{L}_{\hat{\varphi}\xi,T}$.*
The proof uses that if a local system $\mathcal{L}$ on T' is defined by the representation ρ of $\pi_1(T', e)$, the local system $\varphi^*\mathcal{L}$ comes from the composite of ρ and the canonical homomorphism $\pi_1(T, e) \to \pi_1(T', e)$.
2.1.4. **Proposition.** *Let $\mathcal{L}$ be a Kummer local system on T. There is a unique $\xi \in \hat{X}$ such that $\mathcal{L} \simeq \mathcal{L}_\xi$.*
Corollary. *$\xi \mapsto \mathcal{L}_\xi$ defines an isomorphism $\hat{X} \to \mathcal{K}T$*
$\mathcal{L}$ is defined by a one dimensional representation ρ of $\pi_1(T, e)$ whose image lies in the group of m^{th} roots of unity, for some m prime to p. Then Ker ρ defines a Galois covering

$\pi : X \to T$ such that the corresponding extension of function fields $k(X)/k(T)$ is cyclic of degree dividing m. By Kummer theory there is $x \in k(T)$ such that $k(X) = k(T)(x^{\frac{1}{m}})$. Since π is unramified, x cannot have zeros or poles in T, if $X \neq T$. It readily follows that x is a character of T. But then the m-isogeny of T factors:

$$T \to X \xrightarrow{\pi} T$$

and it follows that $\mathcal{L}$ is a constituent of m_*E. The proposition follows from (1).
We record a few easy properties.
2.1.5. **Lemma.** *If $\xi \neq 0$ then $H^{\cdot}(T, \mathcal{L}_\xi) = H^{\cdot}_c(T, \mathcal{L}_\xi) = 0$.*
It follows from (1) that

$$H^{\cdot}(T, E) \simeq H^{\cdot}(T, E) \bigoplus \bigoplus_{\xi \neq 0, m\xi = 0} H^{\cdot}(T, \mathcal{L}_\xi),$$

and similarly for $H^{\cdot}_c$. The lemma follows.
2.1.6. **Lemma.** *The dual $D\mathcal{L}_\xi$ is isomorphic to $\mathcal{L}_{-\xi}[2 \dim T]$.*
Here $\mathcal{L}_\xi$ is viewed as a complex concentrated in dimension zero.

2.2 Weights of torus actions

Let Z be an algebraic variety with a T-action $a : T \times Z \to Z$. If $\mathcal{L}$ is a Kummer local system on T then $\mathcal{L}[d]$ is a perverse sheaf on T, where $d = \dim T$.
2.2.1. **Definition.** *A perverse sheaf $K \in \mathcal{M}Z$ has weight $\mathcal{L}$ for the T-action a if $a^*K[d]$ is isomorphic to $\mathcal{L}[d] \boxtimes K$, as perverse sheaves on $T \times Z$.*
If $\mathcal{L}$ is constant this just means that K is T-equivariant in the sense of 1.4. If $\xi \in \hat{X}$ is such that $\mathcal{L} \simeq \mathcal{L}_\xi$ (2.1.4) we also say that K has weight ξ.
Let m be an integer prime to p. In the previous situation define a T- action a_m on Z by $a_m(t, z) = t^m z (t \in T, z \in Z)$.
2.2.2. **Lemma.** *A semi-simple perverse sheaf K on Z is a_m-equivariant if and only if each irreducible constituent K_i of K has a weight $\xi_i \in \hat{X}$, with $m\xi_i = 0$.*
The a_m-equivariance of K means that

$$(m \times id)^* a^* K[d] \simeq E[d] \boxtimes K,$$

$m : T \to T$ being as before. To prove the only if-part use that for any irreducible perverse sheaf K on $T \times Z$ there is an injective morphism of perverse sheaves

$$K \to (m \times id)_*(m \times id)^* K.$$

The rest of the proof is easy.

Now let G be a connected linear algebraic group and $\varphi : G \to T$ a homomorphism. Let $a : G \times Z \to Z$ be a G- action on Z.
2.2.3. **Definition.** *A perverse sheaf $K \in \mathcal{M}Z$ has weight $\mathcal{L}$ (relative to a and φ) if $a^*K[\dim G] \simeq \varphi^*\mathcal{L}[\dim G] \boxtimes K$.*
If $\mathcal{L} \simeq \mathcal{L}_\xi$, we also say that K has weight ξ.
2.2.4. **Lemma.** *In the previous situation assume that U is a locally closed, smooth, irreducible G-stable subvariety of Z. Let $\mathcal{L}$ be a local system on U such that the perverse sheaf $\mathcal{L}[\dim U]$ on U has weight $\xi \in \hat{X}$. Then the perverse extension $I(\bar{U}, \mathcal{L})$ has weight ξ.*
The notation is as in 1.2.3. The proof of the lemma is straightforward. In the situation of the lemma we shall say that $\mathcal{L}$ has weight ξ.

2.3 Finite ground fields

Assume that k is an algebraic closure of the finite field F_q and that T is defined over F_q. Let F be the Frobenius morphism of T (1.3.1). The fixed point set T^F of F in T is the group $T(\mathsf{F}_q)$ of F_q-rational points.

F operates on the group $\mathcal{K}T$ of Kummer local systems (via F^*), as well as on X and $\hat{X}$.

2.3.1. **Proposition.** *The fixed point group $(\mathcal{K}T)^F$ is canonically isomorphic to the character group* Hom (T^F, E^*) *of* T^F.

By 2.1.3 and 2.1.4 we have that $(\mathcal{K}T)^F$ is isomorphic to the fixed point group $\hat{X}^F$. Using that F has no fixed points on X we see that $\hat{X}^F \simeq X/(F-1)X$. The latter group is well-known to be isomorphic to Hom (T^F, E^*) (the isomorphism comes from the homomorphisms

$$T^F \times X \to \mu \xrightarrow{\psi} E^*,$$

the first one being induced by the pairing $T \times X \to k^*$).

The elements of $(\mathcal{K}T)^F$ are represented by the $\mathcal{L}_\xi$ with $F\xi = \xi$. These $\mathcal{L}_\xi$ come from a sheaf on the scheme T_0 over F_q (see 1.3.1). There is a canonical isomorphism $\varphi : F^*\mathcal{L}_\xi \simeq \mathcal{L}_\xi$.

2.3.2. **Lemma.** *Let $\xi \in \hat{X}^F, m\xi = 0$. If $a \in T(\mathsf{F}_q)$ then φ acts on the stalk $(\mathcal{L}_\xi)_a$ by scalar multiplication with an m^{th} root of unity $\epsilon(\xi)$. We have $\epsilon(-\xi) = \epsilon(\xi)^{-1}$.*

Let $\mathcal{L}_\xi = \mathcal{L}_{x,m}$, as in 2.1.2. We can view $(m_*E)_a$ as the vector space of E-valued functions on $S = \{s \in T \mid s^m = a\}$. The group ${}_mT$ and F act on it. Fix $s_0 \in S$, then $S = {}_mT.s_0$. Define $f \in (\mathcal{L}_\xi)_a$ to be the function $S \to E$ with $f(ts_0) = \psi(x(t))^{-1}$ $(t \in_m T)$. If $t_0 \in_m T$ is defined by $Fs_0 = t_0s_0$ then for $t \in_m T$

$$(\varphi f)(ts_0) = f(F(ts_0)) = f(F(t)t_0s_0) = \psi(x(F(t)t_0))^{-1} = \psi(x(tt_0))^{-1}.$$

It follows that $\varphi f = \psi(x(t_0))^{-1}f$, which proves the first statement. The last point is easy.

3 Some algebraic tools

We shall have to deal with Kummer local systems on a maximal torus in a reductive group. In this situation we require some algebraic tools, to be discussed in the present section.

3.1 Tori in reductive groups

3.1.1. Assume that G is a connected reductive linear algebraic group over k. We assume that the torus T of the previous section is a maximal torus of G.
Denote by R the root system of (G, T) and by $W = N_G T/T$ the Weyl group, $N_G T$ being the normalizer of T in G. Then W acts on T, X and $\hat{X}$.
Let $\mathcal{L}$ be a Kummer local system on T. Put

$$W'_{\mathcal{L}} = \{w \in W \mid w^* \mathcal{L} \simeq \mathcal{L}\}.$$

If $\mathcal{L} = \mathcal{L}_\xi$ $(\xi \in \hat{X})$ then by 2.1.3 we have $w^*\mathcal{L}_\xi = \mathcal{L}_{w^{-1}\xi}$ whence

$$W'_{\mathcal{L}_\xi} = \{w \in W \mid w\xi = \xi\}.$$

We also write W'_ξ for this group. If $\mathcal{L}_\xi = \mathcal{L}_{x,m}$ (2.1.2) then

$$W'_\xi = \{w \in W \mid wx - x \in mX\}. \tag{1}$$

Let Y be the dual of the character group and $<,>$ the pairing $X \times Y \to \mathbb{Z}$. If $\alpha \in R$ denote by $s_\alpha \in W$ the corresponding reflection. So for $\chi \in X$

$$s_\alpha \chi = \chi - < \chi, \alpha^\vee > \alpha,$$

where $\alpha^\vee$ is the coroot defined by α.
With the previous notations, put

$$R_{\mathcal{L}} = R_\xi = \{\alpha \in R \mid < x, \alpha^\vee > \in m\mathbb{Z}\}$$

and let $W_{\mathcal{L}} = W_\xi$ be the subgroup of W'_ξ generated by the s_α with $\alpha \in R_\xi$. If $R_\xi \neq \phi$ it is a root system (in a suitable vector space), with Weyl group W_ξ. We recall the following known lemma.
3.1.2. **Lemma.** *If G has connected center then $W'_\xi = W_\xi$, for all $\xi \in \hat{X}$.*
If $\xi = m^{-1}x + X$, as before, it follows from (1) that the stabilizer of $m^{-1}x \in \mathbb{R} \otimes X$ in the semi-direct product $W \ltimes X$ is isomorphic to W'_ξ. Let Q be the subgroup of X generated by R. If the center of G is connected then Q is a direct summand of X. It follows that then W'_ξ is the stabilizer of an element of $\mathbb{R} \otimes Q$ in the affine Weyl group $W \ltimes Q$. It is well-known that such a stabilizer is a reflection group (see [St2, p.10]). This implies the lemma.
Notice that 3.1.2 implies that

$$W_\xi = \{w \in W \mid wx - x \in mQ\}. \tag{2}$$

It is clear that if $\alpha \in R_\xi$ then $s_\alpha \in W'_\xi$. The converse is not always true, but does hold if $\frac{1}{2}\alpha \notin X$.
3.1.3. **Lemma.** *If $w \in W'_\xi$ then $\mathcal{L}_\xi$ is equivariant for the action of T on itself given by*

$(t,u) \mapsto t^{-1}(w^{-1}t)u$ $(t,u \in T)$.
This is consequence of 2.1.3.
Let $Z = Z_G$ be the center of G. The T-action of the previous lemma induces an action of Z of $\mathcal{L}_\xi$, associated to the trivial map of T. Let $\mathcal{L}_\xi = \mathcal{L}_{x,m}$ be as in 2.1.2. Then $m^{-1}(wx-x)$ is a character y of T and one sees that the Z-action is induced from an action of the finite group Z/Z° (Z° denoting the identity component), given by multiplication in the stalks by $\psi(y(z))$, for $z \in Z$. Notice that $y(z) = 1$ if $z \in Z^\circ$. Also notice that by (2) we have $y(z) = 1$ if $w \in W_\xi$. We have thus attached to $w \in W_\xi^!$ a homomorphism of Z/Z° into E^*.
3.1.4. **Lemma.** *This defines a homomorphism $W_\xi^! \to$ Hom $(Z/Z^\circ, E^*)$ whose kernel is W_ξ. In particular, $W_\xi^!/W_\xi$ is isomorphic to a subgroup of* Hom $(Z/Z^\circ, E^*)$.
We skip the proof of this well-known fact. See [CS, 11.1], or [St 2, no.9] (where closely related results are established).

3.2 The groups W_ξ

The results which follow are based on those of [CS, no.5]. The notations are as before.
3.2.1. Let R^+ be a system of positive roots of the root system R of 3.1.1. We denote by D the corresponding basis of R and by S the generating set of reflections $(s_\alpha)_{\alpha \in D}$. Fix $\xi \in \hat{X}$. Then $R_\xi^+ = R^+ \cap R_\xi$ is a system of positive roots in R_ξ and D_ξ, S_ξ have the obvious meanings. (Notice that these are not necessarily subsets of D resp. S.)
We denote by ℓ, ℓ_ξ the length functions on W resp. W_ξ defined by S resp. S_ξ and by $<, <_\xi$ the corresponding Bruhat orders.
3.2.2. **Lemma.** *Each coset wW_ξ contains a unique element w^* of minimal lenght characterized, by $w^*R_\xi^+ \subset R^+$.*
The easy proof is omitted.
Let W_ξ^* be the set of minimal elements of the lemma. We can then write any $w \in W$ uniquely in the form $w = w^*w_1$, with $w^* \in W_\xi^*$, $w_1 \in W_\xi$. We then put $\ell_\xi(w) = \ell_\xi(w_1)$. Notice that $\ell_\xi(w)$ equals the number of $\alpha \in R_\xi^+$ with $w\alpha \in -R^+$.
Let $\mathbf{s} = (s_1, \ldots, s_r)$ be a sequence of elements in $S \cup \{e\}$. If $s_i \neq e$ write $s_i = s_{\alpha_i}$, where $\alpha_i \in D$. For $i = 1, \ldots, r$ put

$$t_i = s_r \ldots s_{i+1} s_i s_{i+1} \ldots s_r,$$

and let

$$I_{\mathbf{s}} = \{i \in [1,r] \mid t_i \in W_\xi - \{e\}\},$$
$$w = w(\mathbf{s}) = s_1 \ldots s_r.$$

3.2.3. $\ell_\xi(w) \leq \mid I_{\mathbf{s}} \mid$, *with equality if* $\ell(w) = \ell(s_1) + \ldots + \ell(s_r)$.
If $\alpha \in R_\xi^+$ and $w\alpha \in -R^+$ then there is $i \in [1,r]$ such that $s_i \neq e$ and $\alpha = s_r \ldots s_{i+1}\alpha_i$ (compare with [B, p.14]), and then $t_i \in W_\xi$. This gives the asserted inequality, since $\ell_\xi(w)$ equals the number of such roots α. If $\ell(w) = \ell(s_1) + \ldots + \ell(s_r)$ the $t_i \neq e$ are all distinct [loc.cit], which implies the last point.
Let m be the largest number in $I_{\mathbf{s}}$ (assuming $I_{\mathbf{s}} \neq \phi$).
3.2.4. **Lemma.** $t_m \in S_\xi$.
An equivalent statement is : there is only one $\alpha \in R_\xi^+$ with $t_m\alpha \in -R^+$. Assume α has this property. So $s_r \ldots s_{m+1}s_m s_{m+1} \ldots s_r\alpha \in -R^+$. If $s_i s_{i+1} \ldots s_r \alpha \in -R^+$ for some $i > m$ we can find such an i with $s_{i+1} \ldots s_r \alpha = \alpha_i$ (take the largest), whence $t_i \in W_\xi$, a contradiction.

If $s_m s_{m+1} \dots s_r \alpha \in R^+$ there is $i \geq m$ such that $s_i \dots s_{m+1} s_m s_{m+1} \dots s_r \alpha = \alpha_{i+1}$. Then $t_m \alpha = -s_r \dots s_{i+2} \alpha_{i+1} \in -R_\xi$ and we get the contradiction $t_{i+1} \in W_\xi$. It follows that we must have $s_{m+1} \dots s_r \alpha = \alpha_m$ and α is unique.

If $J \subset [1, r]$ is a subset such that $s_i \neq e$ for all $i \in J$ denote by $\mathbf{s}_J$ the sequence obtained from $\mathbf{s}$ by replacing by e all s_i with $i \in J$. Write $I = I_\mathbf{s}$. Write $w = w^* w_1$, as before.
3.2.5. **Lemma.** $w^* = w(\mathbf{s}_I)$.
We prove this by induction on $| I |$, the statement being true for $I = \phi$ by 3.2.3. If $I \neq \phi$ let m be as in the previous lemma and put $\mathbf{s}' = \mathbf{s}_{\{m\}}$, $J = I_{\mathbf{s}'}$. Then $wt_m = w(\mathbf{s}')$ and $| J | < | I |$. We have $w^* = (wt_m)^* = w(\mathbf{s}')^* = w(\mathbf{s}'_J)$, by induction. Since $\mathbf{s}'_J = \mathbf{s}_I$ the assertion follows.

One now constructs the decomposition $w = w^* w_1$ in the following manner. Write $t_m = \sigma(\mathbf{s})$. Let $I = (m_1, \dots, m_a)$, where the m_i are increasing (so $m_a = m$ as in 3.2.4). Write $J_i = (m_{i+1}, \dots, m_a)$ and define $\sigma_i \in S_\xi$ by $\sigma_i = \sigma(\mathbf{s}_{J_i})$
3.2.6. **Proposition.**
(i) $w = w^* w_1$ *with* $w^* = w(\mathbf{s}_I), w_1 = \sigma_1 \dots \sigma_a$;
(ii) *If* $\ell(w) = \ell(s_1) + \dots + \ell(s_r)$ *then* $\ell_\xi(w) = a$.
The first point follows from 3.2.4 and 3.2.5 and the second one from 3.2.3.
Corollary. $\ell(w) \equiv \ell(w^*) + \ell_\xi(w) \pmod 2$.
The next result is another consequence of the preceding lemmas.
3.2.7. **Proposition.** *Let* $s \in S$, $w \in W$.
(i) *If* $w^{-1} s w \notin W_\xi$ *then* $\ell_\xi(sw) = l_\xi(w)$ *and* $(sw)^* = sw^*$;
(ii) *If* $w^{-1} s w \in W_\xi$ *and* $sw > w$ *then* $(sw)^* = w^*$ *and* $\ell_\xi(sw) = \ell_\xi(w) + 1$.
Moreover $(w^*)^{-1} s w^* \in S_\xi$.
To prove (i) we may assume that $sw > w$. Let $\mathbf{s} = (s_1, \dots, s_r)$ be a reduced decomposition of w. Then $\mathbf{s}' = (s, s_1, \dots, s_r)$ is one of sw. If $w^{-1} s w \notin W_\xi$ then (with the previous notations) $| I_\mathbf{s} | = | I_{\mathbf{s}'} |$ and the first part of (i) follows from 3.2.3. The second part follows by applying 3.2.5. The proof of (ii) is quite similar.
3.2.8. **Lemma.** *Let* $\mathbf{s}$ *be as before. If* $w(\mathbf{s}) \in W_\xi^!$ *and* $J \subset I_\mathbf{s}$ *then* $w(\mathbf{s}_J) \epsilon W_\xi^!$.
It suffices to consider the case that J has one element. If $I = (m_1, \dots, m_a)$ as before then $J = \{m_i\}$ and

$$w(\mathbf{s}_J) = w(\mathbf{s}) t_{m_i} \in w(\mathbf{s}) W_\xi,$$

whence the lemma.

3.3 Hecke algebras

We shall now introduce an algebra which generalizes the Hecke algebra of W.
Let $\mathcal{O}$ be a W-orbit in $\hat{X}$. Denote by $\mathcal{K} = \mathcal{K}_\mathcal{O}$ the free $\mathbb{Z}[t, t^{-1}]$ -module on $\mathcal{O} \times W$, with canonical basis $e_{\xi, w}$ $(\xi \in \mathcal{O}, w \in W)$.
3.3.1. **Theorem.** *There exists a unique structure of* $\mathbb{Z}[t, t^{-1}]$- *algebra on* $\mathcal{K}$ *such that for* $\xi, \eta, \in \mathcal{O}$, $x, y \in W, s \in S$
(a) $e_{\xi, x} e_{\eta, y} = 0$ *if* $\xi \neq y\eta$,
(b)
$$\begin{aligned} e_{y\eta, s} e_{\eta, y} &= e_{\eta, sy} \text{ if } sy > y \text{ or } s \notin W_{y\eta}, \\ &= (t^2 - 1) e_{\eta, y} + t^2 e_{\eta, sy} \textit{ if } sy < y \textit{ and } s \in W_{y\eta}, \end{aligned}$$
(c) $\epsilon = \sum_{\xi \in \mathcal{O}} e_{\xi, e}$ *is the identity element.*

Moreover we have

(d) $e_{s\xi,x}e_{\xi s} = e_{\xi,xs}$ if $x, s > x$ or $s \notin W_\xi$.
$= (t^2-1)e_{\xi,x} + t^2 e_{\xi,xs}$ *if* $xs < x$ *and* $s \in W_\xi$.
(e) $e_{y\eta,x}e_{\eta,y} = e_\eta, xy$ *if* $\ell(xy) = \ell(x) + \ell(y)$.
Notice that if $\xi = 0$ the algebra $\mathcal{K}$ is just a Hecke algebra of the Weyl group W (see for example [B, p.55]). The proof of the theorem, which we indicate below, can be given along the lines sketched in [loc.cit].
Introduce endomorphisms $P_{\xi,s}$ and $Q_{\xi,s}$ ($\xi \in \mathcal{O}, s \in S$) of the module $\mathcal{K}$ defined by

$$\begin{aligned} P_{\xi,s}(e_{\eta,y}) &= 0 \text{ if } \xi \neq y\eta, \\ P_{y\eta,s}(e_{\eta,y}) &= e_{\eta,sy} \text{ if } sy > y \text{ or } s \notin W_{y\eta}, \\ &= (t^2-1)e_{\eta,y} + t^2 e_{\eta,sy} \text{ if } sy < y \text{ and } s \in W_{y\eta} \end{aligned}$$

and

$$\begin{aligned} Q_{\xi,s}(e_{\eta,y}) &= 0 \text{ if } \xi \neq s\eta, \\ Q_{s\eta,s}(e_{\eta,y}) &= e_{s\eta,ys} \text{ if } ys > y \text{ or } s \notin W_{s\eta}, \\ &= (t^2-1)e_{s\eta,y} + t^2 e_{s\eta,ys} \text{ if } ys < y \text{ and } s \in W_{s\eta}. \end{aligned}$$

One checks that for $\xi, \eta \in \mathcal{O}, s, t \in S$

(3) $$P_{\xi,s}Q_{\eta,t} = Q_{\eta,t}P_{\xi,s}$$

(this uses the following fact : if $x \in W$ and $sx > x, xt > x$ resp. $sx < x, xt < x$ and $\ell(sxt) = \ell(x)$ then $sx = xt$). Let $x \in W$ and let $\mathbf{s} = (s_1, \ldots, s_r)$ be a reduced decomposition of x, where $s_i \in S$. Then

$$\begin{aligned} P_{s_2\ldots s_r\xi,s_1} \cdots P_{s_r\xi,s_{r-1}}P_{\xi,s_r}(e_{\eta,e}) &= 0 \text{ if } \eta \neq \xi \\ &= e_{\xi,x} \text{ if } \eta = \xi, \end{aligned}$$

and

$$\begin{aligned} Q_{\xi,s_r}Q_{s_r\xi,s_{r-1}} \cdots Q_{s_2\ldots s_r\xi,s_1}(e_{\eta,e}) &= 0 \text{ if } \eta \neq x\xi \\ &= e_{\xi,x} \text{ if } \eta = x\xi. \end{aligned}$$

Putting $\epsilon = \sum_{\xi\in\mathcal{O}} e_{\xi,e}$ we have

(4) $$e_{\xi,x} = P_{s_2\ldots s_r\xi,s_1} \cdots P_{\xi,s_r}(\epsilon) = Q_{\xi,s_r} \cdots Q_{s_2\ldots s_2\xi,s_1}(\epsilon).$$

We may now define for $r = l(x) > 0$

$$P_{\xi,x} = P_{s_2\ldots s_r\xi,s_1} \cdots P_{\xi,s_r}\ , Q_{\xi,x} = Q_{\xi,s_r} \cdots Q_{s_2\ldots s_r\xi,s_1}.$$

To show that, for example, $P_{\xi,x}$ does not depend on the choice of the reduced decomposition s write $e_{\eta,y}$ as a product of $Q's$ applied to ϵ (see (4)) and use (3) and (4).
Define $P_{\xi,e}$ and $Q_{\xi,e}$ by

$$\begin{aligned} P_{\xi,e}(e_{\eta,y}) &= 0 \quad \text{if} \quad \xi \neq y\eta, \\ &= e_{\eta,y} \quad \text{if} \quad \xi = y\eta \end{aligned}$$

and

$$\begin{aligned} Q_{\xi,e}(e_{\eta,y}) &= 0 \quad \text{if} \quad \xi \neq \eta, \\ &= e_{\eta,y} \quad \text{if} \quad \xi = \eta. \end{aligned}$$

Then $e_{\xi,x} = P_{\xi,x}(\epsilon) = Q_{\xi,x}(\epsilon)$ for all $\xi \in \mathcal{O}, x \in W$, moreover $P_{\xi,x}$ and $Q_{\eta,y}$ commute for all $\xi, \eta \in \mathcal{O}, x, y \in W$. Define a product on $\mathcal{K}$ by

$$e_{\xi,x}e_{\eta,y} = P_{\xi,x}(e_{\eta,y}) = Q_{\eta,y}(e_{\xi,x}).$$

This has the required properties.

We shall extend to $\mathcal{K}$ a number of properties of ordinary Hecke algebras.

3.3.2. Lemma.

(i) *Let $\xi \in \mathcal{O}, x \in W$. There exists a unique linear combination $\bar{e}_{\xi,x}$ of the $e_{\eta,y} \in \mathcal{K}$ with $y\eta = x\xi$ such that*

$$e_{x\xi,x^{-1}}\bar{e}_{\xi,x} = e_{\xi,e};$$

(ii) *There is a unique automorphism $u \mapsto \bar{u}$ of $\mathcal{K}$ sending $e_{\xi,x}$ to $\bar{e}_{\xi,x}(\xi \in \mathcal{O}, x \in W)$ and t to t^{-1}.*

If $x \in S \cup \{e\}$ this is proved by a computation. One finds

$$\begin{aligned} \bar{e}_{\xi,e} &= e_{\xi,e}, \\ \bar{e}_{\xi,s} &= e_{\xi,s} \text{ if } s \in S, s \notin W_\xi, \\ &= t^{-2}e_{\xi,s} + (t^{-2}-1)e_{\xi,e} \text{ if } s \in S, s \in W_\xi. \end{aligned}$$

We now proceed by induction on $\ell(x)$. Let $x = sy > y, s \in S$. Then $e_{x\xi,x^{-1}} = e_{y\xi,y^{-1}}e_{x\xi,s}$. Using the previous formulas and the induction hypothesis one finds that $\bar{e}_{\xi,x} = \bar{e}_{y\xi,s}\bar{e}_{\xi,y}$, and (i) follows. We omit the straightforward proof of (ii). For $\xi = 0$ the lemma reduces to a familiar result from [KL].

On the group W_ξ we have the Bruhat order $\leq_\xi$ (3.2.1). We extend it as follows. Let $x, y \in W$ lie in the same right coset of W_ξ. Write $x = x^*x_1, y = x^*y_1$, where $x_1, y_1 \in W_\xi$ (see 3.2). We now write $x <_\xi y$ if $x_1 <_\xi y_1$.

3.3.3. Lemma. *Let $\xi \in \mathcal{O}, x \in W$. There exists polynomials with integral coefficients $R_{\xi yx}(y \in W)$ such that*

$$\bar{e}_{\xi,x} = t^{-2\ell_\xi(x)} \sum_y R_{\xi yx}(t^2)e_{\xi,y}$$

We have $R_{\xi xx} = 1$. Moreover, $R_{\xi yx} \neq 0$ if and only if $y \in xW_\xi$ and $y \leq_\xi x$. In that case $R_{\xi yx}$ has degree $\ell_\xi(x) - \ell_\xi(y)$.

This is similar to results of [KL, §2]. We have the following inductive formulas. Let $x = sv > v$, then

$$\begin{aligned} R_{\xi yx} &= R_{\xi,sy,v} \text{if } v^{-1}sv \notin W_\xi \text{ or } sy < y, \\ &= (1-T)R_{\xi yv} + TR_{\xi,sy,v} \text{ if } v^{-1}sv \in W_\xi \text{ and } sy > y \end{aligned}$$

(T denotes the indeterminate).

We next introduce Kazhdan-Lusztig elements in $\mathcal{K}$.

3.3.4. Theorem. *Given $\xi \in \mathcal{O}$ and $x \in W$ there is a unique element $c_{\xi,x}$ in $\mathcal{K}$ such that $\bar{c}_{\xi,x} = c_{\xi,x}$ and*

$$c_{\xi,x} = t^{-\ell_\xi(x)} \sum_{\substack{y \in xW_\xi \\ y \leq_\xi x}} P_{\xi yx}(t^2)e_{\xi,y},$$

with $P_{\xi yx} \in \mathbb{Z}[T], P_{\xi xx} = 1$ and $\deg P_{\xi yx} \leq \frac{1}{2}(\ell_\xi(x) - \ell_\xi(y) - 1)$ if $y \in xW_\xi, y <_\xi x$.

The proof is similar to the one of Th. 1.1 in [KL]. We omit the details. One has the following formulas, which provide an inductive definition of the elements $c_{\xi,x}$:

$$\begin{aligned} c_{\xi,e} &= e_{\xi,e}, \\ c_{\xi,s} &= e_{\xi,s} \text{ if } s \notin W_\xi, \\ c_{\xi,s} &= t^{-1}(e_{\xi,s} + e_{\xi,e}) \text{ if } s \in W_\xi. \end{aligned}$$

Let $x = sy > y (s \in S)$, then

$$(5) \qquad \begin{cases} c_{\xi,x} = c_{y\xi,s} c_{\xi,y} \text{ if } y^{-1}sy \notin W_\xi, \\ c_{\xi,x} = c_{y\xi,s} c_{\xi,y} - \sum\limits_{\substack{z \in yW_\xi \\ z <_\xi y, sz < z}} \mu_\xi(z,y) c_{\xi,z} \text{ if } y^{-1}sy \in W_\xi, \end{cases}$$

where $\mu_\xi(z,y)$ is the coefficient of $t^{\ell_\xi(y)-\ell_\xi(z)-1}$ in $P_{\xi zy}(t^2)$.
This leads to the following inductive formula for the Kazhdan-Lusztig polynomials $P_{\xi yx}$, where $x = sv > v (s \in S)$:

$$(6) \qquad P_{\xi yx} = P_{\xi, sy, v} \text{ if } v^{-1}sv \notin W_\xi,$$

$$(7) \qquad \begin{cases} P_{\xi yx}(t^2) + \sum\limits_{\substack{z \in vW_\xi \\ z <_\xi v, sz < z}} \mu_\xi(z,v) t^{\ell_\xi(x)-\ell_\xi(z)} P_{\xi yz}(t^2) = \\ = \begin{cases} P_{\xi,sy,v}(t^2) + t^2 P_{\xi yv}(t^2) \\ t^2 P_{\xi,sy,v}(t^2) + P_{\xi yv}(t^2) \end{cases} \text{if } v^{-1}sv \in W_\xi \text{ and } \begin{cases} sy < y \\ sy > y \end{cases}. \end{cases}$$

We also record the formulas

$$\begin{aligned} c_{\xi,x} c_{\eta,y} &= 0 \text{ if } \xi \neq y\eta, \\ c_{y\eta,s} c_{\eta,y} &= (t + t^{-1}) c_{\eta,y} \text{ if } y^{-1}sy \in W_\eta \text{ and } sy < y, \\ c_{\xi,x} c_{\xi,s} &= (t + t^{-1}) c_{\xi,x} \text{ if } s \in W_\xi \text{ and } xs < x. \end{aligned}$$

Fix $\xi \in \mathcal{O}$. We use the notations of 3.2. We shall see that the Hecke algebra $\mathcal{H}_\xi$ of (W_ξ, S_ξ) is a subalgebra of $\mathcal{K}$.

3.3.5. Lemma. *Let* $\sigma \in S_\xi, x \in W$.

(i) $$\begin{aligned} e_{\xi,x} e_{\xi,\sigma} &= e_{\xi,x\sigma} \text{ if } \ell_\xi(x\sigma) > \ell_\xi(x), \\ &= (t^2 - 1) e_{\xi,x} + t^2 e_{\xi,x\sigma} \text{ if } \ell_\xi(x\sigma) < \ell_\xi(x); \end{aligned}$$

(ii) *If* $x \in W_\xi^!$ *then*

$$\begin{aligned} e_{\xi,\sigma} e_{\xi,x} &= e_{\xi,\sigma x} \text{ if } \ell_\xi(\sigma x) > \ell_\xi(x), \\ &= (t^2 - 1) e_{\xi,x} + t^2 e_{\xi,\sigma x} \text{ if } \ell_\xi(\sigma x) < \ell_\xi(x). \end{aligned}$$

Let $\mathbf{s} = (s_1, \ldots, s_r)$ be a reduced decomposition of σ, with $s_i \in S$.
By 3.2.6 there exists a unique $m \in [1, r]$ such that

$$\sigma = s_r \ldots s_{m+1} s_m \ldots s_r,$$

it follows that $s_r \ldots s_{m+1} = s_1 \ldots s_{m-1}$.
We now have

$$e_{\xi,x} e_{\xi,\sigma} = e_{\xi,x} e_{s_2 \ldots s_r \xi, s_1} \cdots e_{s_r \xi, s_{r-1}} e_{\xi, s_r}.$$

By the formulas of 3.3.1 we see that this equals

$$e_{s_m \ldots s_r \xi, x s_1 \ldots s_{m-1}} e_{s_{m+1} \ldots s_r \xi, s_m} \cdots e_{\xi, s_r},$$

which equals

$$e_{s_{m+1}\dots s_r\xi, xs_1\dots s_m} e_{s_{m+2}\dots s_r\xi, s_{m+1}} \dots e_{\xi,s_r} \text{ if } xs_1\dots s_m > xs_1\dots s_{m-1}$$

and

$$\left((t^2-1)e_{s_{m+1}\dots s_r\xi, xs_1\dots s_{m-1}} + t^2 e_{s_{m+1}\dots s_r\xi, xs_1\dots s_m}\right) e_{s_{m+2}\dots s_r\xi, s_{m+1}} \dots e_{\xi,s_r} \text{ if}$$
$$xs_1\dots s_m < xs_1\dots s_{m-1}.$$

Since $s_1\dots s_{m-1} = s_r\dots s_{m+1}$, another application of the formulas shows that

$$e_{\xi,x}e_{\xi,\sigma} = e_{\xi,x\sigma} \text{ resp. } (t^2-1)e_{\xi,x} + t^2 e_{\xi,x\sigma}.$$

The condition $xs_1\dots s_m > xs_1\dots s_{m-1}$ is equivalent to $x\beta > 0$, where $\beta \in R_\xi^+$ corresponds to σ. Hence this condition is equivalent to $l_\xi(x\sigma) > l_\xi(x)$. This proves (i), and (ii) is proved similarly.

The next lemma is proved in a similar fashion (using 3.2.3).

3.3.6. **Lemma.** Let $x^* \in W_\xi^*$. *If $y \in W$ we have*

$$e_{x^*\xi,y}e_{\xi,x^*} = e_{\xi,yx^*}, \quad e_{\xi,x^*}e_{y^{-1}\xi,y} = e_{y^{-1}\xi,x^*y}.$$

In particular, if $x_1 \in W_\xi$ then $e_{\xi,x^*x_1} = e_{\xi,x^*}e_{\xi,x_1}$.

The following result is a consequence of 3.3.5 and 3.3.6.

3.3.7. **Proposition.**

(i) *The elements $e_{\xi,x}$ with $x \in W_\xi$ span a subalgebra $\mathcal{H}_\xi$ of $\mathcal{K}$ which is isomorphic to the Hecke algebra of W_ξ;*

(ii) *The elements $e_{\xi,x}$ with $x \in W_\xi^!$ span a subalgebra $\mathcal{H}_\xi^!$ of $\mathcal{K}$. As a $\mathbb{Z}[t,t^{-1}]$- module it is isomorphic to the tensor product of $\mathcal{H}_\xi$ and the group algebra of the group $W_\xi^* \cap W_\xi^!$.*

Notice that we have $e_{\xi,x}e_{\xi,y} = e_{\xi,xy}$ if $x, y \in W_\xi^!$ and at least one of the elements x, y lies in $W_\xi^* \cap W_\xi^!$

3.3.8. We also see that the automorphism $u \mapsto \bar{u}$ of $\mathcal{K}$ stabilizes $\mathcal{H}_\xi$ (and $\mathcal{H}_\xi^!$) and induces the automorphism of $\mathcal{H}_\xi$ introduced in [KL]. It follows that for $x \in W_\xi$, the element $c_{\xi,x}$ of 3.3.4 is the corresponding Kazhdan-Lusztig element of the Hecke algebra $\mathcal{H}_\xi$.

We state a few properties which readily follow from the preceeding observations.

Let $x, y \in W, y \in xW_\xi$. Put $x = x^*x_1, y = x^*y_1$ with $x^* \in W_\xi^*, x_1, y_1 \in W_\xi$. Then

$$c_{\xi,x} = e_{\xi,x^*}c_{\xi,x_1}$$

and

$$P_{\xi yx} = P_{\xi,y_1,x_1}.$$

It follows that the polynomials $P_{\xi yx}$ can be described by the Kazhdan-Lusztig polynomials for the Hecke algebra $\mathcal{H}_\xi$.

The next lemma will be needed later (5.1.10 and 11.3.17). The proof offers no difficulty.

3.3.9. **Lemma.** Let $y \in W_\xi^*$. *The map $f : e_{\xi,x} \mapsto e_{y\xi,yxy^{-1}}$ defines an isomorphism $\mathcal{H}_\xi^! \to \mathcal{H}_{y\xi}^!$, commuting with the bar automorphism. We have $f(u) = e_{\xi,y}u\bar{e}_{y\xi,y^{-1}}$ ($u \in \mathcal{H}_\xi^!$).*

4 Some perverse sheaves on a reductive group

The results of this section are variants of the ones contained in the first chapter of [L1].

4.1 The perverse sheaves $A_{\xi,\dot{w}}$

4.1.1. We use the notations of 3.1 and 3.2. So G denotes a connected reductive group over k and T a maximal torus in G. Fix a Borel subgroup B of G containing T and assume that the system of positive roots R^+ of 3.2 is the one defined by B. The unipotent radical of B is denoted by U.
For $\alpha \in R$ denote by $x_\alpha : k \to G$ a one parameter addivite subgroup of G associated to α. Put $X_\alpha = \text{ im } x_\alpha$. If $w \in W$ we denote by $\dot{w} \in N_G T$ a representative, for the moment chosen arbitrarily. The subgroup of U generated by the X_α with $\alpha \in R^+$, $-w^{-1}\alpha \in R^+$ is denoted by U_w. By Bruhat's lemma, G is the disjoint union of the locally closed, smooth subsets $G_w = B\dot{w}B$. Also, the closure $\bar{G}_w$ is the union of the G_x with $x \leq w$.
The map $U_w \times T \times U \to G_w$ sending (u, t, u') to $u\dot{w}tu'$ is an isomorphism of varieties. Define a morphism $pr : G_w \to T$ by $pr(u\dot{w}tu') = t$.
If $\mathcal{L} = \mathcal{L}_\xi$ is a Kummer local system on T then $pr^*\mathcal{L} = \mathcal{L}_{\xi,\dot{w}}$ is a (tame) local system on G_w. We denote by $A_{\xi,\dot{w}}$ the perverse extension of $\mathcal{L}_{\xi,\dot{w}}$ to G (see 1.2.3). Its support is contained in $\bar{G}_w$.
It is clear that $A_{\xi,\dot{w}}$ is determined by w up to isomorphism. If we are only interested in its isomorphism class we shall write $A_{\xi,w}$ (and also $\mathcal{L}_{\xi,w}$).
We shall next establish some equivariance properties. The group B operates on G by left and right translations, and also by conjugation.
4.1.2. **Lemma.**
(i) *$A_{\xi,\dot{w}}$ has weight $w\xi$ for left B- action and weight $-\xi$ for right B- action;*
(ii) *If $w\xi = \xi$ then $A_{\xi,\dot{w}}$ is equivariant for the conjugation action of B;*
(iii) $DA_{\xi,w} \simeq A_{-\xi,w}$.
The notion of weight is as in 2.2.3. The assertion (i) follows from 2.2.4, as the local system $\mathcal{L}_{\xi,\dot{w}}$ clearly has the properties required in that lemma. Then (ii) is a formal consequence of (i) and (iii) follows from 2.1.6.
We next consider the restriction to G_x $(x \in W)$ of the cohomology sheaf $H^\cdot(A_{\xi,\dot{w}})$.
4.1.3 **Lemma.**
(i) *$H^\cdot(A_{\xi,\dot{w}})\ |_{G_x} \otimes L_{-\xi,\dot{x}}$ is a constant sheaf;*
(ii) *If $H^\cdot(A_{\xi,\dot{w}})\ |_{G_x} \neq 0$ then $x\xi = w\xi$ and $x \leq w$.*
$\mathcal{S} = H^\cdot(A_{\xi,\dot{w}})\ |_{G_x}$ is a constructible sheaf on $G_x \simeq U_x \times B$ which is U_x-equivariant for left U_x- action and has weight $-\xi$ for right B-action. These facts imply that $\mathcal{S}$ is locally constant and that $\mathcal{S} \otimes \mathcal{L}_{-\xi,\dot{x}}$ is constant, whence (i). They also imply that $\mathcal{S}$ has weight $x\xi$ for left B-action. But by the previous lemma, $\mathcal{S}$ has weight $w\xi$ for B-action. Hence $x\xi = w\xi$. That $\mathcal{S} \neq 0$ implies $x \leq w$ is clear. The lemma is proved.
4.1.4. The Borel group B acts on $G \times G$ by $b(g, h) = (gb^{-1}, bh)$. A quotient $G \overset{B}{\times} G$ exists and the product map $G \times G \to G$ induces a proper morphism $\pi : G \overset{B}{\times} G \to G$. More generally, if V and Z are right resp. left B-stable locally closed subsets of G there exists a similar quotient $V \overset{B}{\times} Z$ and a morphism $V \overset{B}{\times} Z \to G$, also denoted by π. If V and Z are closed this is a proper morphism.
Let $\xi \in \hat{X}, x, y \in W$. It follows from 4.1.2 that the exterior tensor product $A_{\xi,\dot{x}} \boxtimes A_{y^-\xi,\dot{y}}$ is

an irreducible perverse sheaf $G \times G$ which is B-equivariant, for the action just considered. It follows from 1.4.2 that there exists an irreducible perverse sheaf $A_{\xi,\dot{x},\dot{y}}$ (or $A_{\xi,x,y}$) on $G \overset{B}{\times} G$ such that $A_{\xi,\dot{x}} \boxtimes A_{y^{-1}\xi,\dot{y}}$ is the pull-back of $A_{\xi,\dot{x},\dot{y}}$ (up to a dimension shift).
We shall sometimes identify $A_{\xi,\dot{x}} \boxtimes A_{y^{-1}\xi,\dot{y}}$ resp. $A_{\xi,\dot{x},\dot{y}}$ with its restriction to $\bar{G}_x \times \bar{G}_y$ resp. $\bar{G}_x \overset{B}{\times} \bar{G}_y$.
We put

$$\pi_*(A_{\xi,\dot{x},\dot{y}}) = A_{\xi,\dot{x}} * A_{y^{-1}\xi,\dot{y}},$$

this is a semi-simple complex on G, by the decomposition theorem (1.3.2). The product $*$ has the obvious associativity property.
The restriction of $A_{\xi,\dot{x},\dot{y}}$ to $G_x \overset{B}{\times} G_y$ is of the form $\mathcal{L}_{\xi,\dot{x},\dot{y}}[\dim G_x \overset{B}{\times} G_y]$, where $\mathcal{L}_{\xi,\dot{x},\dot{y}}$ is a local system. Its pull-back on $G_x \times G_y$ is $\mathcal{L}_{\xi,\dot{x}} \boxtimes \mathcal{L}_{y^{-1}\xi,\dot{y}}$. If convenient we write $\mathcal{L}_{\xi,x,y}$ for $\mathcal{L}_{\xi,\dot{x},\dot{y}}$.

4.2 The cohomology sheaves of $A_{\xi,\dot{w}}$

4.2.1. We first collect a number of auxiliary results. Let $w \in W$, $s \in S$. So $s = s_\alpha$ with $\alpha \in D$.
We may take $G_s \overset{B}{\times} G_w = X_\alpha \times G_w$ with $\pi(u,g) = u\dot{s}g$ ($u \in X_\alpha, g \in G_w$). There are two cases:
(a) $sw > w$. Then π defines an isomorphism $G_s \overset{B}{\times} G_w \xrightarrow{\sim} G_{sw}$ and one checks that $\pi_*\mathcal{L}_{\xi,\dot{s},\dot{w}} = \mathcal{L}_{w^{-1}\xi,\dot{s}\dot{w}}$ (viewing $\dot{s}\dot{w}$ as a representative of sw).
(b) $sw < w$. Now $\pi(G_s \overset{B}{\times} G_w) = G_w \coprod G_{sw}$. Moreover, with obvious notations, $G_s \overset{B}{\times} G_w \simeq (G_s \overset{B}{\times} G_s) \overset{B}{\times} G_{sw}$. This reduces the analysis of the geometric properties of π to the case $w = s$, in which case the analysis can be carried out in a group of semi-simple rank one. We omit the details. The results are as follows.
If $g \in G_{sw}$ then $\pi^{-1}g \simeq k$ and the restriction of $\mathcal{L}_{\xi,s,w}$ to $\pi^{-1}g$ is constant.
If $g \in G_w$ then $\pi^{-1}g \simeq k^*$. Let $\mathcal{L}_\xi = \mathcal{L}_{x,m}$ (as in 2.1.2) and define $a \in \mathbb{Z}_{(p)}/\mathbb{Z}$ by $a = m^{-1} < x, \alpha^\vee > +\mathbb{Z}$ (notations of 2.1 and 3.1). Then $\mathcal{L}_a$ is a Kummer local system on k^*. Notice that $a = 0$ if and only if $s \in W_\xi$. The restriction of $\mathcal{L}_{\xi,s,w}$ to $\pi^{-1}g \simeq k^*$ is isomorphic to $\mathcal{L}_a$.

We next consider $A_{\xi,s}$.
4.2.2. **Lemma.**
(i) *If $s \in W_\xi$ the local system $\mathcal{L}_{\xi,s}$ on G_s extends to a local system $\bar{\mathcal{L}}_{\xi,s}$ on $\bar{G}_s$ and $A_{\xi,s} = \bar{\mathcal{L}}_{\xi,s}[\dim G_s]$. We have $A_{\xi,s} * A_{\xi,s} = A_{\xi,s}[1] \oplus A_{\xi,s}[-1]$;*
(ii) *If $s \notin W_\xi$ the restriction of $A_{\xi,s}$ to $G_e = \bar{G}_s - G_s$ is zero. We have $A_{s\xi,s} * A_{\xi,s} = A_{\xi,e}$.*
$\bar{G}_s$ is a smooth subvariety of G (it is a parabolic subgroup) and $G_e = B$ is a smooth divisor in $\bar{G}_s$. Applying 1.2.4 (b) we see that $A_{\xi,s}$ can be given by one sheaf, placed in dimension $-\dim G_s$. If the local monodromy of $\mathcal{L}_{\xi,s}$ around the divisor G_e is trivial, $\mathcal{L}_{\xi,s}$ extends to a local system $\bar{\mathcal{L}}_{\xi,s}$ on $\bar{G}_s$ and $A_{\xi,s} = \bar{\mathcal{L}}_{\xi,s}[\dim G_s]$. Otherwise the restriction of $A_{\xi,s}$ to G_e is zero. One checks that the local monodromy of $\mathcal{L}_{\xi,s}$ is trivial if and only if $s \in W_\xi$. To see this one uses that for any $g \in G_e$ there exists in $\bar{G}_s$ a cross section P to G_e passing through g, which is isomorphic to k, such that the restriction of $\mathcal{L}_{\xi,s}$ to $P - \{g\} \simeq k^*$ corresponds to the Kummer local system $\mathcal{L}_b$ on k^*, where $b = m^{-1} < x, \alpha^\vee > +\mathbb{Z} \in \mathbb{Z}_{(p)}/\mathbb{Z}(\mathcal{L}_\xi = \mathcal{L}_{x,m})$.
We have established the first assertions of (i) and (ii).

If $s \in W_\xi$ then $A_{\xi,s,s} = \bar{\mathcal{L}}_{\xi,s,s}[\dim G_s + 1]$, where $\bar{\mathcal{L}}_{\xi,s,s}$ is the extension of $\mathcal{L}_{\xi,s,s}$ to $\bar{G}_s \overset{B}{\times} \bar{G}_s$ (the existence follows from the preceding results). Any fiber of $\pi : \bar{G}_s \overset{B}{\times} \bar{G}_s \to \bar{G}_s$ is isomorphic to the projective line and the restriction of $\bar{\mathcal{L}}_{\xi,s,s}$ to it is constant. If follows that if $g \in \bar{G}_s$ we have, putting $A = A_{\xi,s,s}$, that

$$H^i(\pi_*A)_g = H^i(\pi^{-1}g, A) = H^{i+\dim G_s+1}(\pi^{-1}g, \bar{\mathcal{L}}_{\xi,s,s})$$

is one dimensional for $i = -\dim G_s \pm 1$ and is zero for all other i. Using the decomposition theorem we conclude that

$$\pi_*A = A_{\xi,s}[1] \oplus A_{\xi,s}[-1].$$

If $s \notin W_\xi$ the restriction of $A = A_{s\xi,s,s}$ to the complement of $G_s \overset{B}{\times} G_s$ is zero. It follows that if $g \in \bar{G}_s$

$$H^i(\pi_*A)_g = H^i_c((\pi^{-1}g) \cap (G_s \overset{B}{\times} G_s), A) =$$

$$H_c^{i+\dim G_s+1}((\pi^{-1}g) \cap (G_s \overset{B}{\times} G_s), \mathcal{L}_{s\xi,s,s}).$$

Using the facts stated in 4.2.1 we see that this is zero if $g \in G_s$ or if $g \in G_e$, $i \neq -\dim G_e$ and is one dimensional if $g \in G_e$, $i = -\dim G_e$. It follows that $\pi_*A = A_{\xi,e}$. This proves 4.2.2.

We shall now analyse the perverse sheaves $A_{\xi,w}$, using induction on $\ell(w)$. Let $s \in S, w \in W$ and $v = sw < w$. Consider $\pi : \bar{G}_s \overset{B}{\times} \bar{G}_v \to G$. Its image is $\bar{G}_w$. On $\bar{G}_s \overset{B}{\times} \bar{G}_v$ we have the perverse sheaf $A = A_{v\xi,s,v}$. The inverse image of A on $\bar{G}_s \times \bar{G}_v$ is $A_{v\xi,s} \boxtimes A_{\xi,v}[-\dim B]$. Using 4.2.2 we see that the inverse image of H^iA on $\bar{G}_s \times \bar{G}_v$ is

$$H^{-\dim G_s}(A_{v\xi,s}) \otimes H^{i+1}(A_{\xi,v}). \tag{1}$$

We denote the rank of the local system $H^i(A_{\xi,x})\,|_{G_y}$ (see 4.1.3 (i)) by $n_{y\xi}$.

4.2.3. Theorem.

(i) *If* $H^i(A_{\xi,w})\,|_{G_x} \neq 0$ *then* $x \in wW_\xi$;

(ii) *Let* $g \in G_x$. *Then*

$$\begin{aligned}\dim H^i(\pi_*A)_g &= n_{sx,v,i+1} \text{ if } w^{-1}sw \notin W_\xi, sx < x,\\ &= n_{sx,v,i-1} \text{ if } w^{-1}sw \notin W_\xi, sx > x,\\ &= n_{x,v,i-1} + n_{sx,v,i+1} \text{ if } w^{-1}sw \in W_\xi, sx < x,\\ &= n_{sx,v,i-1} + n_{x,v,i+1} \text{ if } w^{-1}sw \in W_\xi, sx > x;\end{aligned}$$

(iii) $\pi_*A = A_{\xi,w}$ *if* $w^{-1}sw \notin W_\xi$ *and*

$$\pi_*A = A_{\xi,w} \oplus \bigoplus_{sx<x} n_{x,v-\dim G_x-1} A_{\xi,x} \text{ if } w^{-1}sw \in W_\xi;$$

(iv) $A_{\xi,w}$ *is even.*

(We have denoted by nB the direct sum of n copies of the perverse sheaf B.)

We assume that the theorem is true if w is replaced by an element of smaller length. Notice that (i) and (iv) are trivially true if $w = e$.

First assume that $w^{-1}sw \notin W_\xi$. We see using (1) and 4.2.2 (ii) that the restriction of A to $G_e \overset{B}{\times} \bar{G}_v$ is zero and that the inverse image of H^iA on $G_s \times \bar{G}_v$ is $\mathcal{L}_{v\xi,s} \boxtimes H^{i+1}(A_{\xi,v})$. Using 4.1.3 (i) we see that the restriction to $G_s \times G_x$ $(x \leq v)$ of this inverse image is a direct sum of $n_{x,v,i+1}$ copies of $\mathcal{L}_{v\xi,s} \boxtimes \mathcal{L}_{\xi,x}$. If it is non-zero then $x\xi = v\xi$ (4.1.3 (ii)). So the restriction of H^iA to $G_s \overset{B}{\times} G_x$ is a direct sum of $n_{x,v,i+1}$ copies of $\mathcal{L}_{x\xi,s,x}$. Put $V = H_c^i((\pi^{-1}g) \cap (G_s \overset{B}{\times} G_x), A)$.

(a) If $sx > x$ then $G_s \overset{B}{\times} G_x \simeq G_{sx}$ and $\mathcal{L}_{x\xi,s,x} \simeq \mathcal{L}_{\xi,sx}$. If $g \in G_{sx}$ then $(\pi^{-1}g) \cap (G_s \overset{B}{\times} G_x)$ is a point and it follows that $\dim V = n_{x,v,i+1}$. If $g \notin G_{sx}$ then $V = 0$.

(b) If $sx < x$ and $g \in G_{sx}$ then $(\pi^{-1}g \cap (G_s \overset{B}{\times} G_x) \simeq k$, and the restriction of $\mathcal{L}_{x\xi,s,x}$ to it is constant. It follows that now

$$V \simeq H_c^2(k, H^{i-2}A),$$

which is a vector space of dimension $n_{x,v,i-1}$.

If $g \in G_x$ then $(\pi^{-1}g) \cap (G_s \overset{B}{\times} G_x) \simeq k^*$. If the restriction of H^iA to this intersection is non-zero then the restriction of $H^{i+1}(A_{\xi,v})$ to G_x is non-zero and by induction we obtain $x \in vW_\xi$. Since $w^{-1}sw \notin W_\xi$ we have $x^{-1}sx \notin W_\xi$. Now 4.2.1 shows that the restriction of $\mathcal{L}_{x\xi,s,x}$ to $(\pi^{-1}g) \cap (G_s \overset{B}{\times} G_x)$ is a non-trivial Kummer local system. We conclude that now $V = 0$ (see 2.1.5).

Using the decomposition of $\bar{G}_s \overset{B}{\times} \bar{G}_v$ into the locally closed subvarieties $G_e \overset{B}{\times} G_x, G_s \overset{B}{\times} G_x$ $(x \leq v)$ it follows from the preceding observations by standard arguments that if $g \in G_x (x \leq v)$ the vector space $H^i(\pi_*A)_g$ has dimension $n_{sx,v,i+1}$ resp. $n_{sx,v,i-1}$ if $sx < x$ resp. $sx > x$. This proves (ii) if $w^{-1}sw \notin W_\xi$.

Using that $A_{\xi,v}$ is a perverse extension we see that $\dim \operatorname{supp} H^i(\pi_*A) < -i$ if $i \neq -\dim G_w$ (see 1.2.3). The same is true for $H^i(D\pi_*A)$, since π is proper and $DA = A_{-v\xi,s,v}$ by 4.1.2 (iii). Hence π_*A is a perverse extension. Since the restriction of π_*A to G_w is $\mathcal{L}_{\xi,w}[\dim G_w]$ we have $\pi_*A = A_{\xi,w}$. The assertions of the theorem now readily follow.

Next assume that $w^{-1}sw \in W_\xi$. The inverse image of H^iA on $\bar{G}_s \times \bar{G}_v$ is now $\bar{\mathcal{L}}_{v\xi,s} \boxtimes H^{i+1}(A_{\xi,v})$, where $\bar{\mathcal{L}}_{v\xi,s}$ is as in 4.2.2 (i). The variety $\bar{G}_s \overset{B}{\times} \bar{G}_v$ is stratified by the $\bar{G}_s \overset{B}{\times} G_x$, where $x \leq v$. If the restriction of $H^i(A)$ to $\bar{G}_s \overset{B}{\times} G_x$ is non-zero then the restriction of $H^{i+1}(A_{\xi,v})$ to G_x is non-zero and we have, again by induction, that $x \in vW_\xi$, hence $x^{-1}sx \in W_\xi$, and also $i \equiv \dim G_w \pmod 2$.

The restriction of H^iA to $\bar{G}_s \overset{B}{\times} G_x$ is a direct sum of $n_{x,v,i+1}$ copies of a one dimensional local system $\bar{\mathcal{L}}_{x\xi,s,x}$, which extends the local system $\mathcal{L}_{x\xi,s,x}$ on $G_s \overset{B}{\times} G_x$. Put $V = H_c^i((\pi^{-1}g) \cap (\bar{G}_s \overset{B}{\times} G_x), A)$.

(a) If $sx > x$ we have $\pi : \bar{G}_s \overset{B}{\times} G_x \simeq G_{sx} \coprod G_x$ and $\pi_*\bar{\mathcal{L}}_{x\xi,s,x}$ extends $\mathcal{L}_{\xi,sx}$. Now V is a vector space of dimension $n_{x,v,i+1}$ if $g \in G_{sx} \coprod G_x$ and $i \equiv \dim G_w$ (mod2) and is zero otherwise.

(b) If $sx < x$ then$(\pi^{-1}g) \cap (\bar{G}_s \overset{B}{\times} G_x)$ is an affine line if $g \in G_{sx} \coprod G_x$, as follows from the facts stated in 4.2.1, and is empty otherwise. The restriction of $\bar{\mathcal{L}}_{x\xi,s,x}$ to such affine lines is constant. It follows that now $V \simeq H_c^2(k, H^{i-2}A)$, which is a vector space of dimension $n_{x,v,i-1}$ if $g \in G_{sx} \coprod G_x$ and $i \equiv \dim G_w \pmod 2$ and is zero otherwise.

Using the decomposition of $\bar{G}_s \overset{B}{\times} \bar{G}_v$ into the locally closed subvarieties $\bar{G}_s \overset{B}{\times} G_x (x \leq v)$

we obtain that for $g \in G_x$ we have

$$\begin{aligned}\dim H^i(\pi_*A)_g &= 0 \text{ if } i \not\equiv \dim G_w \text{ (mod2)},\\ &= n_{x,v,i-1} + n_{sx,v,i+1} \text{ if } i \equiv \dim G_w \text{ (mod2) and } sx < x,\\ &= n_{sx,v,i-1} + n_{x,v,i+1} \text{ if } i \equiv \dim G_w \text{ (mod2) and } sx > x.\end{aligned}$$

This proves parts (ii) and (iv) of the theorem.
Using that $\dim \operatorname{supp} H^i(A_{\xi,v}) < -i$ if $i > -\dim G_v$ we see that

$$\dim \operatorname{supp} H^i(\pi_*A) \leq -i$$

for all i, and similarly for DA. Hence π_*A is a perverse sheaf. More precisely we find the following: if the restriction of $H^i(\pi_*A)$ to G_x is non-zero then $\dim G_x < -i$ except possibly when $i = -\dim G_x$ and either $x = w$ or $x < v, sx < x$.
By the decomposition theorem, π_*A is a direct sum of simple perverse sheaves and the facts stated above show that π_*A has to be as asserted in part (iii) of the theorem. Parts (ii) and (iv) were already proved, and part (i) also follows readily.
We can now identify the integers n_{xyi}. Put

$$F_{xw}(t^2) = t^{\dim G_x + \ell_\xi(w) - \ell_\xi(x)} \sum_{i \in \mathbf{Z}} n_{xwi} t^i$$

(this makes sense because of parts (i) and (iv) of the theorem and the corollary of 3.2.6). Here ℓ_ξ is as in 3.2.

4.2.4. **Corollary.** *F_{xw} is the Kazhdan - Lusztig polynomial $P_{\xi xw}$ of* 3.3.4.
Let $w = sv$, as in the proof of the theorem.
(a) $w^{-1}sw \notin W_\xi$. By 4.2.3 (i) we may assume that $x \in wW_\xi$. It then follows that $\ell_\xi(w) - \ell_\xi(x) = \ell_\xi(v) - \ell_\xi(sx)$ (use 3.2.7 (i)). We conclude from 4.2.3 (ii), (iii) that $F_{xw} = F_{sx,v}$. By an induction on $\ell(w)$ we may assume that $F_{sx,v} = P_{\xi,sx,v}$. Formula (6) of 3.3 then shows that $F_{xw} = P_{\xi xw}$.
(b) $w^{-1}sw \in W_\xi$. Using 3.2.7 (ii) and parts (ii) and (iii) of 4.2.3 we now obtain that

$$F_{xw}(t^2) + \sum_{sz<z} n_{z,v,-\dim G_z-1} t^{\ell_\xi(w)-\ell_\xi(z)} F_{xz}(t^2) =$$

$$= \begin{cases} F_{sx,v}(t^2) + t^2 F_{xv}(t^2) & \text{if } sx < x \\ t^2 F_{sx,v}(t^2) + F_{xv}(t^2) & \text{if } sx > x. \end{cases}$$

That $F_{xw} = P_{\xi xw}$ follows again by induction, using now formula (7) of 3.3.
Notice that by 3.3.8 the F_{xw} can be expressed in terms of the Kazhdan- Lustzig polynomials for the Weyl group W_ξ.

Next we shall tie up the product $*$ of 4.1.4 with the Hecke algebras of 3.3. Let $\mathcal{O} = W.\xi$ and let $\mathcal{K}$ be as in 3.3. If A is a semi-simple complex on G of the form

$$A = \bigoplus_{\eta \in \mathcal{O}, x \in W} A_{\eta,x}[n_{\eta,x}]$$

we associate to it the element

$$h(A) = \sum_{\eta \in \mathcal{O}, x \in W} t^{-n_{\eta,x}} c_{\eta,x},$$

where $c_{\eta,x}$ is as in 3.3.4.
In particular, $h(A_{\eta,x}) = c_{\eta,x}$.
4.2.5. **Corollary.** *For $\eta \in \mathcal{O}, x, y \in W$ we have*

$$h(A_{y\eta,x} * A_{\eta,y}) = c_{y\eta,x}c_{\eta,y}.$$

We use the notations of the proof of 4.2.3. We then have $\pi_* A = A_{v\xi,s} * A_{\xi,v}$. It follows from 4.2.3 and 4.2.4 that

$$\begin{aligned} h(A_{v\xi,s} * A_{\xi,v}) &= c_{\xi,w} \text{ if } w^{-1}sw \notin W_\xi, \\ &= c_{\xi,w} + \textstyle\sum_{z \in wW_\xi, sz < z} \mu_\xi(x,v) c_{\xi,z} \text{ if } w^{-1}sw \in W_\xi. \end{aligned}$$

The right-hand sides equals $c_{v\xi,s}c_{\xi,v}$ (see formula (5) of 3.3). It follows that the asserted formula holds if $x \in S$ and $xy > y$. If $s \in S$ and $sy < y$ application of the preceding formula for sy and of 4.2.2 readily gives that the asserted formula is true for $x \in S$. Repeated application of this fact and induction on $\ell(x)$ now proves the corollary.
4.2.6. **Corollary.** *Let $\xi, \eta \in \mathcal{O}, x, y \in W$. Then $c_{\xi,x}c_{\eta,y}$ is a linear combination of the $c_{\varsigma,z}(\varsigma \in \mathcal{O}, z \in W)$ whose coefficients are Laurent polynomials with non-negative integral coefficients.*
This follows from the previous corollary and the decomposition theorem (see 4.1.4).

4.3 Finite ground fields

Assume that k is an algebraic closure of the finite field F_q. As before, F denotes a Frobenius morphism.
4.3.1. Assume that G is defined over F_q, and that T and B are defined over F_q and that T is split over F_q. Let $\xi \in \hat{X}$ be F-stable. This means now that $(q-1)\xi = 0$ (since F operates on X by multiplication by q).
We may and shall choose the representatives $\dot{w}$ of the Weyl group elements in $G(\mathsf{F}_q)$. We also assume that for $x, y \in W$ with $\ell(xy) = \ell(x) + \ell(y)$ we have $(xy)^\cdot = \dot{x}\dot{y}$ (which can be arranged, as follows from [Sp1, 11.2.8]). It then follows that the local system $\mathcal{L}_{\xi,\dot{w}}$ of 4.1.1 comes from one on the F_q- scheme G_0 underlying G. The same is then true for the perverse sheaf $A_{\xi,\dot{w}}$. In particular, we have a canonical isomorphism $\varphi : F^* A_{\xi,\dot{w}} \xrightarrow{\sim} A_{\xi,\dot{w}}$.
4.3.2. **Proposition.** *Let $x, w \in W$ be such that $H^i(A_{\xi,\dot{w}})\,|_{G_x} \neq 0$. There exists a $(q-1)$-th root of unity ϵ, depending on $\xi, (w^*)^\cdot$ only such that the eigenvalues of φ on $H^i(A_{\xi,\dot{w}})_{\dot{x}}$ are all $\epsilon q^{\frac{1}{2}(i+\dim G_w)}$.*
Here w^* is as in 3.2.
This emerges from the proof of 4.2.3, if one also takes into account the φ-actions. The root of unity ϵ comes from 2.3.2. We omit the details of the argument. Notice that the q-powers occuring in the proposition are integral and non-negative.

5 Character sheaves, definition and first properties

The notations are as in the preceding section.

5.1 Definition of character sheaves.

5.1.1. Consider the action of B on $G \times G$ defined by

$$b.(g,h) = (gb^{-1}, bhb^{-1}) \ (b \in B, g, h \in G).$$

A quotient $G \times_B G$ exists and the map $(g,h) \mapsto ghg^{-1}$ of $G \times G \to G$ induces a proper morphism $\gamma : G \times_B G \to G$. More generally, if Z is a locally closed subset of G which is stable for B-conjugation there exist a similar quotient $G \times_B Z$ and a morphism $G \times_B Z \to G$, also denoted by γ. It is a proper morphism if Z is closed.
Let $\xi \in \hat{X}$ and $w \in W'_\xi$ (i.e. $w \in W$ and $w\xi = \xi$). By 4.1.2 (ii), the perverse sheaf $A_{\xi,\dot{w}}$ of 4.1 is equivariant for B-conjugation. There exists by 1.4.2 an irreducible perverse sheaf $\tilde{A}_{\xi,\dot{w}}$ on $G \times_B G$ whose pull-back to $G \times G$ is $E \boxtimes A_{\xi,\dot{w}}[\dim G - \dim B]$. Then $\gamma_* \tilde{A}_{\xi,\dot{w}} = C_{\xi,\dot{w}}$ is a semi-simple complex on G. If we are only interested in its isomorphism class we write $C_{\xi,w}$. If necessary we write $C^G_{\xi,\dot{w}}$ or $C^G_{\xi,w}$.
5.1.2. **Definition.** *A character sheaf on G is an irreducible constituent of some* $C_{\xi,\dot{w}}$.
By properties (b) and (c) of 1.4.1 it follows from the definitions that character sheaves are perverse sheaves which are G-equivariant for conjugation action.
In 5.4 we shall discuss some concrete examples. With the notations of [CS] we have $G \times_B \bar{G}_w = \bar{Y}_w$, $G \times_B G_w = Y_w$ and $C_{\xi,w}[-\dim G - \ell(w)] = \bar{K}^{\mathcal{L}_\xi}_{\dot{w}}$ (see [loc. cit., 12.1]).
5.1.3. Similarly to the variety $G \overset{B}{\times} G$ in 4.1.4 one can introduce $V_r = G \overset{B}{\times} \dots \overset{B}{\times} G$ (r factors). The B-action $b(g_1, \dots, g_r) = (bg_1, g_2, \dots, g_r b^{-1})$ on G^r induces a B-action on V_r. There exists a quotient Y_r for the B-action on $G \times V_r$ defined by

$$b(g,v) = (gb^{-1}, b.v)$$

The map $G^{r+1} \to G$ defined by $(g, g_1, \dots, g_r) \mapsto gg_1 \dots g_r g^{-1}$ induces a proper morphism $\gamma_r : Y_r \to G$. The product map $V_r \to G$ induces a morphism $\pi_r : Y_r \to G \times_B G$ such that $\gamma_r = \gamma \circ \pi_r$.
Let $\mathbf{s} = (w_1, \dots, w_r)$ be a sequence of elements in W. Put $w = w_1 \dots w_r$. If $\xi \in \hat{X}$ is such that $w\xi = \xi$, the exterior tensor product on G^r

$$A_{w_2 \dots w_r \xi, w_1} \boxtimes \dots \boxtimes A_{w_r \xi, w_{r-1}} \boxtimes A_{\xi, w_r}$$

is a perverse sheaf on G^r, which up to a shift $[-(r-1)\dim B]$ is the pull-back of a perverse sheaf A on V_r. There is a perverse sheaf $\tilde{A}_{\xi,\mathbf{s}}$ on Y_r whose pull-back to $G \times V_r$ is $E \boxtimes A[\dim G - \dim B]$. We put $C_{\xi,\mathbf{s}} = (\gamma_r)_* \tilde{A}_{\xi,\mathbf{s}}$.
We introduce (obvious notations) $\bar{Y}_\mathbf{s} = G \times_B (\bar{G}_{w_1} \overset{B}{\times} \dots \overset{B}{\times} \bar{G}_{w_r})$, this is a closed subvariety of Y_r and supp $\tilde{A}_{\xi,\mathbf{s}} = \bar{Y}_\mathbf{s}$. We have a proper morphism $\gamma^G_\mathbf{s} = \gamma_\mathbf{s} = \bar{Y}_\mathbf{s} \to G$, induced by γ_r. Then $C_{\xi,\mathbf{s}}$ is the extension by zero of $(\gamma_\mathbf{s})_*(\tilde{A}_{\xi,\mathbf{s}} |_{\bar{Y}_\mathbf{s}})$ and $C_{\xi,\mathbf{s}}$ is a semi-simple complex on G. Similarly, we can introduce $Y_\mathbf{s} = G \times_B (G_{w_1} \overset{B}{\times} \dots \overset{B}{\times} G_{w_r})$
If all w_i lie in $S \cup \{e\}$ then $\bar{Y}_\mathbf{s}$ is smooth and then $C_{\xi,\mathbf{s}}[-\dim B - \ell(w_1) - \dots - \ell(w_r)]$ is the complex $\bar{K}^{\mathcal{L}_\xi}_\mathbf{s}$ of [CS, 2.8].

5.1.4. **Lemma.** *The character sheaves are also the irreducible constituents of the various complexes $C_{\xi,\mathbf{s}}$, where $\xi \in \hat{X}, \mathbf{s} = (s_1, \ldots, s_r)$ with $s_i \in S \cup \{e\}$ and $s_1 \ldots s_r \in W_\xi^!$.*
Let A be a character sheaf, which is an irreducible constituent of $C_{\xi,w}$, for some $\xi \in \hat{X}$ and $w \in W_\xi^!$. Let $\mathbf{s} = (s_1, \ldots, s_r)$ be a reduced decomposition of w. It follows from 4.2.5, using the multiplication rules in the algebra $\mathcal{K}$ of 3.3, that $A_{\xi,w}$ is a constituent of

$$(1) \qquad A_{s_2\ldots s_r\xi,s_1} * \ldots * A_{s_r\xi,s_{r-1}} * A_{\xi,s_r}.$$

It follows from the definitions that $C_{\xi,w} = \gamma_* \tilde{A}_{\xi,w}$ is a direct summand of $C_{\xi,\mathbf{s}} = \gamma_*((\pi_r)_* \tilde{A}_{\xi,\mathbf{s}})$, hence A is a constituent of $C_{\xi,\mathbf{s}}$.
On the other hand, if $\mathbf{s}$ is an arbitrary sequence in $S \cup \{e\}$ with $s_1 \ldots s_r \in W_\xi^!$, it follows from 4.2.5 that all irreducible constituents of (1) are of the form $A_{\xi,w}$, for some $w \in W$ with $w\xi = \xi$. We conclude that all irreducible constituents of $C_{\xi,\mathbf{s}}$ are character sheaves. The lemma is a conjunction of part of [CS, 2.9] and [CS, 12.7].

Let $\mathbf{s} = (w_1, \ldots, w_r)$ be a sequence in W and $\xi \in \hat{X}$ be such that $w_1 \ldots w_r \in W_\xi^!$. Put $\mathbf{s}' = (w_r, w_1, \ldots, w_{r-1})$. Then $w_r w_1 \ldots w_{r-1} \in W_{w_r\xi}^!$.
5.1.5. **Lemma.** $C_{\xi,\mathbf{s}} = C_{w_r\xi,\mathbf{s}'}$.
The map $(g, g_1, \ldots, g_r) \mapsto (gg_r^{-1}, g_r, g_1, \ldots, g_{r-1})$ induces an isomorphism $\varphi : Y_r \to Y_r$ such that $\varphi_* \tilde{A}_{\xi,\mathbf{s}} = \tilde{A}_{w_r\xi,\mathbf{s}'}$ and that $\gamma_r \circ \varphi = \gamma_r$. The lemma follows from these observations.

5.1.6. Let $\mathbf{K}G$ be the Grothendieck group of the category of perverse sheaves on G. If $A = \oplus A_i[n_i]$ is a semi-simple complex on G, with $A_i \in \mathcal{M}G$, we associate to it the element

$$\chi^G(A) = \chi(A) = \sum_i t^{-n_i}[A_i]$$

of $\mathbb{Z}[t, t^{-1}] \otimes \mathbf{K}G$. We denote by a bar the automorphism of this tensor product induced by the automorphism of the first factor sending t to t^{-1}.
Now let $\mathcal{O}$ be a W-orbit in $\hat{X}$. Define a $\mathbb{Z}[t, t^{-1}]$-linear map τ (or τ^G) of the algebra $\mathcal{K}$ to $\mathbb{Z}[t, t^{-1}] \otimes \mathbf{K}G$ by

$$\begin{aligned} \tau(c_{\xi,w}) &= \chi(C_{\xi,w}) \quad \text{if} \quad w\xi = \xi, \\ &= 0 \qquad\qquad \text{if} \quad w\xi \neq \xi. \end{aligned}$$

Notice that if $\mathbf{s} = (w_1, \ldots, w_r)$ is a sequence in W and $w_1 \ldots w_r \in W_\xi^!$ we have

$$\tau(c_{w_2\ldots w_r\xi,w_1} \ldots c_{\xi,w_r}) = \chi(C_{\xi,\mathbf{s}})$$

The next proposition gives the basic properties of τ. The second one shows that it is a generalized trace function on the associative algebra $\mathcal{K}$. The proposition is similar to [CS, 6.2].
5.1.7. **Proposition.**
(i) *For $u \in \mathcal{K}$ we have $\tau(\bar{u}) = \overline{\tau(u)}$;*
(ii) *If $u, v \in \mathcal{K}$ then $\tau(uv) = \tau(vu)$.*
In (i) $\bar{u}$ is as in 3.3. It suffices to prove (i) for $u = c_{\xi,w}$, with $\xi \in \mathcal{O}$. The assertion is then a direct consequence of the relative hard Lefschetz theorem [BBD, p.114, p.165], applied to the projective morphism $\gamma : G \times_B G \to G$. It suffices to prove (ii) for $u = c_{\xi,x}, v = c_{\xi,y}$ with $\xi \in \mathcal{O}$, $x, y \in W_\xi^!$. The assertion follows from 5.1.5, applied with $\mathbf{s} = (x, y)$.

Let $\mathcal{H}'_\xi$ be as in 3.3.7 (ii) ($\xi \in \hat{X}$).

5.1.8. **Corollary.** *If* $x \in W'_\xi$, $\sigma \in S_\xi$ *then*

$$\begin{aligned}\tau(e_{\xi,\sigma x\sigma}) &= (t^2-1)\tau(e_{\xi,\sigma x}) + t^2\tau(e_{\xi,x}) \quad \text{if} \quad \ell_\xi(\sigma x\sigma) > \ell_\xi(x),\\ &= \tau(e_{\xi,x}) \quad \text{if} \quad \ell_\xi(\sigma x \sigma) = \ell_\xi(x).\end{aligned}$$

In the first case we have $e_{\xi,\sigma x\sigma} = e_{\xi,\sigma}e_{\xi,x}e_{\xi,\sigma}$ and

$$\tau(e_{\xi,\sigma x\sigma}) = \tau(e^2_{\xi,\sigma}e_{\xi,x}) = (t^2-1)\tau(e_{\xi,\sigma}e_{\xi,x}) + t^2\tau(e_{\xi,x}),$$

by 3.3.5 and 5.1.7 (ii). The asserted formula follows.

To prove the second formula observe that if $\ell_\xi(\sigma x) > \ell_\xi(x)$, $\ell_\xi(\sigma x\sigma) = \ell_\xi(x)$ we have $e_{\xi,\sigma x\sigma} = e_{\xi,\sigma}e_{\xi,x}e^{-1}_{\xi,\sigma}$ and apply 5.1.7 (ii).

The corollary can be viewed as a version of [CS, 11.2 (b)].

5.1.9. **Lemma.** *Let* $w \in W'_\xi, y \in W^*_\xi$. *Then* $C_{\xi,w} = C_{y\xi,ywy^{-1}}$.

We have $\chi(C_{y\xi,ywy^{-1}}) = \tau(e_{\xi,y}c_{\xi,w}\bar{e}_{y\xi,y^{-1}}) = \tau(\bar{e}_{y\xi,y^{-1}}e_{\xi,y}c_{\xi,w}) = \tau(c_{\xi,w}) = \chi(C_{\xi,w})$, by 3.3.9, 5.1.7 (ii) and 3.3.2. This implies the assertion.

The following result is a consequence of 4.1.2 (iii).

5.1.10. **Lemma.** $DC_{\xi,w} = C_{-\xi,w}$.

5.2 Some invariants of character sheaves

We denote by $\hat{G}$ the set of isomorphism classes of character sheaves of G and by $\hat{G}(\xi)(\xi \in \hat{X})$ those coming from some $C_{\xi,w}$ with $w \in W'_\xi$.

5.2.1. **Lemma.** *Let* $\xi, \eta \in \hat{X}$ *with* $\eta \notin W\xi$. *For* $x \in W'_\xi, y \in W'_\eta$ *we have*

$$H^{\cdot}_c(G, C_{\xi,x} \otimes C_{-\eta,y}) = 0.$$

Write $\bar{Y}_u = G \times_B \overline{G_u}, Y_u = G \times_B G_u$. We have $\gamma_u : \bar{Y}_u \to G$, the restriction of γ. Let $\bar{Y}_{x,y} \subset \bar{Y}_x \times \bar{Y}_y$ (resp. $Y_{x,y} \subset Y_x \times Y_y$) be the inverse image for (γ_x, γ_y) of the diagonal in $G \times G$. The assertion of the lemma is equivalent to

$$H^{\cdot}_c(\bar{Y}_{x,y}, \tilde{A}_{\xi,x} \boxtimes \tilde{A}_{-\eta,y}) = 0.$$

Now $\bar{Y}_{x,y}$ is stratified by the $Y_{u,v}$ with $u \le x, v \le y$ and it suffices to show that

$$H^{\cdot}_c(Y_{u,v}, \tilde{A}_{\xi,x} \boxtimes \tilde{A}_{-\eta,y}) = 0$$

for such u, v. By 4.1.3 (ii) the restriction of $\tilde{A}_{\xi,x} \boxtimes \tilde{A}_{-\eta,y}$ to $Y_{u,v}$ is non-zero only if $u\xi = x\xi, v\eta = y\eta$. Using also 4.1.3 (i) we see that it suffices to prove that

$$H^{\cdot}_c(Y_{x,y}, \tilde{\mathcal{L}}_{\xi,x} \boxtimes \tilde{\mathcal{L}}_{-\eta,y}) = 0,$$

ξ, η, x, y being as in the lemma, $\tilde{\mathcal{L}}_{\xi,x}$ denoting the restriction of $\tilde{A}_{\xi,x}$ to Y_x etc. Projection on the first factor determines a morphism $G \times G \to G/B$, whence a morphism $\alpha : Y_{x,y} \to G/B \times G/B$. It suffices to prove for $a \in G/B \times G/B$ that

$$H^{\cdot}_c(\alpha^{-1}a, \tilde{\mathcal{L}}_{\xi,x} \boxtimes \tilde{\mathcal{L}}_{-\eta,y}) = 0$$

and, moreover, it suffices to do this for representatives a of the G- orbits in $G/B \times G/B$, so for $a = (B, \dot{w}B)$, where $w \in W$. Then

$$\alpha^{-1}a = \{(g,h) \in G_x \times G_y \mid g = \dot{w}h(\dot{w})^{-1}\}.$$

But now $\alpha^{-1}a$ is a product of the maximal torus T and another variety, such that the restriction of $\tilde{\mathcal{L}}_{\xi,x} \boxtimes \tilde{\mathcal{L}}_{-\eta,y}$ to $\alpha^{-1}a$ corresponds to the exterior tensor product of the Kummer local system $\mathcal{L}_{\xi-w^{-1}\eta}$ on T and a constant sheaf. Application of 2.1.5 now gives the desired vanishing of cohomology.

5.2.2. **Theorem.** *Let $\xi, \eta \in \hat{X}$.*
(i) *If $\eta \in W\xi$ then $\hat{G}(\xi) = \hat{G}(\eta)$;*
(ii) *If $\eta \notin W\xi$ then $\hat{G}(\xi)$ and $\hat{G}(\eta)$ are disjoint.*
Let $s \in S, s \notin W_\xi^!, w \in W_\xi^!$. By 5.1.5 we have

$$C_{\xi,(s,s,w)} = C_{s\xi,(s,w,s)}$$

It follows from 4.2.5 and 4.2.2 (ii) that

$$C_{\xi,(s,s,w)} = C_{\xi,w}.$$

Using again 4.2.5 we see that any constituent of $C_{s\xi,(s,w,s)}$ lies in $\hat{G}(s\xi)$. It follows that $\hat{G}(\xi) \subset \hat{G}(s\xi)$. This is clearly also true for $s \in W_\xi^!$. We then conclude that $\hat{G}(\xi) \subset \hat{G}(w\xi)$ for all $w \in W$ and (i) follows.
(ii) is a direct consequence of 5.2.1, 5.1.9 and 1.2.5(ii).

5.2.3. Let A be a character sheaf on G, which is an irreducible constituent of $C_{\xi,w}$, where $\xi \in \hat{X}, w \in W_\xi^!$. Let Z be the center of G and Z° its identity component. Then Z acts on G by left translations (or right translations, which is the same). It follows readily from the definitions that A has a weight for this Z°- action in the sense of 2.2.1. The weight is the image of ξ in $\hat{X}(Z^\circ)$.
5.2.4. In the previous situation, Z acts trivially on G by conjugation. It follows (see property (a) in 1.4) that there is a homomorphism of the finite group Z/Z° into the automorphism group of our character sheaf A, which is E^*. In other words, there is a homomorphism $\gamma : Z/Z^\circ \to E^*$ such that the action of Z/Z° on the stalks of H^iA is given by γ.
If A is a constituent of $C_{\xi,w}$ as before it follows from the definitions that γ is as described in 3.1 (before 3.1.4).
5.2.5. **Proposition.** *Let $\xi \in \hat{X}$. There is a map $\hat{G}(\xi) \to W_\xi^!/W_\xi$ such that the image of the isomorphism class of the character sheaf A is wW_ξ whenever A is a constituent of $C_{\xi,w} (w \in W_\xi^!)$.*
We have an injective homomorphism $\alpha : W_\xi^!/W_\xi \to \mathrm{Hom}\ (Z/Z^\circ, E^*)$ (see 3.1.4). The preceding remarks lead to a map $\beta : \hat{G}(\xi) \to \mathrm{Hom}\ (Z/Z^\circ, E^*)$ and one shows that im $\beta \subset$ im α. So we can define a map $\hat{G}(\xi) \to W_\xi^!/W_\xi$, it has the required properties.
5.2.6. **Corollary.** *Let $x \in W$. Then $\hat{G}(x\xi) = \hat{G}(\xi)$ and the map of* 5.2.5. *for $x\xi$ is the composite of the map for ξ with the bijection $W_{x\xi}^!/W_{x\xi} \to W_\xi^!/W_\xi$ defined by $w \mapsto x^{-1}wx$.*
See [CS, no.11] for results of this kind.

5.3 Finite ground fields

We now assume that we are in the situation of 4.3.1. We write G^F for $G(\mathbb{F}_q)$ etc.
5.3.1. If $\xi \in \hat{X}, F\xi = \xi$ then the complex $C_{\xi,\dot{w}}$ of 5.1.1, where $\dot{w}$ is as in 4.3.1, comes from a complex on the $\mathbb{F}_q$-scheme G_0. By a theorem of Deligne [D, p.248], $C_{\xi,\dot{w}}$ is pure of weight

$\dim G + \ell(w)$.
We denote by $\gamma_{\xi,\dot{w}}$ the characteristic function of $C_{\xi,\dot{w}}$ (see 1.3.4), which is a class function on the finite group G^F. It can be described in terms of an explicit representation of G^F and the Hecke algebra $\mathcal{H}^!_\xi$. We shall review this description, without going into the details of the proofs (for which we refer to [CS, no.13] and [HK]).
5.3.2. The element $\xi \in \hat{X}$ defines a character φ of T^F, according to 2.3.1. If $\xi = (q-1)^{-1}x + X$ then for $t \in T^F$

$$\varphi(t) = \psi(x(t)),$$

with ψ as in 2.1.2.
Consider the vector space V_φ of functions $f : G^F \to E$ such that for $g \in G^F, t \in T^F, u \in U^F$

$$f(gtu) = f(g)\varphi(t)^{-1}.$$

Then G^F acts on V_φ by left translations. The representation ρ of G^F thus obtained is the one induced by the character $B^F \to T^F \xrightarrow{\varphi} E^*$ of B^F. If $n \in (N_G T)^F$ has image in $W^!_\xi$ define the endomorphism θ_n of V_φ by

$$(\theta_n f)(g) = | U^F |^{-1} \sum_{h \in G^F,\, g^{-1}h \in UnU} f(h) \quad (g \in G^F).$$

Then θ_n commutes with left translations. In fact the $\theta_{\dot{w}}$ $(w \in W^!_\xi)$ span the commuting algebra of ρ.
Let $w = w^* w_1 (w^* \in W^*_\xi, w_1 \in W_\xi)$, as in 3.2.
5.3.3. **Proposition.** *There exists a choice of representatives* $(\dot{w})$ *in* G^F *such that for* $g \in G^F$

$$\gamma_{\xi,\dot{w}}(g) = (-1)^{\dim G + \ell(w)} \sum_{y \in wW_\xi,\, y \le_\xi w} P_{\xi yw}(q) q^{\frac{1}{2}(\ell(w)-\ell(y)-\ell_\xi(w)+\ell_\xi(y))} Tr(g\theta_{\dot{w}^*\dot{y}}, V_\varphi).$$

Here $P_{\xi yw}$ is as in 3.3.4.
Let $Y_w = G \times_B G_w, \gamma_w : Y_w \to G$ be as before. The restriction of $\tilde{A}_{\xi,\dot{w}}$ to Y_w is a local system $\mathcal{L}$, up to a shift. One first shows that the characteristic function of the complex $\gamma_! \mathcal{L}$ is given by

$$g \mapsto Tr(g\theta_{\dot{w}}, V_\varphi)$$

(see [CS, 13.4]). Then one uses the stratification of $\bar{Y}_w$ given by the Y_y $(y \le w)$ and 4.3.2 to obtain the proposition.

5.4 Some examples.

The easy proof of the following reduction results is omitted.
5.4.1. **Lemma.** *Let G be a product $G_1 \times G_2$ of two connected reductive groups. The character sheaves on G are the complexes of the form $A_1 \boxtimes A_2$, where $A_i \in \hat{G}_i (i = 1, 2)$.*
Next let $\varphi : G \to G'$ be a central isogeny of connected reductive groups. If T is a maximal torus of G then $T' = \varphi T$ is one in G' and we can identify the Weyl groups of (G, T) and (G', T'). Moreover, φ induces a surjective homomorphism $\hat{\varphi} : \hat{X}(T') \to \hat{X}(T)$ with finite kernel.
5.4.2. **Lemma.**

(i) *If* $\xi \in \hat{X}(T'), w \in W$ *and* $w\xi = \xi$ *then* $\varphi_* C^G_{\hat{\varphi}\xi,w} = \oplus_{\eta \in \ker \hat{\varphi}} C^{G'}_{\xi+\eta,w}$ *and* $\varphi^* C^{G'}_{\xi,w} = C^G_{\hat{\varphi}\xi,w}$;
(ii) *If* $A \in \hat{G}$ *is such that the action of* $\ker \varphi$ *on* A *induced by the action of* 5.2.4 *is trivial then* $\varphi_* A$ *is a direct sum of character sheaves of* G';
(iii) *If* $A' \in \hat{G}'$ *then* $\varphi^* A$ *is a direct sum of character sheaves of* G.
(One should recall here the properties of 1.2.6 (a)).

5.4.3. **Lemma.** *The character sheaves on a torus are the Kummer local systems.*

The previous properties can be used to reduce the study of character sheaves to the case that G is a connected, quasi-simple, simply connected linear algebraic group.

5.4.4. **The case** $w = e$.

We have $G_e = \tilde{G}_e = B$ and for $\xi \in \hat{X}(T)$ the complex $A_{\xi,e}$ is $\mathrm{pr}^* \mathcal{L}_\xi[\dim B]$, where pr is the projection map $B \to T$. Now $\gamma : G \times_B B \to B$ is Grothendieck's simultaneous resolution map, studied at length in [Sl]. The map $(g, b) \mapsto \mathrm{pr}\, b$ induces a morphism $\phi : G \times_B B \to T$ and $\tilde{A}_{\xi,e} = \pi^* \mathcal{L}_\xi[\dim G]$.

It is known that in this situation γ is a small map in the sense of Goresky-Mac Pherson. This means that there exists a stratification of G by locally closed, irreducible, smooth subvarieties $(S_i)_{1 \leq i \leq n}$ such that for $x \in S_i$ we have

$$\dim \gamma^{-1} x < \frac{1}{2}(\dim G - \dim S_i),$$

if S_i is not dense in G. It then follows from the definition of perverse sheaves that $C_{\xi,e} = \gamma_* \tilde{A}_{\xi,e}$ is perverse, hence a direct sum of character sheaves (see [Sp 2]). Taking $\xi = 0$ we deduce (using the description of irreducible perverse sheaves in 1.2.3) that $E[\dim G]$ is a character sheaf.

In the general case we can describe in the following manner the character sheaves which occur.

Let G_{reg} be the open subset of G consisting of the regular semi-simple elements and put $T_{\mathrm{reg}} = T \cap G_{\mathrm{reg}}$. We can identify $\gamma^{-1}(G_{\mathrm{reg}})$ with $G/T \times T_{\mathrm{reg}}$, such that γ is the map $(gT, t) \mapsto gtg^{-1}(g \in G, t \in T_{\mathrm{reg}})$. This shows that $\gamma : \gamma^{-1}(G_{\mathrm{reg}}) \to G_{\mathrm{reg}}$ is a Galois covering with group W, this group acting by $w(gT, t) = (g\dot{w}^{-1}T, wt)$. Now $\mathcal{S} = \gamma_*(E \boxtimes \mathcal{L}_\xi \mid_{T_{\mathrm{reg}}})$ is a local system on G_{reg}, and $C_{\xi,e}$ is its perverse extension to G (in the sense of 1.2.3). The local system $\mathcal{S}$ is a sum of irreducible ones (as will be shown presently), and the irreducible constituents of $C_{\xi,e}$ are the perverse extensions of the irreducible constituents of $\mathcal{S}$. We can describe $\mathcal{S}$ as follows. Let $\mathcal{L}_\xi = \mathcal{L}_{x,m}$ (see 2.1.2). Consider the map $\gamma_m : G/T \times T_{\mathrm{reg}} \to G_{\mathrm{reg}}$ with $\gamma_m(gT, t) = gt^m g^{-1}$. then γ_m is a Galois covering whose group is the semi-direct product $\Gamma = W \times_m T$ of W and the group ${}_mT$ of elements of T of order dividing $m(W$ operates in the natural way). As in 2.1.2, ξ defines a character χ of ${}_mT$. Then $\mathcal{S}$ is the local system on G_{reg} defined by the representation of Γ induced by χ. In particular we see that if $\xi = 0$, the representation in question is the regular representation of W. If $W'_\xi = \{e\}$ the representation is irreducible, by familiar results about representations of semi-direct products.

Since the representation is semi-simple, we see that $\mathcal{S}$ is semi- simple, as asserted above. In the case $\xi = 0$, the irreducible constituents of $C_{\xi,e}$ correspond to the irreducible representations of W.

5.4.5. **The case of a Coxeter element** w.

We use the notations of 3.1.1 and 3.2.1. Write $S = (s_1, \ldots. s_r)$ and let $w = s_1 \ldots s_r$. So w is a Coxeter element of W. We first record the following result.

5.4.6. Lemma.
(i) *The map* $\pi : \overline{G_{s_1}} \overset{B}{\times} \overline{G_{s_2}} \overset{B}{\times} \ldots \overset{B}{\times} \overline{G_{s_r}} \to \overline{G_w}$ *induced by the product map* $G^r \to G$ *is bijective*;
(ii) $\overline{G_w}$ *is smooth.*
The statement (ii) is equivalent to the smoothness of the Schubert variety $\overline{G_w}/B$. (This is well-known). It is a consequence of (i). We sketch a proof of (i).
$\overline{G_{s_1}}$ is a parabolic subgroup of G and the $\overline{G_{s_i}}(2 \le i \le r)$ are contained in a parabolic subgroup P of G such that $P \cap \overline{G_1} = B$ (as a consequence of [B, p.27-28, Th.3]). Now let $g_i, g_i' \in \overline{G_{s_i}}(1 \le i \le r)$ and assume that $g_1 \ldots g_r = g_1' \ldots g_r'$. Then $g_1^{-1}g_1' \in P \cap \overline{G_1} = B$ and similarly $g_i^{-1}g_i' \in B(1 \le i \le r)$. This implies (i).

Now $\pi(G_{s_1} \overset{B}{\times} \ldots \overset{B}{\times} G_{s_r}) = G_w$ and $\mathcal{L}_{\xi,w}$ determines a local system on $G_{s_1} \overset{B}{\times} \ldots \overset{B}{\times} G_{s_r}$, which is an open subset of the smooth variety $\overline{G_{s_1}} \overset{B}{\times} \ldots \overset{B}{\times} \overline{G_{s_r}}$, whose complement is a union of smooth divisors crossing normally. These divisors are $\overline{G_{s_1}} \overset{B}{\times} \ldots \overset{B}{\times} \overline{G_{s_{i-1}}} \overset{B}{\times} B \overset{B}{\times} \overline{G_{s_{i+1}}} \overset{B}{\times} \ldots \overset{B}{\times} \overline{G_{s_r}} = D_i$, say. Let $\pi D_i = E_i(1 \le i \le r)$. The result of 1.2.4 (b) shows that $A_{\xi,w}$ is a complex $\bar{\mathcal{L}}_{\xi,w}[r + \dim B]$ concentrated in one dimension, where $\bar{\mathcal{L}}_{\xi,w}$ is a constructible sheaf on $\overline{G}_w$. To describe it according to 1.2.4(b), we use the following result, which is proved by a straightforward verification, using 4.2.2. We put $\mathcal{L}_\xi = \mathcal{L}_{x,m}$.
5.4.7. **Lemma.** *The local monodromy of* $\mathcal{L}_{\xi,w}$ *around* E_i *is trivial if and only if* $< x, s_r \ldots s_{i+1}\alpha_i^\vee > \in m\mathbb{Z}$, *for* $i = 1, \ldots, r$.
5.4.8. We now consider a particular case. Take $G = SL_n$ and let T be the torus of diagonal matrices, B the subgroup of upper trangular matrices. Now $X = \mathbb{Z}^n/\mathbb{Z}(1, \ldots, 1), X^\vee = \{(x_1, \ldots, x_n) \in \mathbb{Z}^n / x_1 + \ldots + x_n = 0\}$. The roots are the images in X of the $e_i - e_j$ $(i \ne j)$, where (e_i) is the canonical basis.
$W = S_n$, operating in the obvious way. We now take $s_i = (i, i+1)(1 \le i \le n-1)$, a transposition. Then $w = s_1 \ldots s_{n-1}$ is the cyclic permutaion $i \mapsto i + 1 (\mathrm{mod}\ n)$. Assume that the characteristic of k does not divide n. We now take $\xi = n^{-1}ax + X$, where x is the image in X of $0.e_1 + e_2 + \ldots + (n-1)e_n$ and a is prime to n. It is easy to see that $w\xi = \xi$ and that $W_\xi = \{e\}$.
In this situation we conclude from 5.4.7 that $\bar{\mathcal{L}}_{\xi,w}$ is the extension by zero of $\mathcal{L}_{\xi,w}$.
Now consider the morphism $\gamma : G \times_B G_w \to G$ of 5.1.1. It is clear from the preceding observations that, $\tilde{\mathcal{L}}_{\xi,w}$ denoting the restriction of $\tilde{A}_{\xi,w}$ to $G \times_B G_w$ (this is a local system),

$$C_{\xi,w} = \gamma_! \tilde{\mathcal{L}}_{\xi,w},$$

in particular we see that the right-hand side is semi-simple. We shall briefly describe it more precisely. Recall that an element g of a connected reductive group G is called *regular* if its centralizer $Z(g)$ has dimension equal to the rank of G.
5.4.9. Lemma.
(i) *All elements of* G_w *are regular*;
(ii) *Two elements of* G_w *which are conjugate in* G *are conjugate by an element of* B;
(iii) *The centralizer of an element of* G_w *intersects* B *in the center of* G.
(i) is a result due to Steinberg, stated in [St1, 8.8]. (ii) follows readily from [loc. cit. 7.6, 7.16a and 8.9]. These results hold for arbitrary semi-simple simply connected groups. To prove (iii) we may in the present case $G = SL_n$ assume the element of G_w to have the normal form described in [loc. cit. 7.4b)]. The assertion of (iii) then follows by a direct

matrix computation.

5.4.10. **Proposition.** $(G = SL_n)$. *$H^i(C_{\xi,w})_g \neq 0$ only if g is a regular unipotent element times an element of the center of G and $i = -n^2 + n$.*

One knows that the centralizer of a regular element of SL_n is abelian. It suffices to prove the proposition under the assumption that $Z(g) \subset B$. We have

$$H^i(C_{\xi,w})_g = H^i_c(\gamma^{-1}g, \tilde{\mathcal{L}}_{\xi,w}),$$

where $\tilde{\mathcal{L}}_{\xi,w}$ is the restriction of the complex $\tilde{A}_{\xi,w}$ to $G \times_B G_w$. It now follows from 5.4.8 that $\gamma^{-1}g \simeq Z(g)/Z$ (Z denoting the center of G). If g is not unipotent modulo Z then $Z(g)$ contains a non-trivial torus and the restriction of $\tilde{\mathcal{L}}_{\xi,w}$ to it is non-trivial. By 2.1.5 we have $H^i(C_{\xi,w})_g = 0$. If g is unipotent modulo Z then $\gamma^{-1}g$ is an affine space of dimension $(n-1)$ and we have $H^i_c(\gamma^{-1}g, \tilde{\mathcal{L}}_{\xi,w}) = E$ if $i = -n^2 + n$ and $= 0$ otherwise.

Now let U be set of regular unipotent elements of G. It is a conjugacy class in G. If $g \in U$ then $Z(g) = Z(g)^\circ \times Z$. It follows that for any injective character $\chi : Z \to E^*$ there exists a G-equivariant local system (for conjugation) $\mathcal{L}_\chi$ on U, which is a direct summand of the direct image of the constant sheaf under the map $G/Z(g)^\circ \to G/Z(g) = U$ (a Galois covering with group Z).

5.4.11. **Corollary.** *The restriction of $C_{\xi,w}$ to $\bar{U}$ is the perverse extension $I(\bar{U}, \mathcal{L}_\chi)$.*

This follows from 5.4.8. We find a similar result for the varieties $zU (z \in Z)$.

5.4.12 **The case of SL_2.**

Now consider the special case $G = SL_2$, when char $k \neq 2$. We now have $\hat{X} = \mathbb{Q}/\mathbb{Z}$ (resp. $\mathbb{Z}_{(p)}/\mathbb{Z}$). If $\xi \in \hat{X}$ and $2\xi \neq 0$ then $C_{\xi,e}$ is a character sheaf, as follows from the results of 5.4.4. The same results give that if $2\xi = 0$ we have that $C_{\xi,e}$ is a sum of two character sheaves. By 5.4.10 for $\xi \neq 0, 2\xi = 0$ the complex $C_{\xi,s}$ is a sum of two character sheaves, concentrated on the set of regular unipotent elements and its negative.

Finally, $C_{0,s} = E[4] \oplus E[2] = A[1] \oplus A[-1]$, where A is the character sheaf $E[3]$.

Any character sheaf on SL_2 is isomorphic to one of those just reviewed.

6 Parabolic restriction, cuspidal perverse sheaves

We keep the notations of the preceding sections. The center of G is denoted by $Z(G)$ and its identity component by $Z(G)^{\circ}$. We denote by P a parabolic subgroup, by $U(P)$ its unipotent radical and by L a Levi subgroup of$P$. We then say that L is a Levi subgroup of G.
P is the semi-direct product of L and $U(P)$. We denote by π_P the canonical homomorphism $P \to L$ sending ℓu to ℓ ($\ell \in L, u \in U(P)$).

6.1 Parabolic restriction

6.1.1. We define a functor res $= \mathrm{res}_P^G : \mathcal{D}G \to \mathcal{D}L$ by

$$\mathrm{res}\, K = (\pi_P)_! i^* K \qquad (K \in \mathcal{D}G)$$

where i is the inclusion $P \to G$. If G, P, L are defined over a finite field F_q and K comes from a complex on the F_q-scheme G_0, so that we have $\varphi : F^*K \simeq K$ (see 1.3.1.) then a Tate twist $(\dim U(P))$ is added.
In this situation, res K inherits $\psi : F^*\mathrm{res}\, K \simeq$ res K and one shows that we have the following relation between characteristic functions (see 1.3.4):

$$\chi^L_{\mathrm{res}\, K,\psi}(\ell) = \mid U(P)^F \mid^{-1} \sum_{u \in U(P)^F} \chi^G_{K,\varphi}(\ell u) \ (\ell \in L^F).$$

The right-hand side is well-known in the character theory of finite groups of Lie type (see for example [Ca, p.263]). If $\chi^G_{K,\varphi}$ is a character of G^F then $\chi^L_{\mathrm{res} K,\psi}$ is the character of L^F obtained by parabolic restriction (called "truncation" in loc.cit.]).
We shall call the functor res just defined *parabolic restriction.*
6.1.2 **Definition.** *A perverse sheaf $K \in \mathcal{M}G$ is cuspidal if*
(a) *K has a weight for the action of the connected center $Z(G)^{\circ}$ by left (= right) translations*;
(b) *K is equivariant for the conjugation action of G*;
(c) *For any proper parabolic subgroup P with Levi group L we have* $\mathrm{res}_P^G K \in \mathcal{D}L^{<0}$.
$K \in \mathcal{M}G$ is strongly cuspidal if we have (a), (b) *and*
(c)$'$ *For any proper parabolic subgroup P with Levi group L we have* $\mathrm{res}_P^G K = 0$.
For weights of torus actions see 2.2.1. The condition $\mathrm{res}_P^G K \in \mathcal{D}L^{<0}$ of (c) can also be written as ${}^pH^i(\mathrm{res}_P^G K) = 0$ if $i \geq 0$ (see 1.2).
The definition of cuspidal perverse sheaves adopted here is slightly stronger than the one of [CS, 7.1.1]. We shall analyse cuspidal perverse sheaves, following [CS] and [L2].

6.2 A stratification of G

6.2.1. **Isolated classes.** It is well-known that if G is semi-simple the number of conjugacy classes of connected, semi-simple, closed subgroups of G with rank equal to that of G is finite. The center of such a subgroup is finite (cf. [BS]).
A conjugacy class in G is *isolated* if it contains an element g whose semi-simple part g_s lies in such a center. Alternatively, a conjugacy class C in G is isolated if for $g \in C$ the centralizer $Z_G(g_s)$ has semi-simple rank equal to the rank of G.

If G is an arbitrary connected reductive group we call isolated class in G the inverse image of an isolated class in the semi-simple group $G/Z(G)^\circ$. It is clear that the number of isolated classes is finite.

6.2.2. Now let L be a Levi subgroup in G and $\sum$ an isolated class in L. We put

$$\Sigma_{\mathrm{reg}} = \{x \in \Sigma \mid Z_G(x_s)^\circ \subset L\}.$$

This is a non-empty open subset of $\sum$. If $x \in \sum_{\mathrm{reg}}$ then L is the smallest Levi subgroup of G containing $Z_G(x_s)^\circ$. (These facts readily follow from the explicit description of Levi subgroups.)
We put

$$N(L,\Sigma) = \{g \in G \mid gLg^{-1} = L, g\Sigma g^{-1} = \Sigma\}, W(L,\Sigma) = N(L,\Sigma)/L,$$

and

$$Y_{(L,\sum)} = \bigcup_{g\in G} g\Sigma_{\mathrm{reg}}g^{-1}.$$

Let $G \times_L \sum_{\mathrm{reg}}$ be the quotient of $G \times \sum_{\mathrm{reg}}$ for the L-action $\ell(g,x) = (g\ell^{-1}, \ell x\ell^{-1})$ $(\ell \in L, g \in G, x \in \sum_{\mathrm{reg}})$. The map $(g,x) \mapsto gxg^{-1}$ induces a morphism

$$\gamma : G \times_L \sum_{\mathrm{reg}} \to Y_{(L,\sum)}.$$

6.2.3. Lemma.
(i) $Y_{(L,\sum)}$ *is a locally closed, smooth, irreducible subvariety of* G, *of dimension* dim G - dim L+ dim $\sum$. *If* $g \in G$ *we have* $Y_{(L,\sum)} = Y_{(gLg^{-1}, g\sum^{-1} g)}$;
(ii) $\gamma : G \times_L \sum_{\mathrm{reg}} \to Y_{(L,\sum)}$ *is a Galois covering with group* $W(L,\sum)$;
(iii) $G = \bigcup_{L,\sum} Y_{(L,\sum)}$.
Most of this is straightforward. To prove (iii), let $g \in G$. There is a smallest Levi subgroup L containing $Z_G(g_s)^\circ$ (viz. the centralizer of the connected center of $Z_G(g_s)^\circ$). If $\sum$ is the product of $Z(L)^\circ$ and the conjugacy class of g in L then $g \in Y_{(L,\sum)}$.
6.2.4. **The closure of** $Y_{(L,\sum)}$. Assume that L is a Levi subgroup of the parabolic group P. As above, we define $G \times_P \overline{\sum}U(P)$, and a morphism $\delta : G \times_P \overline{\sum}U(P) \to G$, which is proper ($\bar{\sum}$ denotes the closure of $\sum$ in L). The inclusion $G \times \sum \to G \times \overline{\sum}U(P)$ induces a morphism $\alpha : G \times_L \sum_{\mathrm{reg}} \to G \times_P \bar{\sum}U(P)$, which is easily seen to be an open imbedding. Also, im δ is the closure $\bar{Y}_{(L,\sum)}$ and it will follow from 6.2.7 (iii) that $\delta^{-1}Y_{(L,\sum)} =$ im α.
6.2.5. **Lemma.** *The set of semi-simple parts of elements of* $\bar{\sum}$ *coincides with the set of semi-simple parts of elements of* $\sum$.
It suffices to proves this statement for the semi-simple group $L/Z(L)^\circ$, in which case it follows from the known fact that the set of elements whose semi-simple part is conjugate to a given element is closed (see [St1, 6.6, 6.11]).
6.2.6. **Lemma.** *Let* $x = \ell u$ *be an element of* P $(\ell \in L, u \in U(P))$ *Then* x_s *is conjugate to* ℓ_s *by an element of* $U(P)$.
Let L' be a Levi subgroup of P containing x_s. There is $u' \in U(P)$ with $L' = u'L(u')^{-1}$. Write $x_s = u'\ell'(u')^{-1}$. Then $x_s \in \ell'U(P)$. But also $x_s \in \ell_sU(P)$. Hence $\ell' = \ell_s$, whence the lemma.
6.2.7. **Lemma.** *Let* $(L,\sum)$ *be as before and let* $\sum_1$ *be an isolated class in a Levi group* L_1. *Assume that* $x \in \bar{\sum}U(P)$ *is conjugate in* G *to an element of* $(\sum_1)_{\mathrm{reg}}$.

(i) *L is conjugate to a subgroup of L_1;*
(ii) $Y_{(L_1,\sum_1)} \subset \bar{Y}_{(L,\sum)}$;
(iii) *If $L_1 = L, \sum_1 = \sum$ then x is conjugate in P to an element of $(\sum_1)_{\text{reg}}$.*
If H is a connected reductive group we write $r(H)$ for its radical $Z(H)^\circ$. By the previous lemmas there exists $y \in \sum$ such that x_s is conjugate to y_s by an element of $U(P)$, say $x_s = uy_su^{-1}$. We then have $r(Z_G(x_s)^\circ) = ur(Z_G(y_s)^\circ)u^{-1} \subset ur(Z_L(y_s)^\circ)u^{-1} = ur(L)u^{-1}$, by the definiton of isolated classes.
Let $g \in G$ and $\ell \in (\sum_1)_{\text{reg}}$ be such that $x = g\ell g^{-1}$. Then $r(Z_G(x_s)^\circ) = gr(Z_G(\ell_s)^\circ)g^{-1} = gr(L_1)g^{-1}$, since $Z_G(\ell_s)^\circ \subset L_1$. We conclude that

$$gr(L_1)g^{-1} \subset ur(L)u^{-1},$$

whence $gL_1g^{-1} \supset uLu^{-1}$, and (i) follows. It also follows that $\sum_1 \subset \bar{Y}_{(L,\sum)}$, which implies (ii). If $L_1 = L, \sum_1 = \sum$ we must have $u^{-1}g \in N(L,\sum)$, whence (iii).
We denote by $(Y_i)_{i\in I}$ the finite set of varieties of the form $Y_{(L,\sum)}$, for any Levi group L and isolated class $\sum$ of L. (Notice that $Y_{(gLg^{-1},g\sum g^{-1})} = Y_{(L,\sum)}$, if $g \in G$.) We state the following consequence of the results established above.
6.2.8. **Proposition.**
(i) *$(Y_i)_{i\in I}$ is a stratification of G by a finite number of locally closed, smooth irreducible subvarieties which are stable under conjugation;*
(ii) *The closure $\bar{Y}_i$ is a union of certain Y_j.*
The varieties $Y_{(L,\sum)}$ will be referred to as *strata*. Notice that there is one open stratum, namely the set of regular semi-simple elements.
We shall see later (8.3.1) that if A is a character sheaf the restriction to a stratum of H^iA is a locally constant sheaf, for all $i \in \mathbb{Z}$.

6.3 Cuspidal perverse sheaves

We shall now establish some basic properties of irreducible cuspidal perverse sheaves, after [CS, no.7] and [L2].
6.3.1. **Theorem.** *Let K be an irreducible cuspidal perverse sheaf.*
(i) *There is a unique isolated class Σ in G and a local system $\mathcal{L}$ on Σ, unique up to isomorphism, such that $K = I(\sum, \mathcal{L})$;*
(ii) *If P is a proper parabolic subgroup of G , with Levi group L, then for $\ell \in L, i \geq \dim \sum - \dim Z(G)^\circ - \dim L + \dim Z_L(\ell)$ we have*

$$H^i_c(\ell U(P) \cap \Sigma, \mathcal{L}) = 0;$$

(iii) *If $g \in \sum$ then $Z_G(g)^\circ/Z(G)^\circ$ is a unipotent group.*
Recall that K has a weight for the action of $Z(G)^\circ$. Also, it is clear that the local system $\mathcal{L}$ of (i) is G-equivariant. In the proof we need some auxiliary results. For the first one, let P and L be as before. Let C be a conjugacy class in L and put $\Gamma = CZ(L)^\circ$,

$$Z = \{(g, x_1P, x_2P) \in G \times G/P \times G/P \mid x_i^{-1}gx_i \in \Gamma U(P), i = 1,2\}.$$

If $\mathcal{O}$ is a G-orbit in $G/P \times G/P$ we denote by $Z_{\mathcal{O}}$ the piece of Z defined by adding the condition $(x_1P, x_2P) \in \mathcal{O}$. Define Z' similarly to Z, with $\Gamma U(P)$ replaced by $CU(P)$ and let $Z'_{\mathcal{O}} = Z' \cap Z_{\mathcal{O}}$.
6.3.2. **Proposition.** *Let C' be a conjugacy class in G.*

(i) *If* $\ell \in C$ *then*

$$\dim(\ell U(P) \cap C') \leq \frac{1}{2}(\dim C' - \dim C);$$

(ii) *If* $g \in C'$, $\ell \in C$ *then*

$$\dim\{xP \in G/P \mid x^{-1}gx \in \Gamma U(P)\} \leq \frac{1}{2}(\dim Z_G(g) - \dim Z_L(\ell));$$

(iii) *If* $\mathcal{O}$ *is as above then*

$$\dim Z_{\mathcal{O}} \leq \dim G - \dim L + \dim \Gamma.$$

If equality holds then for any $(x_1P, x_2P) \in \mathcal{O}$ *the parabolic groups* $x_1Px_1^{-1}$ *and* $x_2Px_2^{-1}$ *have a common Levi group;*
(iv) $\dim Z'_{\mathcal{O}} \leq \dim G - \dim L + \dim C$.
This result (part of which we need in the proof of 6.3.1) is established in [L2, §1]. We refer to the proof given there.
6.3.3. **Lemma.** *For all* $\ell \in L$ *the intersection* $U(P) \cap Z_G(\ell)$ *is connected.*
The following proof is due to Spaltenstein [HS]. Let S be a maximal torus in $Z_L(\ell)$. Choose a Borel subgroup B of P such that $S \subset B$ and that $\ell \in B$. Then $L \supset Z_G(S)$ (since $Z_G(L)^\circ \subset S$), hence S is a maximal torus of $Z_G(\ell)$ and also of $Z_B(\ell)$. Therefore every irreducible component of $Z_B(\ell)$ contains an element normalizing S. But such an element centralizes S, since it belongs to B. Hence every irreducible component of $Z_B(\ell)$ meets L. We have $B = (B \cap L)U(P)$, so

$$Z_B(\ell) = (L \cap Z_B(\ell))(U(P) \cap Z_G(\ell)),$$

and it follows that $U(P) \cap Z_G(\ell)$ is connected.

We can now prove 6.3.1. Choose a locally closed, smooth, irreducible subvariety V of G and an irreducible local system $\mathcal{L}$ on V such that : V is dense in supp$K$, V is stable under conjugation by G and multiplication by $Z(G)^\circ$, $\mathcal{L}$ has the same $Z(G)^\circ$-weight as K and is G-equivariant, $K|_V$ is isomorphic to $\mathcal{L}[\dim V]$.
This is possible by 1.2.3. There is a stratum $Y_{(L,\Sigma)}$ which intersects V in a dense open subset. We may therefore assume that $V \subset Y_{(L,\Sigma)}$ (replacing V by a smaller set).
Choose $g \in V \cap \Sigma_{\text{reg}}$. Let P be a parabolic subgroup with Levi group L. We claim that

(1) $$\{ugu^{-1} \mid u \in U(P)\} = gU(P).$$

The set of the left-hand side is irreducible and closed (being an orbit of a connected unipotent group acting on an affine variety) and is contained in $gU(P)$. The isotropy group of g in $U(P)$ (acting by conjugation) is $U(P) \cap Z_G(g) \subset U(P) \cap Z_G(g_s)$. Since $g \in \Sigma_{\text{reg}}$, the latter group is trivial (see 6.2.2) and (1) follows.
The restriction of $\mathcal{L}$ to $gU(P)$ is a $U(P)$-equivariant local system, which must be constant (and non-zero). It follows that $H_c^{2d}(gU(P), \mathcal{L}) \neq 0$, where $d = \dim U(P)$. This means that $H^i(\text{res } A)_g \neq 0$, if $i = 2d - \dim V$.
Hence

$$\text{supp } H^i(\text{res } A) \supset V \cap \Sigma_{\text{reg}} \text{ if } i = 2\dim U(P) - \dim V.$$

It follows from 6.2.3 (ii) that $\dim V \cap {}_{\text{reg}} = \dim V + \dim L - \dim G$. Since $\dim G = \dim L + 2\dim U(P)$ we see that there is an i such that

$$\dim \operatorname{supp} H^i(\operatorname{res} A) \geq -i.$$

Since K is cuspidal we must have $P = G = L$, and $\sum$ is an isolated class in G. We have proved (i).
Let P be as in (ii). Since $\operatorname{res}_P^G K \in \mathcal{D}L^{<0}$ we have for $i \in \mathbb{Z}$

$$\dim \{\ell \in L \mid H_c^i(\ell U(P), K) \neq 0\} < -i.$$

Using the equivariance of K it follows that for $\ell \in L$

$$H_c^i(\ell U(P), K) = 0 \text{ if } i \geq \dim Z_L(\ell) - \dim L - \dim Z(G).$$

By 6.3.2 we have

$$\text{(2)} \qquad \dim(\ell U(P) \cap \Sigma) \leq \frac{1}{2}(\dim \Sigma + \dim Z_L(\ell) - \dim L - \dim Z(G)^\circ).$$

It follows that the assertion of (ii) is true for $i > \dim \sum - \dim Z(G)^\circ - \dim L + \dim Z_L(\ell)$. Let e be the number in the right-hand side. The assertion of (ii) for $i = e$ will follow from the exact sequence

$$\text{(3)} \qquad H_c^{e-\dim \sum -1}(\ell U(P) \cap (\bar{\Sigma} - \Sigma), K) \xrightarrow{\delta} H_c^e(\ell U(P) \cap \Sigma, \mathcal{L}) \to$$
$$\to H_c^{d-\dim \Sigma}(\ell U(P), K)(= 0),$$

if we show that the map δ is zero.
To prove this we may by a reduction argument ([BBD, no.6]) assume that k is the algebraic closure of a finite field and that K is obtained from a complex K_0 as in 1.3.1. We may assume that $\mathcal{L}$ has weight zero. If the vector space $H_c^e(\ell U(P) \cap \sum, \mathcal{L})$ is non-zero, we must have $e = 2\dim(\ell U(P) \cap \sum)$ (by (2)). Then $H_c^e(\ell U(P) \cap \sum, \mathcal{L})$ is pure of weight e (in the sense of 1.3.1). We shall prove that $H_c^{e-\dim \sum -1}(\ell U(P) \cap \bar{\sum} - \sum, K)$ is a vector space of weight $< e$. It suffices to prove that for any $\sum' \subset \bar{\sum} - \sum$ which is the inverse image of a conjugacy class in $G/Z(G)^\circ$, the vector space $H_c^{e-\dim \sum -1}(\ell U(P) \cap \sum', K)$ has a similar property.
Let $\sum'$ be such and put $j = \dim \sum'$. We have a spectral sequence

$$H_c^p(\ell U(P) \cap \Sigma', H^q K) \Rightarrow H_c^{\cdot}(\ell U(P) \cap \Sigma', K).$$

If the left-hand side is non-zero, we must have

$$p \leq 2\dim(\ell U(P) \cap \Sigma') \leq e - \dim \Sigma + j,$$

by 6.3.2 (i). Also $q < -j$, since K is a perverse extension (see 1.2.3). If also $p + q = d - \dim \sum -1$ it follows that $p = 2\dim(\ell U(P) \cap \sum')$ and $q = -j-1$. Since K is pure of weight $\dim \sum$, we have that $H^{-j-1}K$ is pure of weight $\dim \sum - j - 1$ (see 1.3.2 for these properties). Then $H_c^p(\ell U(P) \cap \sum', H^{-j-1}A)$ has weight $\leq p + \dim \sum - j - 1 = e - 1$, by Deligne's theorem ([D, p.247]).
It now follows that in the exact sequence (3) the first term has weight $< e$ and the second

has weight e. It follows that $\delta = 0$, as asserted. This concludes the proof of (ii).
Let $g \in \sum$ and choose a maximal torus S in $Z_G(g)$. Let $L = Z_G(S)$ and choose a parabolic subgroup P with Levi group L. Now

$$V = \{ugu^{-1} \mid u \in U(P)\}$$

is a closed subvariety of $gU(P)$ on which $U(P)$ acts transitively. The isotropy group of g in $U(P)$ is connected by 6.3.3. It follows that the restriction of the local system $\mathcal{L}$ to V is constant, whence $H^{2\dim V}(V, \mathcal{L}) \neq 0$.
We claim that

$$\text{(4)} \qquad \dim V = \frac{1}{2}(\dim \textstyle\sum - \dim Z(G)^\circ - \dim L + \dim Z_L(g)) =$$

$$= \frac{1}{2}(\dim G - \dim L - \dim Z_G(g) + \dim Z_L(g)).$$

If this has been established it will follow from (ii) that P must be G itself. Hence $S = Z(G)^\circ$ and (iii) follows. To prove the asserted equality for dim V we introduce a parabolic subgroup P' with Levi group L which is opposite to P, i.e. such that $P \cap P' = L$. Let V' be defined as V relative to P'. One knows that the product map defines an isomorphism of $U(P) \times L \times U(P')$ onto an open subset of G. It follows that

$$\dim Z_G(g) \geq \dim Z_L(g) + \dim Z_G(g) \cap U(P) + \dim Z_G(g) \cap U(P') =$$

$$= \dim Z_L(g) + 2\dim U(P) - \dim V - \dim V',$$

whence

$$\dim V + \dim V' \geq \dim G - \dim L - \dim Z_G(g) + \dim Z_L(g).$$

It follows from 6.3.2 (i) that dim V and dim V' are majorized by the asserted value (4). The last inequality then shows that (4) does hold. This concludes the proof of 6.3.1.

Remarks.

(a) The results of 6.3.1 are contained in [CS, nos.3,6] and [L2, §2]. The proof of (ii) is not given in [loc.cit], it was communicated to us by Lusztig.
(b) In [L2] several other properties of cuspidal perverse sheaves are established, which are used to give a complete classification. We shall not enter into this here.

Example. Let $G = SL_n$. Using 6.3.1(iii) it is not hard to see that if K is an irreducible cuspidal perverse sheaf on G, $\sum$ must be the class of an element $g = g_s g_n$ where g_s lies in the center and g_n is a regular unipotent element. Examples are the character sheaves of 5.4.11 (they are, in fact, the only possible ones, by [L2],p.246).

6.3.4. **Definition** *A cuspidal perverse sheaf K on G is clean if it is a perverse extension $I(\bar{\sum}, \mathcal{L})$, where $\sum$ is the inverse image of a conjugacy class in $G/Z(G)^\circ$ such that the restriction of K to $\bar{\sum} - \sum$ is zero.*

It will be seen in 9.2.15 that the perverse sheaves of the previous example are strongly cuspidel. Hence they are clean, by 5.4.10.
For later use we record a result on clean sheaves. Let $K = I(\bar{\sum}, \mathcal{L})$ be a clean irreducible cuspidal perverse sheaf on G. Assume that $\mathcal{E}$ is a G-equivariant non-constant local system, which has a weight for $Z(G)^\circ$.

6.3.5. **Lemma.** $H_c^\cdot(\sum, \mathcal{E}) = 0$.

Put $Z = Z(G)^\circ$ and fix $a \in$. We can choose a morphism $\varphi : G/Z \times Z \to$ of the form $\varphi(g, Z, z) = gag^{-1}z^n$ ($g \in G, z \in Z$) such that $\varphi^*\mathcal{E}$ is constant. Let

$$G_1 = \{g \in G \mid gag^{-1} \in aZ\}.$$

Then $G_1^\cdot/Z$ is a connected unipotent group by 6.3.1 (iv).
We factor φ

$$G/Z \times Z \xrightarrow{\varphi_1} \tilde{\Sigma} \xrightarrow{\tilde{\varphi}} \Sigma,$$

where φ_1 is a fibering with fibers G_1°/Z and $\tilde{\varphi}$ is a finite Galois covering such that $\tilde{\varphi}^*\mathcal{E}$ is trivial with group G_1/G_1°.
It suffices to prove that G_1/G_1° acts trivially on all $H_c^i(\tilde{\Sigma}, E)$. But

$$H_c^i(\tilde{\Sigma}, E) = H_c^{i-2a}(G/Z \times Z, E)$$

where $a = \dim G_1^\circ/Z$ and the action of G_1/G_1° on the cohomology group of the left-hand side is the restriction of an action of the connected group G/Z on the one of the right-hand side. The latter action is trivial, and the assertion follows.
A direct consequence of the previous lemma is the following one ([CS, 7.8]).
6.3.6. **Lemma.** *Let K and K' be two clean irreducible perverse sheaves on G such that K' is not isomorphic to DK. Then $H_c^\cdot(G, K \otimes K') = 0$.*

7 Parabolic induction

G, P and L are as in no.6.

7.1 Parabolic induction

7.1.1. Let $G \times_P P$ be defined as in 6.2.4, i.e. the quotient of $G \times P$ by the P-action $x(g,y) = (gx^{-1}, xyx^{-1})$ $(g \in G, x, y \in P)$. We have a proper morphism $\delta : G \times_P P \to G$ induced by $(g,y) \mapsto gyg^{-1} (g \in G, y \in P)$. Consider the diagram

$$L \xleftarrow{\alpha} G \times P \xrightarrow{\beta} G \times_P P \xrightarrow{\delta} G,$$

where $\alpha(g,y) = \pi_P y$ and β is the canonical map. If K is a perverse sheaf on L which is L-equivariant for conjugation then by property (b) of 1.2.6 we have that $\alpha^* K[\dim G + \dim U(P)]$ is a perverse sheaf on $G \times P$. It is P-equivariant and 1.4.2 shows that there exists a perverse sheaf $\tilde{K}$ on $G \times_P P$ such that

$$\beta^* \tilde{K} = \alpha^* K[2 \dim U(P)],$$

moreover $\tilde{K}$ is G-equivariant for the action via left translations on G.
We put

$$\text{ind } K = \text{ind}_P^G K = \delta_* \tilde{K}.$$

By property (c) of 1.4.1 the ${}^pH^i(\text{ind } K)$ are G-equivariant (for conjugation).
If K is irreducible then so is $\tilde{K}$ and it follows from the decomposition theorem that ind K is a semi-simple complex, so is G-equivariant (see 1.4.1).
Examples.
(a) Assume that everything is defined over a finite field $\mathbb{F}_q$ and that k is an algebraic closure of $\mathbb{F}_q$. Assume that we have $\varphi : F^*K \xrightarrow{\sim} K$. Then ind K inherits $\psi : F^*(\text{ind } K) \xrightarrow{\sim} \text{ind } K$ and one checks from the definitions that the characteristic function $\chi_{\psi, \text{ind } K}$ on G^F is the class function induced by the class function $\chi_{\varphi, K}$ on L^F, in the sense of parabolic induction of finite groups of Lie type (i.e. lifting from L^F to P^F and induction from P^F to G^F in the sense of Frobenius).
(b) Assume that $P = B$, our Borel group and $L = T$. Take $K = \mathcal{L}_\xi[\dim T]$, where $\xi \in \hat{X}$. The definitions show that

$$\text{ind}_B^G \mathcal{L}_\xi = C_{\xi,e}^G.$$

We review some properties of induction.
Let $Q \subset P$ be another parabolic subgroup, with Levi group M contained in L. Then $Q \cap L$ is a parabolic subgroup of L, with Levi group M.
7.1.2. **Proposition.** *Let K be a perverse sheaf on M which is M-equivariant for conjugation such that* $\text{ind}_{Q \cap L}^L K$ *is perverse. Then*

$$\text{ind}_Q^G K = \text{ind}_P^G \text{ind}_{Q \cap L}^L K.$$

This is transitivity of induction. The straightforward proof is omitted (see [CS, 4.2]).
The next result is of the Frobenius duality type.
7.1.3. **Proposition.** *Let K_1 (resp. K) be a perverse sheaf on G (resp. L) which is equivariant for conjugation. Assume that* $\text{res}_P^G K_1 \in \mathcal{D}L^{\leq 0}$. *Then*

$$\text{Hom}_{\mathcal{D}G}(K_1, \text{ind}_P^G K) = \text{Hom}_{\mathcal{D}L}(\text{res}_P^G K_1, K).$$

Consider the diagram

$$\begin{array}{ccccccc} & G \times P & \xrightarrow{\beta} & G \times_P P & \xrightarrow{\delta} & G \\ \swarrow^{\alpha} & \downarrow \text{id} \times \pi_P & & \downarrow \gamma & & \\ L \xleftarrow{\alpha'} & G \times L & \xrightarrow{\beta'} & G \times_P L & & \end{array},$$

where $G \times_P L$ is the quotient of $G \times L$ by the P-action

$$x(g,l) = (gx^{-1}, \pi_P(x) l \pi_P(x)^{-1}) (g \in G, x \in P, l \in L).$$

The maps α, β, δ are as before and α', β' are defined in an obvious way. The map γ is such that the square is cartesian. Since res $K_1 \in \mathcal{D}L^{\leq 0}$ we have (see [BBD, 1.3])

$$\text{Hom}\ (\text{res}\ K_1, K) = \text{Hom}\ ({}^pH^0(\text{res}\ K_1), K) =$$
$$= \text{Hom}\ ((\alpha')^*\ {}^pH^0(\text{res}\ K_1), (\alpha')^*K) = \text{Hom}\ ((\alpha')^*\ (\text{res} K_1), (\alpha')^*K),$$

here we also use property (b) of 1.2.6.
Now we have a diagram

$$\begin{array}{ccccc} G \times P & \xrightarrow{\text{pr}} & P & \xrightarrow{i} & G \\ \downarrow \text{id} \times \pi_P & & \downarrow \pi_P & & \\ G \times L & \xrightarrow{\alpha'} & L & & \end{array}$$

and, by the definition of restriction in 6.1.1,

$$(\alpha')^*(\text{res}\ K_1) = (\alpha')^*((\pi_P)_! i^* K_1) = (\text{id} \times \pi_P)_!((\text{pr})^* i^* K_1).$$

From the G-equivariance of K_1 it follows that $(\text{pr})^* i^* K_1$ is isomorphic to $\beta^* \delta^* K_1$, whence

$$(\alpha')^*(\text{res} K_1) = (\beta')^* \gamma_! \delta^* K_1.$$

It follows that $\gamma_! \delta^* K_1 \in \mathcal{D}(G \times_P L)^{\leq \dim U(P)}$.

As in 7.1.1 there is a perverse sheaf $\tilde{K}'$ on $G \times_P L$ such that $(\beta')^* \tilde{K}'[\dim P] = (\alpha')^* K[\dim G]$ and one sees easily that $\gamma^* \tilde{K}'[\dim U(P)] = \tilde{K}$.
We now have

$$\text{Hom}(\text{res} K_1, K) = \text{Hom}\ ((\alpha')^*(\text{res}\ K_1), (\alpha')^* K) =$$
$$= \text{Hom}((\beta')^* \gamma_! \delta^* K_1, (\beta')^* \tilde{K}'[-\dim U(P)]) =$$
$$= \text{Hom}\ (\gamma_! \delta^* K_1, \tilde{K}'[-\dim U(P)]) = \text{Hom}\ (\delta^* K_1, \gamma^! \tilde{K}'[-\dim U(P)]) =$$
$$= \text{Hom}\ (\delta^* K_1, \gamma^* \tilde{K}'[\dim U(P)]) = \text{Hom}\ (\delta^* K_1, \tilde{K}) =$$
$$= \text{Hom}\ (K_1, \delta_* \tilde{K}) = \text{Hom}\ (K_1, \text{ind}\ K),$$

which proves 7.1.3.

The preceding results about induction are contained in [CS, no.4]. There is also a Mackey type formula, which we use for character sheaves (to be established in 10.1).

7.2 Induction of cuspidal perverse sheaves

7.2.1. With the previous notations let K be an irreducible cuspidal perverse sheaf on L. By 6.3.1 (i) there is an isolated class Σ in L and an irreducible local system $\mathcal{L}$ on Σ such that $K = I(\bar{\Sigma}, \mathcal{L})$. Moreover, $\mathcal{L}$ is equivariant for conjugation and has a weight for translation action of $Z(L)^\circ$.
Let $Y = Y_{(L,\Sigma)}$, as in 6.2.2. With the notations introduced there we have a diagram

$$\begin{array}{ccccccc} \Sigma & \stackrel{\alpha_1}{\leftarrow} & G \times \Sigma_{\mathrm{reg}} & \stackrel{\beta_1}{\rightarrow} & G \times_L \Sigma_{\mathrm{reg}} & \stackrel{\gamma}{\rightarrow} & Y \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \bar{\Sigma} & \stackrel{\alpha}{\leftarrow} & G \times \bar{\Sigma}U(P) & \stackrel{\beta}{\rightarrow} & G \times_P \bar{\Sigma}U(P) & \stackrel{\delta}{\rightarrow} & \bar{Y} \end{array} .$$

Here γ and δ are as in 6.2.2 resp. 6.2.4. The horizontal maps are the evident ones and the vertical maps are imbeddings (use 6.2.7(iii) for the map $G \times_L \Sigma_{\mathrm{reg}} \to G \times_P \bar{\Sigma}U(P)$).
There is a local system $\tilde{\mathcal{L}}$ on $G \times_L \Sigma_{\mathrm{reg}}$ such that $\alpha_1^* \mathcal{L} = \beta_1^* \tilde{\mathcal{L}}$. Since γ is a Galois covering (6.2.3(ii)) we have that $\gamma_* \tilde{\mathcal{L}}$ is a semi-simple local system on Y.
7.2.2. **Theorem.** $\mathrm{ind}_P^G K = I(\bar{Y}, \gamma_* \tilde{\mathcal{L}})$.
(This result is prop. 4.5 of [L2].)
It follows from the definition of induction that the restriction of $\mathrm{ind}_P^G K$ to Y is $\gamma_* \tilde{\mathcal{L}}[\dim Y]$.
Put $X = G \times_P \bar{\Sigma}U(P)$. The image of $(x, y) \in G \times \bar{\Sigma}U(P)$ in X is denoted by $x * y$. It is clear that ind K is zero outside $\bar{Y}$ and that the restriction of ind K to $\bar{Y}$ is $\delta_* \tilde{K}$, where $\tilde{K}$ is the irreducible perverse sheaf on X with

$$\beta^* \tilde{K}[\dim P] = \alpha^* K[\dim G + \dim U(P)].$$

To prove the theorem it suffices to show that

$$\text{(1)} \qquad \dim \mathrm{supp} H^i(\delta_* \tilde{K}) < -i \text{ if } i > -\dim Y$$

and a similar assertion with $\mathcal{L}$ replaced by the dual $\mathcal{L}^\vee$ (which follows if (1) is established).
We have a partition $\bar{\Sigma} = \coprod S_j$, the S_j being the orbits of $L \times Z(L)^\circ$ in $\bar{\Sigma}$ (L acting by conjugation and $Z(L)^\circ$ by translation). Write $S_0 = \Sigma$.
Let $X_j = \{x * y \in X \mid y \in S_j U(P)\}$. Then $X = \coprod X_j$ is a stratification of X, with smooth strata. Also, X_0 is open dense and the closure of a stratum is a union of strata. We have if $g \in \bar{Y}$

$$H^i(\delta_* \tilde{K})_g = H^i(\delta^{-1} g, \tilde{K}).$$

If this is non-zero, we have $H_c^i(\delta^{-1} g \cap X_j, \tilde{K}) \neq 0$ for some j. Using the spectral sequence

$$H_c^p(\delta^{-1} g \cap X_j, H^q \tilde{K}) \Rightarrow H_c^{\cdot}(\delta^{-1} g \cap X_j, \tilde{K})$$

we see that in this situation there are p, q with $p + q = i$ such that the left-hand side of the last formula is non-zero. Then $p \leq 2 \dim(\delta^{-1} g \cap X_j)$. Moreover, since $\tilde{K}$ is irreducible we have $q < -\dim X_j$ if $j \neq 0$ and $q \leq -\dim X_j$ if $j = 0$. We conclude that

$$i \leq 2 \dim(\delta^{-1} g \cap X_j) - \dim X_j.$$

Now since $\dim X_j = \dim Y - \dim \Sigma + \dim S_j$, we see that

$$\dim(\delta^{-1} g \cap X_j) \geq \tfrac{1}{2}(i - \dim \Sigma + \dim S_j + \dim Y),$$

with strict inequality if $j \neq 0$.
The support inequality now follows from the following lemma, which is a consequence of 6.3.2.

7.2.3. Lemma.
(i) *If* $j \neq 0$ *then*

$$\dim\{g \in \bar{Y} \mid \dim(\delta^{-1}g \cap X_j) > \tfrac{1}{2}(i - \dim \Sigma + \dim S_j)\} < \dim Y - i;$$

(ii) $\dim\{g \in \bar{Y} \mid \dim(\delta^{-1}g \cap X_0) \geq \frac{1}{2}i\} < \dim Y - i$ *if* $i > 0$.
We have $\delta^{-1}g \cap X_j = \{x * y \in X \mid y \in S_jU(P), xyx^{-1} = g\}$. This is isomorphic to the variety $\{xP \in G/P \mid x^{-1}gx \in S_jU(P)\}$. Let $Z_j = \{(g, x_1P, x_2P) \in G \times G/P \times G/P \mid x_i^{-1}gx_i \in S_jU(P), i = 1,2\}$. By 6.3.2(iii) we have $\dim Z_j \leq \dim G - \dim L + \dim S_j = \dim Y - \dim \sum + \dim S_j$. The assertion of (i) now follows readily from these facts. We also see that for all j the left-hand side of (i) is majorized by the right-hand side.
Now assume that i is such that the inequality of (ii) does not hold. It follows that the two sides of the formula of (ii) are equal. Let Z and $Z_{\mathcal{O}}$ be as in 6.3.2 (with $\Gamma = \sum$). We have $\dim Z_{\mathcal{O}} \leq \dim G - \dim L + \dim \sum = \dim Y$ for all G-orbits $\mathcal{O}$ in $G/P \times G/P$. It follows that Z has an irreducible component of dimension $\geq \dim Y$, whose projection onto G has dimension $\leq \dim Y - i$. This component contains some $Z_{\mathcal{O}}$ with $\dim Z_{\mathcal{O}} = \dim Y$. It follows from 6.3.2(iii) that the orbit contains an element (P, nP), with $n \in N_GL$. We then must have $n \sum n^{-1} = \sum$ and it follows that the projection of $Z_{\mathcal{O}}$ on G contains Y. Since this projection has dimension $\leq \dim Y - i$ we must have $i = 0$ and (ii) follows.
This concludes the proof of 7.2.2.

7.2.4. Corollary. *If K is as in* 7.2.2 *then* $\mathrm{ind}_P^G K$ *is semi-simple and perverse.*
The next result is a complement to 7.1.3.

7.2.5. Lemma. *Let K_1 be a perverse sheaf on G which is equivariant for conjugation. Assume that*

$$\mathrm{res}_P^G K_1 = \bigoplus K_\lambda[n_\lambda],$$

where the K_λ are perverse sheaves on L such that $\mathrm{ind}_P^G K_\lambda$ *is perverse. Then* $\mathrm{res}_P^G K_1 \in \mathcal{D}L^{\leq 0}$.
We use the notations of the proof of 7.1.3. We define $\tilde{K}'_\lambda$ similar to $\tilde{K}'$ in that proof. We have

$$\mathrm{Hom}(K_1, (\mathrm{ind}\ K_\lambda)[s]) = \mathrm{Hom}\ (\gamma_!\delta^* K_1, \tilde{K}'_\lambda[s - \dim U(P)])$$

for all $s \in \mathbb{Z}$. Since K_1 and ind K_λ are perverse Hom $(K_1, (\mathrm{ind}\ K_\lambda)[s])$ is zero if $s < 0$, whence

(2) $$\mathrm{Hom}\ (\gamma_!\delta^* K_1, \tilde{K}'_\lambda[s]) = 0 \text{ if } s < -d,\ d = \dim U(P).$$

Put $C = \gamma_!\delta^* K_1$. Then $(\beta')^* C = (\alpha')^*\ \mathrm{res}\ K_1 = \oplus(\alpha')^*\ K_\lambda[n_\lambda] = \oplus(\beta')^* \tilde{K}'_\lambda[n_\lambda - d]$, hence, by 1.2.6 (b), ${}^pH^iC = \bigoplus_{n_\lambda = d-i} \tilde{K}'_\lambda$.
It follows, using (2), that, if $C \in \mathcal{D}^{\leq i}$ and $i > d$, then the canonical morphism $C \to {}^pH^iC[-i]$ is zero. So we have $C \in \mathcal{D}^{\leq d}$ and $\mathrm{res} K_1 \in \mathcal{D}^{\leq 0}$ by 1.2.6 (b).

Remark. The condition of the lemma is satisfied when the sheaves K_λ are irreducible and cuspidal (7.2.2) and, more generally when they are admissible, see section 8.

7.2.6. Let $\mathcal{A} = \mathrm{End}\ (\mathrm{ind}\ K)$ be the algebra of all endomorphisms of the perverse sheaf ind K of 7.2.2. It is a finite dimensional semi-simple E-algebra. It follows from 7.2.2 that it is also the endomorphism algebra of the local system $\gamma_* \tilde{\mathcal{L}}$ on Y.
Recall that $\gamma : G \times_L \sum_{\mathrm{reg}} \to Y$ is a Galois covering with group $N(L, \sum)/L = W(L, \sum)$.

Consider the subgroup $W(L,\ \ ,\mathcal{L}) = N(L,\ \ ,\mathcal{L})/L$, where

$$N(L,\Sigma,\mathcal{L}) = \{n \in N(L,\Sigma) \mid n^*\mathcal{L} \simeq \mathcal{L}\}$$

(n acts on $\sum$ by conjugation). For $w \in W(L,\sum,\mathcal{L})$ we put

$$\mathcal{A}_w = \text{Hom}\,(\tilde{\mathcal{L}}, \dot{w}^*\tilde{\mathcal{L}}),$$

where $\dot{w}$ represents w. This is a one dimensional vector space. Since $\gamma_*\tilde{\mathcal{L}} = \gamma_*\dot{w}^*\tilde{\mathcal{L}}$ we have a natural imbedding

$$\mathcal{A}_w \rightarrow \text{End}\,(\gamma_*\tilde{\mathcal{L}}) = \mathcal{A}.$$

Now $\mathcal{A} = \oplus\mathcal{A}_w$ and the multiplication in $\mathcal{A}$ maps $\mathcal{A}_w \times \mathcal{A}'_w$ onto $\mathcal{A}_{ww'}$. In particular, we see that all endomorphisms of $\gamma_*\tilde{\mathcal{L}}$ are G- endomorphisms.

7.2.7. Now assume that k is an algebraic closure of the finite field F_q and that G is defined over F_q. Assume that $FL = L, F\sum = \sum$ and that we are given an isomorphism of local systems $\varphi : F^*\mathcal{L} \xrightarrow{\sim} \mathcal{L}$. The varieties $G \times \sum_{\text{reg}}, G \times_L \sum_{\text{reg}}, Y$ are defined over F_q and φ induces isomorphisms $F^*\tilde{\mathcal{L}} \xrightarrow{\sim} \tilde{\mathcal{L}}$, $F^*\gamma_*\tilde{\mathcal{L}} \xrightarrow{\sim} \gamma_*\tilde{\mathcal{L}}$. The latter isomorphism determines an isomorphism $\varphi : F^*K' \xrightarrow{\sim} K'$, where $K' = I(\bar{Y}, \gamma_*\tilde{\mathcal{L}})$. Note that P does not enter the picture here. In general, there need not exist a parabolic subgroup with Levi subgroup L which is defined over F_q.

8 Admissible perverse sheaves

G is as before.

Definition. *A perverse sheaf on G is admissible if it is an irreducible constituent of a perverse sheaf* $\mathrm{ind}_P^G K$ *as in 7.2.2.*

It follows from transitivity of induction that if K is an admissible perverse sheaf on the Levi group L of a parabolic subgroup P then $\mathrm{ind}_P^G K$ is a direct sum of admissible perverse sheaves. We shall prove later that character sheaves are admissible (see 9.3.2).

In [CS] and [L2] admissible perverse sheaves are called "admissible complexes".

8.1 Some properties of cuspidal perverse sheaves

8.1.1. Let K be an irreducible cuspidal perverse sheaf on G. We have $K = I(\Sigma, \mathcal{L})$, where Σ and $\mathcal{L}$ are as in 6.3.1.

Let Σ_s be the set of semi-simple parts of elements of $\bar{\Sigma}$ (or Σ, see 6.2.5). Fix $s \in \Sigma_s$ and put $H = Z_G(s)^0$, this is a connected reductive subgroup of G. Since Σ is isolated we have $Z(H)^\circ = Z(G)^\circ$ (6.2.1).

The set of elements in Σ with semi-simple part s is sC , where C is the set of unipotent elements u in H with $su \in \Sigma$. Then C is an orbit of the group

$$\{g \in G \mid gsg^{-1} \in sZ(G)^\circ\},$$

which has H as its identity component. It follows that C is a union of finitely many unipotent conjugacy classes of H, all of the same dimension (as a matter of fact, C itself is a conjugacy class in H, see 8.1.4). Put $\Sigma' = CZ(G)^\circ$ and define $\alpha = \overline{\Sigma'} \to \bar{\Sigma}$ by $\alpha x = sx$. We put

$$K' = \alpha^* K[\dim \Sigma\,' - \dim \Sigma].$$

8.1.2. **Proposition.** *K' is a cuspidal perverse sheaf on H.*

Let $\mathcal{L}'$ be the pull-back under α of the local system $\mathcal{L}$ on Σ, it is a local system on $\Sigma\,'$. We claim that

$$K' = I(\bar{\Sigma}', \mathcal{L}').$$

to see this consider the fibration $\bar{\Sigma} \to \Sigma_s/Z(G)^\circ$. The fibers are all isomorphic to $\overline{\Sigma'}$ and the basis is smooth. One checks without difficulty the support conditions of 1.2.3, for K' and its dual. So K' is perverse.

Now let Q be a proper parabolic subgroup of H and choose a proper parabolic subgroup P of G with $Q = P \cap H$. Also choose Levi groups, L, M of P resp. Q with $M = L \cap H$. Notice that $U(Q) = U(P) \cap H$. Put res $K' =$ $\mathrm{res}_Q^H K'$. We have to show that for all i we have

$$\dim \operatorname{supp} H^i(\operatorname{res} K') < -i.$$

Let $g \in M$ be such that $H^i(\operatorname{res} K')_g = H_c^i(gU(Q), K')$ is non-zero. Then $sg \in \bar{\Sigma}$ and

$$H_c^i(gU(Q), K') = H_c^{i-a}(sgU(Q) \cap \bar{\Sigma}, K),$$

where $a = \dim \Sigma - \dim \Sigma\,'$.

Consider the map

$$\beta : sgU(P) \to (sU(P)Z(G)^\circ \cap \Sigma_s)/Z(G)^\circ,$$

sending $x \in sgU(P)$ to the coset modulo $Z(G)^\circ$ of x_s. Then β is $U(P)$-equivariant for the actions induced by conjugation. The action of $U(P)$ on $X = (sU(P)Z(G)^\circ \cap \sum_s)/Z(G)^\circ$ is transitive and $U(Q)$ is an isotropy group. It follows that X is an affine space and that the $U(P)$-equivariant local system $H^j(\beta_! K)$ on X is constant for all j. For $j = i - a$ this local system is non-zero, since the stalk of $H^{i-a}(\beta_! K)$ over the image of s is $H_c^{i-a}(sgU(Q) \cap \bar{\sum}, K)$. We have a spectral sequence

$$E_2^{p,q} = H_c^p(X, H^q(\beta_! K)) \Rightarrow H_c^{\cdot}(sgU(P), K).$$

We see from the preceding remarks that $E_2^{p,q} = 0$ if $p \neq 2d$ and that $E_2^{2d,i-a} \neq 0$, where $d = \dim X$. Consequently, $H_c^{i-a+2d}(sgU(P), K) \neq 0$. Let A be the conjugacy class of sg in L and A' the conjugacy class of g in M. We conclude that

$$AZ(G)^\circ \subset \operatorname{supp} H^{i-a+2d}(\operatorname{res} K).$$

Since K is cuspidal we have

$$\dim AZ(G)^\circ < -i + a - 2d$$

Now observe that

$$\dim AZ(G)^\circ - a + 2d = \dim A'Z(G)^\circ.$$

We conclude that

$$\dim A'Z(G)^\circ < -i.$$

Since supp $H^i(\operatorname{res} K')$ is a union of finitely many sets of the form $A'Z(G)^\circ$, we conclude that res K' is cuspidal, as asserted.

The proof also gives

8.1.3. **Corollary.** *If K is strongly cuspidal then K' is strongly cuspidal.*

Another consequence is the following result.

8.1.4. **Corollary.** *C is a single unipotent class.*

This follows from the following result, which we do not prove here.

8.1.5. **Lemma.** *Let G be semi-simple and simply connected, and let $s \in G$ be an isolated semi-simple element. Assume that C_1 and C_2 are two unipotent classes in $Z_G(s)$ such that*

(a) *there exists a local system $\mathcal{L}_i$ on C_i such that $I(\bar{C}_i, \mathcal{L}_i)$ are cuspidal perverse sheaves on $Z_G(s)(i = 1, 2)$,*

(b) *there is $g \in G$ with $gC_1g^{-1} = C_2, gsg^{-1}s^{-1} \in Z(G)$. Then $C_1 = C_2$.*

The proof uses the classification of cuspidal perverse sheaves given in [L2], see [CS, 7.12].

8.1.6. **Corollary.** *K' is clean if and only if K is clean.*

See 6.3.4 for the definition of clean cuspidal sheaves. The corollary follows from the proof of 8.1.2, using 8.1.4.

8.2 Restriction of an induced character sheaf to the centralizer of a semi-simple element

We fix a parabolic subgroup P of G, with Levi subgroup L. We also fix a semi-simple element $s \in G$. Put $H = Z_G(s)^\circ$. Let $\sum$ be an isolated class in L. Denote by $\sum_s$ the set of semi-simple parts of the elements of $\sum$.

The set $S = \{g \in G \mid g^{-1}sg \in \sum_s\}$ is a union of finitely many double cosets Hg_iL

$(1 \leq i \leq h)$ and the set $\{g \in G \mid g^{-1}sg \in \ \ _sU(P)\}$ is the disjoint union of the double cosets Hg_iP.
If $g \in S$ then $gPg^{-1} \cap H$ is a parabolic subgroup of H, with Levi group $gLg^{-1} \cap H$. If $g = g_i$ we write P_i resp. L_i for these subgroups.
Define

$$C_i = \{h \in H \mid h \text{ unipotent}, \ sh \in g_i \Sigma g_i^{-1}\},$$

and let $\sum_i = C_i Z(L_i)^\circ \mid 1 \leq i \leq h)$.
8.2.1. **Proposition.** *There exists an open neighbourhood U of s in H with the following properties:*
(a) *U is stable under conjugation by elements of H;*
(b) *$h \in U$ if and only if $h_s \in U$;*
(c) *If $g \in G$ and $U \cap g \sum_s g^{-1} \neq \emptyset$ then $s \in g \sum_s g^{-1}$ and $U \cap g \sum_s g^{-1} \subset sZ(gLg^{-1})^\circ$;*
(d) *If $g \in G$ and $h \in U \cap g\bar{\sum} U(P) g^{-1}$ then $s \in g \sum_s U(P) g^{-1}$ and $h_s \in sgZ(L)^\circ U(P) g^{-1}$.*
Put $H_{\text{reg}} = \{h \in H \mid Z_G(h_s)^\circ \subset H\}$. Then H_{reg} is an open subset of H which contains s.
8.2.2. **Lemma.** *Let $g \in G$.*
(i) *If $H_{\text{reg}} \cap g \sum g^{-1} \neq \emptyset$ (resp. $H_{\text{reg}} \cap g \sum_s g^{-1} \neq \emptyset$) then s lies in an isolated class of gLg^{-1};*
(ii) *If $h \in H_{\text{reg}} \cap g \sum g^{-1}$ (resp. $H_{\text{reg}} \cap g \sum_s g^{-1}$) then h lies in an isolated class of $H \cap gLg^{-1}$.*
We may take $g = 1$. Let $h \in H_{\text{reg}} \cap \sum$. Take a maximal torus T of L containing h_s. Then T is also a maximal torus of H (since $Z_G(h_s)^\circ \subset H$) whence $s \in T$. From $Z_L(h_s)^\circ \subset H \cap L = Z_L(s)^\circ$ and the fact that $\sum$ is isolated in L we see that s lies in an isolated class of L and that h lies in an isolated class of $H \cap L$.
If $\sum$ is replaced by $\sum_s$ the same argument can be given.

Let $\sum_1, \ldots, \sum_m$ be the set of semi-simple isolated classes in L, where $\sum_1 = \sum_s$. Put $M = \{x \in G \mid x^{-1}sx \in \bigcup_{j \geq 2} \sum_j\}$. Let H' be the subset of H_{reg} obtained by removing the elements h such that $h_s \in x \sum_1 x^{-1}$ for some $x \in M$. It follows from 8.2.2(i) that if $H' \cap g \sum g^{-1} \neq \emptyset$ for $g \in G$, then $s \in g \sum_1 g^{-1}$. The same conclusion holds if $\sum$ is replaced by $\sum_1$. Also, $s \in H'$.
Next we show that H' is open in H_{reg}. Since $H \backslash M / L$ is finite it is enough to prove that if a is a fixed element in G such that $a^{-1}sa$ is isolated in L, the set

$$\{h \in H_{\text{reg}} \mid h_s \in \bigcup_{x \in H} xa\Sigma_1 a^{-1} x^{-1}\} \tag{1}$$

is closed in H_{reg}. If $h \in H_{\text{reg}}, x \in H$ and $x^{-1}h_s x \in a \sum_1 a^{-1}$ then $y^{-1}h_s y$ belongs to an isolated class of $H \cap aLa^{-1}$, by 8.2.2 (ii). It follows that the set (1) is the intersection with H_{reg} of finitely many sets of the form

$$\{h \in H \mid h_s \in \bigcup_{x \in H} xbZ(aLa^{-1})^\circ x^{-1}\},$$

b being a given element of $H \cap aLa^{-1}$. Since such sets are closed, the set (1) is closed in H_{reg}.
Now let U be the subset of H' obtained by removing the elements h such that $h_s \in x \sum_1 x^{-1} - sZ(xLx^{-1})^\circ$ for some $x \in G$ with $x^{-1}sx \in \sum_1$. As in the case of H', one shows that U is open. It contains s. The properties (a), (b), (c) of the proposition hold.
To prove property (d) let $x \in \bar{\sum} U(P)$ and $g \in G$ be such that $gxg^{-1} \in U$. Using 6.2.6 we

deduce (d) from (c).
This concludes the proof of 8.2.1.
8.2.2. Corollary. *If G is defined over a finite field, with Frobenius morphism F, then if $FL = L, F\sum = \sum, Fs = s$ we can take U to be such that $FU = U$.*

Next assume that K is an irreducible cuspidal perverse sheaf on L, of the form $I(\bar{\sum}, \mathcal{L})$, where $\sum$ is as above. With the notations introduced in the beginning of 8.2, let K_i be the inverse image of K for the map $\sum_i \to \sum$ sending x to $g_i^{-1}sxg_i$, shifted by $\dim \sum_i - \dim \sum = \dim L_i - \dim L$. By 8.1.4 we know that C_i is a single unipotent class in L_i and by 8.1.2, K_i is a cuspidal perverse sheaf on L_i with support $\overline{\sum_i}$.
Let U be as in 8.2.1.
8.2.3. Proposition. $s^*\left((\text{ind}_P^G K)\mid_U\right) \cong \bigoplus_{i=1}^{h}(\text{ind}_{P_i}^H K_i)\mid_{s^{-1}U}[\dim G - \dim H]$.
s^* denotes the inverse image for the map $x \mapsto sx$.
We use the notations of the proof of 7.2.2. So let $\delta : X = G \times_P \bar{\sum} U(P) \to \bar{Y}$. Put $X_U = \delta^{-1}(\bar{Y} \cap U)$ and $X_{U,i} = \{x * y \in X_U \mid x \in Hg_iP\}$. Then X_U is the disjoint union of the $X_{U,i}$, by property (d) of 8.2.1. Moreover these sets are open and closed in X_U (since the image of $Hg_iP \times_P P$ under the canonical map $G \times_P P \to G/P$ is the closed subvariety H/P_i of G/P).
For each i we have a commutative diagram

$$\begin{array}{ccccccc} \bar{\sum}_i & \leftarrow & H \times \bar{\sum}_i U(P_i) & \rightarrow & H \times_{P_i} \bar{\sum}_i U(P_i) & \xrightarrow{\delta_i} & \bar{Y}_i \\ \downarrow_{\tau_i} & & \downarrow & & \downarrow_{\sigma_i} & & \downarrow_s \\ \bar{\sum} & \leftarrow & G \times \bar{\sum} U(P) & \rightarrow & G \times_P \bar{\sum} U(P) & \xrightarrow{\delta} & \bar{Y}. \end{array}$$

The horizontal rows are as in the proof of 7.2.2. Moreover, $\tau_i(x) = g_i^{-1}sxg_i$ and $\sigma_i(x * y) = (xg_i * g_i^{-1}(sy)g_i)$. There is a perverse sheaf $\tilde{K}$ on X such that ind $K = \delta_*\tilde{K}$. Similarly, $\text{ind}K_i = \delta_i\tilde{K}_i$, where $\tilde{K}_i$ is perverse on $H \times_P \bar{\sum}_i U(P_i)$.
Now $\text{ind}K \mid_{\bar{Y}\cap U} = \delta_!(\tilde{K}\mid_{X_*})$ and $\text{ind}K_i \mid_{\bar{Y}_i \cap s^{-1}U} = (\delta_i)_!(\tilde{K}_i \mid_{\delta_i^{-1}(s^{-1}U)})$.
It follows from property (d) of 8.2.1 that the restriction of σ_i to $\delta_i^{-1}(s^{-1}U)$ is an isomorphism of that set onto $X_{U,i}$. Under this isomorphism the restriction of $\tilde{K}_i$ to $\delta_i^{-1}(s^{-1}U)$ corresponds to the restriction of $\tilde{K}$ to $X_{U,i}$, shifted by $\dim H - \dim G$, as follows from the definitions of $\tilde{K}$ and $\tilde{K}_i$ (see the proof of 7.2.2 and 8.1.1). This implies the theorem.

8.2.4. We next give some complements to 8.2.3. With the notations of 7.2.1 we have a commutative diagram

$$\begin{array}{ccccc} & G \times_L \sum_{\text{reg}} & \xrightarrow{\gamma} & Y & \\ & \curvearrowright & & \curvearrowright & . \\ X = & G \times_P \bar{\sum} U(P) & \xrightarrow{\delta} & \bar{Y} & \end{array}$$

Put $\tilde{Y} = G \times_L \sum_{\text{reg}}$, $\tilde{Y}_U = \tilde{Y} \cap X_U$, $\tilde{Y}_{U,i} = \tilde{Y}_U \cap X_{U,i}$, where U is as before. Then $\tilde{Y}_{U,i}$ is open and closed in $\tilde{Y}_U$ since $X_{U,i}$ is open and closed in X_U (see the proof of 8.2.3).
8.2.5. Lemma.
(i) *$\tilde{Y}_{U,i}$ is non-empty;*
(ii) *$\tilde{Y}_{U,i}$ is open and dense in $X_{U,i}$, $\tilde{Y}_U$ is open and dense in X_U and $Y \cap U$ is open and dense in $\bar{Y} \cap U$.*
To prove (i) we have to show that $U \cap g_i \sum_{\text{reg}} g_i^{-1}$ is not empty, for if x lies in that set then $g_i * g_i^{-1}xg_i \in \tilde{Y}_{U,i}$. Property (b) of 8.2.1 implies that U contains sC_i. Hence $U \cap s\sum_i$ is open

dense in $s \quad_i$. Also, $g_i \quad_{\text{reg}} g_i^{-1} \cap s \quad_i$ is open dense in $s \quad_i$. Hence $U \cap s \quad_i \cap g_i \quad_{\text{reg}} g_i^{-1} \neq \emptyset$. It follows from the proof of 8.2.3 that $X_{U,i}$ is isomorphic to an open subset of the irreducible variety $H \times_{P_i} \sum_i U(P_i)$. Since $\tilde{Y}_{U,i}$ is non-empty by (i), we have the first point of (ii). The rest of (ii) then easily follows.

Now put $Y_{U,i} = \gamma(\tilde{Y}_{U,i})$. Then $Y \cap U = \cup_i Y_{U,i}$ and $Y_{U,i}$ is irreducible, and closed in $Y \cap U$. Also, $Y_{U,i} = Y_{U,j}$ if and only if $g_j \in Hg_i N_G(L, \sum)$, otherwise $Y_{U,i} \cap Y_{U,j} = \emptyset$. We write $Y_i = \cup_{h \in H} h(\sum_i)_{\text{reg}} h^{-1}$.

8.2.6. **Lemma.**

(i) *$Y_{U,i}$ is an open subset of sY_i;*

(ii) *Y_i is a smooth variety, of dimension* $\dim H - \dim L + \dim \sum$.

(ii) is a direct consequence of (i). To prove (i) it suffices to show that if $g_i * x \in G \times_L \sum_{\text{reg}}$ and $g_i x g_i^{-1} \in U$ then $g_i x g_i^{-1} \in sY_i$. By property (c) of 8.2.1 we have $g_i x_s g_i^{-1} \in sZ(L_i)^\circ$, hence $g_i x g_i^{-1} \in s \sum_i$. It follows from the definitions that $s \sum_i \cap g_i \sum_{\text{reg}} g_i^{-1} \subset s \sum_{i,\text{reg}}$. This implies (i).

Let $K = I(\bar{\sum}, \mathcal{L})$ be as in 8.2.3. Then $K_i = I(\bar{\sum}_i, \mathcal{L}_i)$, where $\mathcal{L}_i$ is the pull-back of $\mathcal{L}$ under the map $x \mapsto g_i^{-1} s x g_i$ of $\sum_i$ to $\sum$. We have $\tilde{K} = I(X, \tilde{\mathcal{L}})$ where $\tilde{\mathcal{L}}$ is a local system on $\tilde{Y}$ (see 7.2.1) and $\operatorname{ind}_P^G K = I(\bar{Y}, \gamma_* \tilde{\mathcal{L}})$ by 7.2.2. Similarly, $\operatorname{ind}_{P_i}^H K_i = I(\bar{Y}_i, (\gamma_i^* \tilde{K}_i)$, where $\gamma_i : H \times_{L_i} (\sum_i)_{\text{reg}} \to Y_i$.

From the lemmas just proved and the proof of 8.2.3 we now obtain

8.2.7. **Proposition.** $s^*(\gamma_* \tilde{\mathcal{L}} \mid_{Y \cap U}) \simeq \bigoplus_{i=1}^{h} ((\gamma_i)_* \tilde{\mathcal{L}}_i) \mid_{s^{-1}(Y \cap U)}$.

Notice that $(\gamma_i)_* \tilde{\mathcal{L}}_i$ is a local system on Y_i. By its restriction to $s^{-1}(Y \cap U)$ we understand its restriction to $Y_i \cap s^{-1}(Y \cap U)$, extended by zero on the other components of $s^{-1}(Y \cap U)$. The isomorphism of the proposition can also be formulated as an isomorphism

$$s^*(\operatorname{ind}_P^G K) \mid_{Y \cap U} \simeq \bigoplus_{i=1}^{h} (\operatorname{ind}_{P_i}^H K_i) \mid_{s^{-1}(Y \cap U)} [\dim G - \dim H],$$

this is the restriction of the isomorphism of 8.2.3 to $s^{-1}(Y \cap U)$.

8.2.8. Now assume that k is an algebraic closure of the finite field F_q. With the usual notations assume $FL = L$, $F\sum = \sum$, $Fs = s$, $FU = U$ and assume given $\varphi : F^* \mathcal{L} \xrightarrow{\sim} \mathcal{L}$. The $\mathcal{L}_i$ inherit isomorphisms $\varphi_i : F^* \mathcal{L}_i \xrightarrow{\sim} \mathcal{L}_i$. It follows from the arguments used to prove 8.2.7 that the isomorphism of 8.2.7 is compatible with the actions induced by φ in the left-hand side and the set (φ_i) in the right-hand side. As in 7.2.6 we can conclude from this a similar compatibility for the isomorphism of 8.2.3. Again, the parabolic subgroup P does not enter the picture.

8.3 Applications

We give here some applications of the preceding results.

8.3.1. **Theorem.** *Let K be an admissible perverse sheaf on G. The restriction of a cohomology sheaf $H^i K (i \in \mathbb{Z})$ to a stratum $Y_{(L,\sum)}$ of the stratification of* 6.2.8 *is a local system with finite monodromy.*

Consider the diagram

$$\Sigma_{\text{reg}} \overset{\alpha_1}{\leftarrow} G \times \Sigma_{\text{reg}} \overset{\beta_1}{\to} G \times_L \Sigma_{\text{reg}} \overset{\gamma}{\to} Y$$

of 7.2.1. The restriction of H^iK to Σ_{reg} is a G- equivariant constructible sheaf. From the G-equivariance we see that

$$\alpha_1^*(H^iK\mid_{\sum_{\text{reg}}}) = \beta_1^*\gamma^*(H^iK\mid_Y).$$

Since γ is a Galois covering and β_1 a principal fibration with group L it will be sufficient to show that the restriction of H^iK to $\sum_{\text{reg}}$ (which is L-equivariant) is a local system with finite monodromy.
Consider the fibration $\varphi : \Sigma_{\text{reg}} \to \sum_s /Z(L)^\circ (x \mapsto$ coset of $x_s)$. It is compatible with the L-actions (by conjugation). Since the action on $\sum_s /Z(L)^\circ$ is transitive, it suffices to prove that the restriction of H^iK to a fiber of φ is a local system with finite monodromy.
Let s be the semi-simple part of a fixed element of $\sum_{\text{reg}}$. Put $H = Z_G(s)^\circ$, as before. Then $H \subset L$ and $Z(H)^\circ = Z(L)^\circ$. The fiber of φ over $sZ(L)^\circ$ is

$$Z = \{x \in \Sigma_{\text{reg}} \mid x_s \in sZ(L)^\circ\},$$

which is an open dense subset of

$$\begin{aligned} Z' &= \{x \in \textstyle\sum \mid x_s \in sZ(L)^\circ\} = \\ &= \{syv \mid y \in Z(L)^\circ, v \in H, v \text{ unipotent}, sv \in \textstyle\sum\}. \end{aligned}$$

By 8.2.3 there is an open neighbourhood U of s in H such that we have an isomorphism

$$s^*(K\mid_U) \simeq \bigoplus_j K_j \mid_{s^{-1}U} [\dim G - \dim H],$$

the K_j being certain admissible perverse sheaves on H, which are H- equivariant for conjugation and have a weight for $Z(H)^\circ$.
Now $s^{-1}Z'$ is a union of finitely many open sets of the form $CZ(H)^\circ$, where H is a conjugacy class for H. It follows that the restriction of H^iK_j to $s^{-1}(Z' \cap U)$ is a local system with finite monodromy. The same is then true for the restriction of H^iK to $Z \cap U$.
Replacing s by sx with $x \in Z(L)^\circ$ such that $Z_G(sx)^\circ \subset L$ we get a similar open neighbourhood of sx. Since such open sets cover Z we can conclude that the restriction of H^iK to Z is also a local system with finite monodromy.
Theorem 8.3.1 is part (a) of [CS, 14.2].

8.3.2. We next review briefly another application, given in [CS, no.8]. Assume G to be defined over the finite field F_q , with Frobenius map F.
Let L be a Levi subgroup in G and $\sum$ an isolated class in L. Assume there is a unipotent class C in L such that $\sum = CZ_G(L)^\circ$ and that $\mathcal{E}$ is a local system on C. Moreover assume that $FL = L, F\sum = \sum$ and that we are given an isomorphism $\psi : F^*\mathcal{E} \xrightarrow{\sim} \mathcal{E}$.
Suppose we can extend $\mathcal{E}$ to a local system $\mathcal{L}$ on $\sum$ and ψ to an isomorphism $\varphi : F^*\mathcal{L} \xrightarrow{\sim} \mathcal{L}$, such that $K = I(\bar{\sum}, \mathcal{L})$ is an irreducible cuspidal perverse sheaf on L. Denote by $K_1 = \text{ind}_P^G K$ the induced perverse sheaf on G. We have an isomorphism $\varphi : F^*K_1 \xrightarrow{\sim} K_1$ (as in 7.2.7).
We define a function $Q = Q_{G,L,C,\mathcal{E},\psi}$ on the set of unipotent elements of G^F by

$$Q(u) = \sum_i (-1)^i \text{Tr}\,(\varphi, (H^iK_1)_u).$$

This definition makes sense, because it is independent of the choice of the extension $(\mathcal{L}, \varphi)$ of $(\mathcal{E}, \psi)$ ([CS, 8.3.2.], this fact follows by observing that the restriction of the perverse sheaf $\delta_* \tilde{K}$ of the proof of 7.2.2 to the set of unipotent elements of G depends only on $\mathcal{E}$, and not on the extension $\mathcal{L}$).

Let $\sum_s$ be the set of semi-simple parts of elements of $\sum$. If $x, s \in G$ and $x^{-1}sx \in \sum_s$ then $L_x = xLx^{-1} \cap Z_G(s)^\circ$ is a Levi group in the reductive group $Z_G(s)^\circ$. Let C_x be the set of unipotent elements v in $Z_G(s)^\circ$ with $sv \in x \sum x^{-1}$. Let $\mathcal{E}_x$ be the pull-back of $\mathcal{L}$ under the map $C_x \to \sum$ sending v to $x^{-1}svx$. By 8.1.4 , C_x is a unipotent class in L_x.
If $x, s \in G^F$ then $FL_x = L_x$ and we have an induced isomorphism $\psi_x : F^* \mathcal{E}_x \xrightarrow{\sim} \mathcal{E}_x$.
8.3.3. **Theorem.** *With the previous notations assume that su is the Jordan decomposition of an element of G^F. We then have, $\chi_{K_1,\varphi}$ denoting the characteristic function of* 1.3.4

$$\chi_{K_1,\varphi}(su) = \sum_{x \in G^F, x^{-1}sx \in \sum_s} | L^F |^{-1} | Z_G(s)^{\circ,F} |^{-1} | L_x^F | \, Q_{Z_G(s)^\circ, L_x, C_x, \mathcal{E}_x, \psi_x}(u).$$

The proof is a fairly straightforward consequence of the results of 8.2 (see [CS, 8.8]).
The functions Q introduced above are the *generalized Green functions*. They are further studied in [CS, no.9], where among other things orthogonality relations are proved for generalized Green functions. These are also discussed in Shoji's contribution to this volume [Sh].

9 Restriction and induction of character sheaves

9.1 Preliminaries

We use the notations of 3.1 and 3.2.

9.1.1. Let I be a subset of the generating set S of the Weyl group W and denote by W_I the subgroup of W generated by I. There is a unique parabolic subgroup $P = P_I$ of our connected reductive group G containing the Borel group B such that, L denoting the Levi group of P containing T, the Weyl group of (L, T) is W_I. As before $\pi = \pi_P$ is the canonical map $P \to L$.

We write $B_I = B \cap L = \pi B$, a Borel group of L. Denote by R_I the root system of (L, T) (which may be empty). Then B_I defines a system of positive roots R_I^+ in R_I, with basis D_I and $I = \{s_\alpha \mid \alpha \in D_I\}$.

The group W_I is an instance of a group W_ξ considered in 3.2. So we have the results of that section. However, here we need left cosets $W_I w$, instead of right ones.

Let $W^* = W_I^*$ be the set of minimal length left coset representatives (see 3.2.2). If $v \in W^*$ put $O_v = PvB$. Then G is the disjoint union of the $O_v (v \in W^*)$. We have

$$O_v = \coprod_{w \in W_I} G_{wv}.$$

Also, if $v \in W^*$ then $\pi(\dot{v}B\dot{v}^{-1} \cap P) = B_I$ ($\dot{v}$ denoting a representative as usual).

9.1.2. Let $\mathcal{O}$ be a W-orbit in $\hat{X}$ and denote by $\mathcal{K} = \mathcal{K}_{\mathcal{O}}$ the algebra introduced in 3.3. Let $\mathcal{O} = \coprod \mathcal{O}_i$ be the decomposition of $\mathcal{O}$ into W_I-orbits. We denote by $\mathcal{K}_i$ the algebra $\mathcal{K}_{\mathcal{O}_i}$, relative to W_I. Let $\mathcal{K}_I$ be the direct sum of the algebras $\mathcal{K}_i$. We shall view $\mathcal{K}_I$ as the subalgebra of $\mathcal{K}$ with basis $(e_{\xi,w})_{\xi \in \mathcal{O}, w \in W_I}$.

9.1.3. **Lemma.** *$\mathcal{K}$ is a free left $\mathcal{K}_I$-module, with basis*

$$u_v = \sum_{\xi \in \mathcal{O}} e_{\xi,v} \quad (v \in W_I^*).$$

This follows from property (e) of 3.3.1.

9.1.4. If $u \in \mathcal{K}$ we define $\mathrm{tr}_I u \in \mathcal{K}_I$ to be the trace of right multiplication by u in $\mathcal{K}$, viewed as a left $\mathcal{K}_I$-module, with respect to the basis (u_v). So, if

$$u_v u = \sum_{x \in W^*} a_{xv} u_x \quad (v \in W^*),$$

then

$$\mathrm{tr}_I u = \sum_{v \in W^*} a_{vv}.$$

This definition depends on a choice of basis.

Now let f be a $\mathbb{Z}[t, t^{-1}]$-linear function on $\mathcal{K}$, with values in some $\mathbb{Z}[t, t^{-1}]$-module, such that $f(uu') = f(u'u)$ for all $u, u' \in \mathcal{K}$. Then $f(\mathrm{tr}_I u)$ is independent of a choice of basis, as follows by familiar argumencs.

In this situation, we shall need the following result (in which the bar automorphism of $\mathcal{K}$ is as in 3.3).

9.1.5. **Lemma.** *For all $u \in \mathcal{K}$ we have $f(\mathrm{tr}_I \bar{u}) = f(\overline{\mathrm{tr}_I u})$.*

The lemma follows readily from the definitions, observing that $\mathcal{K}_I$ is stable under the bar automorphism of $\mathcal{K}$.

9.2 Restriction of $C_{\xi,w}$

9.2.1. Let $i : P \to G$ be the inclusion map. Recall that the restriction functor res $= \mathrm{res}_P^G$: $\mathcal{D}G \to \mathcal{D}L$ is defined by res $= \pi_! i^*$ (with a Tate twist $(\dim U(P))$ if this makes sense), see no.6.

Fix $\xi \in \hat{X}, w \in W_\xi^!$ and let $C_{\xi,w} = C_{\xi,w}^G$ be as in 5.1. The following result, which is the main one of no.9, allows one to deal with parabolic restriction of character sheaves.

Let $\mathcal{O}$ be the orbit $W\xi$. We use the function $\tau = \tau^G$ on the algebra $\mathcal{K}$ defined in 5.1.6 and its analog τ^L for $\mathcal{K}_I$. Notice that τ^L is the restriction of τ^G to $\mathcal{K}_I$. The maps χ^G and χ^L are as in 5.1.6.

9.2.2. **Theorem.**

(i) $\mathrm{res} C_{\xi,w}^G$ *is a semi-simple complex on* L*, of the form*

$$\bigoplus_{\eta \in \mathcal{O}, x \in (W_I)_\eta^!} C_{\eta,x}^L[n_{\eta,x}];$$

(ii) $\chi^L(\mathrm{res} C_{\xi,w}^G) = \tau^L(\mathrm{tr}_I c_{\xi,w})$.

9.2.3. To prove the theorem we shall establish results like (i) and (ii) for the complex $C_{\xi,\mathbf{s}}$ of 5.1.3, where $\mathbf{s} = (s_1, \ldots, s_r)$ is a reduced decomposition of w. The analogue of (ii) for $C_{\xi,\mathbf{s}}$ is

(1) $$\chi^L(\mathrm{res} C_{\xi,\mathbf{s}}) = \tau^L\left(\mathrm{tr}_I(c_{s_2\ldots s_r\xi,s_1} \cdots c_{s_r\xi,s_{r-1}} c_{\xi,s_r})\right).$$

The statements of the theorem follow from their analogues for $C_{\xi,\mathbf{s}}$, by arguments similar to those of the proof of 5.1.4.

We first discuss some auxiliary results. Fix $v_1, v_2 \in W^*$ and $s = s_\alpha \in S$. Put

$$Z = \bar{G}_s \cap \dot{v}_1^{-1} P \dot{v}_2 \ , \ Z_s = Z \cap G_s, \ Z_e = Z \cap G_e.$$

Then Z is a closed subset of $\bar{G}_s$. Define $\varphi : Z \to L$ by

$$\varphi g = \pi(\dot{v}_1 g \dot{v}_2^{-1}),$$

and denote by φ_s, φ_e the restrictions of φ to Z_s resp. Z_e. Since $\pi(\dot{v}_i B \dot{v}_i^{-1} \cap P) = B_I$ ($i = 1, 2$, see 9.1.1), the image of φ is a union of double cosets $B_I x B_I$.

On G_s resp. G_e we have the local systems $\mathcal{L}_{\xi,s} = \mathcal{L}_{\xi,s}^G$ and $\mathcal{L}_{\xi,e} = \mathcal{L}_{\xi,e}^G$ of 4.1.1, where $\xi \in \hat{X}$.

9.2.4. **Lemma.** *Assume* $Z \neq \emptyset$. *Then* $v_1 = v_2$ *or* $v_1 s = v_2$. *We have several cases :*

(a) $v_1 s \notin W^*$. *Then* $v_1 = v_2 < v_1 s$ *and* $\sigma = v_1 s v_1^{-1} \in I$. *We have* $\varphi Z = \bar{L}_\sigma$, $\varphi_s Z_s = L_\sigma$ *and* $\varphi_s^* \mathcal{L}_{\xi,\sigma}^L = \mathcal{L}_{v_1^{-1}\xi,s}^G |_{Z_s}$;

(b) $v_1 s \in W^*$ *and* $\varphi Z = L_e (= B_I)$, *with the following subcases.*

(bi) $v_1 = v_2 < v_1 s$. *Then* $Z = Z_e$ *and* $\varphi_e^* \mathcal{L}_{\xi,e}^L = \mathcal{L}_{v_1^{-1}\xi,e}^G |_{Z_e}$;

(bii) $v_1 = v_2 > v_1 s$. *Then* $\varphi_s Z_s = L_e$ *and* $\varphi_s^* \mathcal{L}_{\xi,e}^L \simeq \mathcal{L}_{v_1^{-1}\xi,s}^G |_{Z_s}$ *if* $s \in W_{v_1^{-1}\xi}$;

(biii) $v_1 s = v_2$. *Then* $\varphi_s Z_s = L_e$ *and* $\varphi_s^* \mathcal{L}_{\xi,e}^L = \mathcal{L}_{v_2^{-1}\xi,s}^G |_{Z_s}$.

If $Z \neq \emptyset$ there exists $g \in P$ with $g\dot{v}_2 \in \dot{v}_1 \bar{G}_s$. Assume $g \in G_x$, where $x \in W_I$. Then $g\dot{v}_2 \in G_{xv_2}$. Moreover, $\dot{v}_1 \bar{G}_s \subset G_{v_1} \coprod G_{v_1 s}$. It follows that $xv_2 = v_1$ or $xv_2 = v_1 s$.

If $x \neq e$ we must have $v_1 = v_2$ and $x \in I$ (see 3.2.7, so we have case(a)). It is not hard to see that now $\varphi Z = \bar{L}_x, \varphi_s Z_s = L_x$.

Next assume we are in case (b), then $x = e$ and $\varphi Z = L_e$. If $v_1 < v_1 s$ we have $G_{v_1} G_s = G_{v_1 s}$. This implies that in the case that $v_1 = v_2$ we must have $Z = Z_e$ (if $v_1 s = v_2$ then $Z = Z_s$).

It remains to prove the assertions about inverse images of local systems. In case (a) we have a diagram of morphisms (for a suitable choice of representatives $\dot v_1, \dot v_2, \dot s$)

$$\begin{array}{ccccc} G_s & \overset{i}{\leftarrow} & Z_s & \overset{\varphi_s}{\to} & L_\sigma \\ & \searrow^{\mathrm{pr}^G} & & & \downarrow \mathrm{pr}^L \\ & & T & \overset{v_1}{\to} & T \end{array},$$

where pr^G and pr^L are as in 4.1.1 for G resp. L, i is inclusion and the lower horizontal map is given by the Weyl group action, such that

$$\mathrm{pr}^L \circ \varphi_s = v_1 \circ \mathrm{pr}^G \circ i.$$

It follows that $\varphi_s^* \mathcal{L}^L_{\xi,\sigma} = \mathcal{L}^G_{v_1^{-1}\xi,s}$. In the cases (bi) and (biii) a similar argument gives the asserted result.
Now consider case (bii). The first assertion of (b i), applied for $v_1 s$ shows that

$$\dot s^{-1} Z \dot s = B \cap \dot s \dot v_1^{-1} B (\dot v_1 \dot s),$$

whence

$$Z = \dot s B \dot s^{-1} \cap \dot v_1^{-1} B (\dot v_1).$$

Also, $Z_e = B \cap \dot v_1^{-1} B \dot v_1$.
We denote by α the simple root of R such that $s = s_\alpha$. Also denote by $X_{-\alpha}$ the one parameter subgroup defined by $-\alpha$ (see 4.1.1) and by $\alpha^\vee$ the coroot defined by α, a homomorphism $k^* \to T$.
The unipotent radical of B is denoted by U. We put $U' = U \cap \dot v_1^{-1} U \dot v_1$. We then have

$$Z_s = T.(X_{-\alpha} - \{e\}).U'.$$

More precisely, the product map

$$T \times (X_{-\alpha} - \{e\}) \times U' \to Z_s$$

is an isomorphism of varieties.
It follows from familiar properties (see for example [Sp1, p.238]) that there is an isomorphism $\varphi : X_{-\alpha} - \{e\} \tilde{\to} k^*$ such that (for suitable $\dot s$)

$$\mathrm{pr}^G u = \alpha^\vee(\varphi u) \ \ (u \in X_{-\alpha} - \{e\}).$$

Hence $\psi : tuu' \mapsto \varphi u \ \ (t \in T, u \in X_{-\alpha} - \{e\}, u' \in U')$ defines a morphism $Z_s \to k^*$.
As before, we have a diagram of morphisms, which is now

$$\begin{array}{ccccc} G_s & \overset{i}{\leftarrow} & Z_s & \overset{\varphi_s}{\to} & L_e \\ & \searrow^{\mathrm{pr}^G} & & & \downarrow \mathrm{pr}^L \\ & & T & \overset{v_1 s}{\to} & T \end{array}.$$

If $g = tuu' \in Z_s$ then $\mathrm{pr}^L(\varphi_s g) = v_1(t)$ and $\mathrm{pr}^G(ig) = s(t)\alpha^\vee(\psi g)$. This can be reformulated as follows. Let $\mu : T \times k^* \to T$ be defined by $\mu(t,a) = t.v_1 s(\alpha^\vee(a))$ and $\rho : Z_s \to T \times k^*$ by $\rho = (\mathrm{pr}^L \circ \varphi_s, \psi)$. Then

$$(v_1 s) \circ \mathrm{pr}^G \circ i = \mu \circ \rho.$$

It follows that

$$\mathcal{L}^G_{v_1^{-1}\xi,s}\,|_{Z_s} = \mathcal{L}^G_{s^{-1}v_1^{-1}\xi,s}\,|_{Z_s} = (\mu \circ \rho)^* \mathcal{L}_\xi,$$

where $\mathcal{L}_\xi$ is as in 3.1. Since $s \in W_{v_1^{-1}\xi}$, i.e. $v_1 s \in W_\xi$, we have $((v_1 s) \circ \alpha^\vee)^*$ $\mathcal{L}_\xi \cong E$, the constant sheaf, and

$$(\mu \circ \rho)^* \mathcal{L}_\xi = \rho^*(\mathcal{L}_\xi \boxtimes E) = (\rho^* \circ \mathrm{pr}_1^*)(\mathcal{L}_\xi),$$

where $\mathrm{pr}_1 : T \times k^* \to T$ is projection. Since $\mathrm{pr}_1 \circ \rho = \mathrm{pr}^L \circ \varphi_s$, we obtain the asserted relation $\varphi_s^* \mathcal{L}^L_{\xi,e} = \mathcal{L}^G_{v_1^{-1}\xi,s}\,|_{Z_s}$.

With the notations of 9.2.4, we have the following complementary result. The straightforward proof is omitted.

9.2.5. **Lemma.** *The morphism $\varphi : Z \to \overline{L}_\sigma$ (case (a)) resp. $\varphi : Z \to L_e$ (case (b)) is a locally trivial fibration by affine spaces.*

9.2.6. Let $\mathbf{s} = (s_1, \ldots, s_r)$ be as in 9.2.3. We shall need the variety $\bar{Y}^G_{\mathbf{s}} = \bar{Y}_{\mathbf{s}} = G \times_B (\bar{G}_{s_1} \overset{B}{\times} \ldots \overset{B}{\times} \bar{G}_{s_r})$ introduced in 5.1.3. The image of $\bar{Y}_{\mathbf{s}}$ of an element $(g_0, \ldots, g_r) \in G^{r+1}$ will be denoted by $(g_0, \ldots, g_r)^*$.
Let $\mathbf{v} = (v_0, \ldots, v_r)$ be a sequence of elements in W^*, with $v_0 = v_r$. Define a locally closed subvariety $Z_{\mathbf{v}}$ of $\bar{Y}_{\mathbf{s}}$ by

$$Z_{\mathbf{v}} = \{(g_0, \ldots, g_r)^* \in \bar{Y}_{\mathbf{s}} \mid g_0 \ldots g_i \in O_{v_i} \text{ for } i \in [0, r] \text{ and } g_0 g_1 \ldots g_r g_0^{-1} \in P\}.$$

It follows from 9.2.4 that $Z_{\mathbf{v}} = \emptyset$ unless $v_{i-1}^{-1} v_i \in \{e, s_i\}$ for $i \in [1, r]$. For such i define $\sigma_i \in W_I$ by $\sigma_i = e$ unless $v_{i-1} = v_i$ and $v_{i-1} s_i v_{i-1}^{-1} \in W_I$, in which case we put $\sigma_i = v_{i-1} s_i v_{i-1}^{-1}$.
Let $\mathbf{t} = (\sigma_1, \ldots, \sigma_r)$, a sequence in $I \cup \{e\}$.
If $(g_0, \ldots, g_r)^* \in Z_{\mathbf{v}}$ choose $p_i \in P$ with

$$g_0 \ldots g_i \in p_i v_i B \; (0 \le i \le r), p_r p_0^{-1} = g_0 g_1 \ldots g_r g_0^{-1}.$$

Put $\ell_0 = \pi p_0, \ell_i = \pi(p_{i-1}^{-1} p_i)$ $(1 \le i \le r)$. It follows from 9.2.4 that $\ell_i \in \bar{L}_{\sigma_i}$. We define a morphism

$$\rho : Z_{\mathbf{v}} \to \bar{Y}^L_{\mathbf{t}} \text{ by } \rho(g_0, \ldots, g_r)^* = (\ell_0, \ldots, \ell_r)^*.$$

9.2.7. **Lemma.** *ρ is a locally trivial fibration. The fibres are affine spaces of dimension*

$$d(\mathbf{v}) = \dim U(P) + \mathrm{card}\{i \in [1, r] \mid v_i s_i \in W^* \text{ and } v_i s_i < v_i\}.$$

This is a consequence of 9.2.5.
Now consider the perverse sheaf $\tilde{A}_{\xi,\mathbf{s}}$ of 5.1.3, which we identify with the restriction to its support $\bar{Y}_{\mathbf{s}}$. Denote the restriction to $Z_{\mathbf{v}}$ by $A_{\mathbf{v}}$. Put

$$I_{\mathbf{s}} = \{i \in [1, r] \mid s_r \ldots s_{i+1} s_i s_{i+1} \ldots s_r \in W_\xi\}$$

(see 3.2) and

$$J_{\mathbf{v}} = J = \{i \in [1, r] \mid v_{i-1} = v_i \ , \ v_{i-1} s_i \in W^*\}.$$

9.2.8. **Lemma.** *If $J \not\subset I_{\mathbf{s}}$ then $\rho_! A_{\mathbf{v}} = 0$.*
Let $i \in J - I_{\mathbf{s}}$ and fix $z \in Z_{\mathbf{v}}$.

(a) $v_{i-1} < v_{i-1}s_i$. For all points $(g_0, \dots, g_r)^*$ in the fiber $F_z = \rho^{-1}(\rho z)$ we have $g_i \in B$ (9.2.4). Since $i \notin I_s$ the complex A_v is zero in $(g_0, \dots, g_r)^*$, by 4.2.2 (ii). Hence $H^{\cdot}(\rho_! A_v)_z = H_c^{\cdot}(F_z, A_v) = 0$.
(b) $v_{i-1}s_i < v_{i-1}$. For all $(g_0, \dots, g_r)^* \in F_z$ we have that either $g_i \in B$, in which case again A_v is zero in z or $g_i \in G_{s_i}$ (9.2.4). In the latter case it follows that $H_c^{\cdot}(F_z, A_v)$ decomposes into a sum of cohomology spaces of suitable varieties, each of which has a tensor factor of the form $H_c^{\cdot}(S, \mathcal{L})$, where S is a torus and $\mathcal{L}$ a non- constant Kummer local system on S. We conclude from 2.1.5 that $H_c^{\cdot}(F_z, A_v) = 0$. The lemma follows.

Next let $J \subset I_s$. Then $v_{i-1}^{-1}\sigma_i v_i = s_i$ if $i \notin J$ and $v_{i-1}^{-1}\sigma_i v_i = e$ if $i \in J$. We conclude, using 3.2.8, that $v_0^{-1}\sigma_1 \dots \sigma_r v_0 \in W'_\xi$ or $\sigma_1 \dots \sigma_r \in W'_{v_0\xi}$. On $\bar{Y}_t^L$ we then have the perverse sheaf $A = \tilde{A}^L_{v_0\xi,\sigma}$ (as in 5.1.3).
9.2.9. **Lemma.** *Let $J \subset I_s$.*
(i) $A_v = \rho^* A[\dim \bar{Y}_s^G - \dim \bar{Y}_t^L]$
(ii) $(\gamma_t^L)_! \rho_! A_v = C^L_{v_0\xi,t}[\dim \bar{Y}_s^G - \dim \bar{Y}_t^L - 2d(v)]$.
Here γ_t^L is as in 5.1.3. The assertion (ii) follows from (i) and the definiton of $C^L_{v_0\xi,t}$ (5.1.3), taking into account that ρ is a fibering by affine spaces of dimension $d(v)$ (which implies that $\rho_! \rho^* = \text{id}\,[-2d(v)]$. So it suffices to prove (i).
Put $H_0 = \{i \in [1,r] \mid \sigma_i \neq e\}$. then $H_0 \cap J = \emptyset$. If $H \subset H_0$ denote by $Z'_H \subset \bar{Y}_t^L$ the locally closed subvariety of the points $(\ell_0, \dots, \ell_r)^*$ with $\ell_i \in L_{\sigma_i}$ for $i \in H_0 - H$, $\ell_i \in L_e$ for $i \in H$. The $Z'_H (H \subset H_0)$ form a partition of $\bar{Y}_t^L$. Put $Z_{v,H} = \rho^{-1}(Z'_H)$.
Put

$$I_t = \{i \in [1,r] \mid \sigma_i \neq e \text{ and } \sigma_r \dots \sigma_{i+1}\sigma_i\sigma_{i+1} \dots s_r \in W_{v_0\xi}\},$$

then $I_t = I_s \cap H_0$.
The restriction of A to $\cup_{H \subset I_t} Z'_H$ is a shifted local system of rank one, as follows from the definition of the complexes like A in 5.1.3 and from 4.2.2. Moreover, the restriction of A to the complement of $\cup_{H \subset I_t} Z'_H$ is zero (see 4.2.2). If $K \subset [1,r]$ denote by S_K the sequence obtained from s by replacing by e all s_i with $i \in K$ (as in 3.2).
Put

$$J_1 = \{i \in J \mid v_{i-1}s_i > v_{i-1}\}.$$

Then

$$Z_{v,H} = \bigcup_{H \cup J_1 \subset K \subset H \cup J} (Y_{s_K} \cap Z_v),$$

with Y_{s_K} as in 5.1.3.
The subset $\bigcup_{H \subset I_t} Z_{v,H}$ of Z_v is open and smooth. The restriction of A_v to it is a shifted local system of rank one and the restriction of A_v to the complement is zero. It suffices to prove the assertions of (i) for the restrictions of the complexes in question to the dense open subset $Y_{s_{J_1}} \cap Z_v$ of $\bigcup_{H \subset I_t} Z_{v,H}$.
We shall apply 9.2.4. Put $s'_i = s_i$ if $i \notin J_1$ and $s'_i = e$ if $i \in J_1$. Let $Z_i = G_{s'_i} \cap \dot{v}_{i-1}^{-1} P \dot{v}_i$ and let $\varphi_i : Z_i \to L_{\sigma_i}$ be as in 9.2.4 $(1 \leq i \leq r)$. We have a commutative diagram

$$\begin{array}{ccc} P \times Z_1 \times \dots \times Z_r & \xrightarrow{\varphi} & L \times L_{\sigma_1} \times \dots \times L_{\sigma_r} \\ \psi \downarrow & & \downarrow \\ Z_v \cap Y_{s_{J_1}} & \xrightarrow{\rho} & Y_t^L \end{array},$$

where $\varphi = \pi_P \times \varphi_1 \times \ldots \times \varphi_r$ and $\psi(p, g_1, \ldots, g_r) = (pv_0, g_1, \ldots, g_r)^*$. By 9.2.4 we have that

$$\varphi_i^* \mathcal{L}^L_{\sigma_{i+1}\ldots\sigma_r v_0 \xi, \sigma_i} \simeq \mathcal{L}^G_{s_{i+1}\ldots s_r \xi, s_i'} \ (1 \leq i \leq r).$$

Here we use that $J_1 \cup J_2 \subset I_{\mathbf{s}}$, which implies in particular that $v_{i-1}s_i \ldots s_r\xi = \sigma_i \ldots \sigma_r v_0 \xi$. Also notice that $s_{i+1} \ldots s_r \xi = s'_{i+1} \ldots s'_r \xi$. Hence

$$\varphi^*(E \boxtimes \mathcal{L}^L_{\sigma_2\ldots\sigma_r v_0\xi,\sigma_1} \boxtimes \ldots \boxtimes \mathcal{L}^L_{v_0\xi,\sigma_r}) \simeq E \boxtimes \mathcal{L}^G_{s_2'\ldots s_r'\xi,s_1'} \boxtimes \ldots \boxtimes \mathcal{L}^G_{\xi,s_r'}.$$

The asserted isomorphism between local systems on $Z_{\mathbf{v}} \cap Y_{\mathbf{s}_{J_1}}$ follows.

9.2.10. Lemma. *$\mathrm{res}C^G_{\xi,\mathbf{s}}$ is a semi-simple complex, isomorphic to*

$$\bigoplus_{J_{\mathbf{v}} \subset I_{\mathbf{s}}} C^L_{v_0\xi,\sigma}[\dim \bar{Y}^G_{\mathbf{s}} - \dim \bar{Y}^L_{\mathbf{t}} - 2d(\mathbf{v})],$$

the sum being taken over sequences $\mathbf{v}$ *as in* 9.2.6.
The sets $J_{\mathbf{v}}$ and $I_{\mathbf{s}}$ where defined above.
We have a commutative diagram of morphisms

$$\begin{array}{ccccc} & & Z & \xrightarrow{i'} & \bar{Y}_{\mathbf{s}} \\ & \swarrow^{\rho} & \downarrow & & \downarrow \gamma_{\mathbf{s}} \\ L & \xleftarrow{\pi} & P & \xrightarrow{i} & G \end{array},$$

where $Z = (\gamma_{\mathbf{s}})^{-1}P$. The definition of the restriction functor shows that

$$\mathrm{res}C_{\xi,\mathbf{s}} = \mathrm{res}(\gamma_{\mathbf{s}})_! \tilde{A}_{\xi,\mathbf{s}} = \rho_!(\tilde{A}_{\xi,\mathbf{s}} \mid_Z).$$

By a reduction argument as in [BBD, no.6] it suffices to prove the proposition when k is the algebraic closure of a finite field F_q and everything is defined over F_q. We shall apply the lemma of 1.3.3 to the variety Z, which is the disjoint union of the $Z_{\mathbf{v}}$ of 9.2.6. We put $Z_a = \coprod_{\dim Z_{\mathbf{v}} \leq a} Z_{\mathbf{v}}$. The complex in question will be $\tilde{A}_{\xi,\mathbf{s}} \mid_Z$. Its restriction to $Z_{\mathbf{v}}$ is $A_{\mathbf{v}}$, as before. Since complexes like $C_{\xi,w}$ are pure (see 5.3.1) we conclude that the condition of 1.3.4 is satisfied (taking into account the appropriate Tate twist in 9.2.9 (ii)). The lemma follows.

9.2.11. We already noticed that 9.2.10 implies part (i) of 9.2.2. It will also imply part (ii), as we shall establish now.

If $\mathbf{v}$ is as before we denote by $n_+(\mathbf{v})(n_-(\mathbf{v}))$ the number of $i \in [1, r]$ with $v_{i-1}s_i \in W^*$ and $v_{i-1}s_i > v_{i-1}$ (resp. $v_{i-1}s_i < v_{i-1}$). By $n_{+,\xi}(\mathbf{v})(n_{-,\xi}(\mathbf{v}))$ we denote the similar number of $i \in I_{\mathbf{s}}$ (i.e. such that $s_r \ldots s_{i+1}s_is_{i+1} \ldots s_r \in W_\xi$).

It follows from 9.2.10 that

$$(2) \qquad \chi^L(\mathrm{res}C^G_{\xi,\mathbf{s}}) = \sum_{J_{\mathbf{v}} \subset I_{\mathbf{s}}} t^{-n_+(\mathbf{v})+n_-(\mathbf{v})} \tau^L(c_{\sigma_2\ldots\sigma_r v_0\xi,\sigma_1} \ldots c_{v_0\xi,\sigma_r}).$$

Recall the basis elements u_v ($v \in W^*$) of 9.1.3. With the notations of 3.3 it follows from the formulas established there that if $v \in W^*, s \in S$

$$\begin{aligned} u_v c_{\xi,s} &= c_{vs\xi,e} u_{vs} && \text{if} \quad vs \in W^*, s \notin W_\xi, \\ &= t^{-1} c_{v\xi,e}(u_v + u_{vs}) && \text{if} \quad vs \in W^*, s \in W_\xi, vs > v, \\ &= t c_{v\xi,e}(u_v + u_{vs}) && \text{if} \quad vs \in W^*, s \in W_\xi, vs < v, \\ &= c_{v\xi,\sigma} u_v && \text{if} \quad vsv^{-1} = \sigma \in I. \end{aligned}$$

It follows from these formulas that

$$(3)\qquad \mathrm{tr}_I(c_{s_2\dots s_r\xi,s_1}\dots c_{\xi,s_r}) = \sum_{J_{\mathbf{v}}\subset I_{\mathbf{s}}} t^{-n_{+,\xi}(\mathbf{v})+n_{-,\xi}(\mathbf{v})} c_{\sigma_2\dots\sigma_r v_0\xi,\sigma_1}\dots c_{v_0\xi,\sigma_r}.$$

Formula (1), which is equivalent to the assertion of 9.2.2 (ii), will follow if we show that for all $\mathbf{v}$ with $J_{\mathbf{v}} \subset I_{\mathbf{s}}$ the exponents of t in the right-hand sides of (2) and (3) are equal. This will follow from a combinatorial result.

9.2.12. Let $\mathbf{s} = (s_1,\dots,s_r)$ be a sequence in S and $\mathbf{v} = (v_0,\dots,v_r)$ a sequence in W^* such that for $i \in [1,r]$ we have $v_{i-1}^{-1}v_i \in \{e, s_i\}$ and that $v_r = v_0$.

Put $t_i = s_r \dots s_{i+1}s_i s_{i+1}\dots s_r$ and let

$$\begin{aligned} I_{\mathbf{s}} &= \{i \in [1,r] \mid t_i \in W_\xi\},\\ J &= \{i \in [1,r] \mid v_{i-1} = v_i \text{ and } v_{i-1}s_i \in W^*\},\\ H_0 &= \{i \in [1,r] \mid v_{i-1}s_i v_{i-1}^{-1} \in I\}. \end{aligned}$$

If $i \in H_0$ we put $\sigma_i = v_{i-1}s_i v_{i-1}^{-1}$. Then $v_i = v_{i-1}$. If $i \notin H_0$ we put $\sigma_i = e$. The complement of H_0 decomposes into four subsets, described as follows.

$$\begin{aligned} J_1 &: \quad v_i = v_{i-1},\ v_{i-1}s_i > v_{i-1},\\ J_2 &: \quad v_i = v_{i-1},\ v_{i-1}s_i < v_{i-1},\\ J_3 &: \quad v_i = v_{i-1}s_i, v; v_{i-1}s_i > v_{i-1},\\ J_4 &: \quad v_i = v_{i-1}s_i,\ v_{i-1}s_i < v_{i-1}. \end{aligned}$$

Then $J = J_1 \cup J_2$. We have in the situation of 9.2.11

$$n_+(\mathbf{v}) - n_-(\mathbf{v}) = \mid J_1 \mid - \mid J_2 \mid + \mid J_3 \mid - \mid J_4 \mid,$$

$$n_{+,\xi}(\mathbf{v}) - n_{-,\xi}(\mathbf{v}) = \mid J_1 \mid - \mid J_2 \mid + \mid J_3 \cap I_{\mathbf{s}} \mid - \mid J_4 \cap I_{\mathbf{s}} \mid .$$

Since $v_r = v_0$ we have $\mid J_3 \mid = \mid J_4 \mid$. The result we need to establish the equality of (2) and (3) is the following one.

9.2.13. **Lemma.** *If $J \subset I_{\mathbf{s}}$ and $s_1\dots s_r \in W_\xi^!$ then $\mid J_3 \cap I_{\mathbf{s}} \mid = \mid J_4 \cap I_{\mathbf{s}} \mid$.*

Notice that in the situation of 9.2.11 the hypotheses of this lemma are fulfilled.

Let $i \in I_{\mathbf{s}}$ be such that $v_i = v_{i-1}$. By cancelling s_i and v_i from $\mathbf{s}$ resp. $\mathbf{v}$ we obtain sequences satisfying the same conditions (recall 3.2.8), and $\mid J_3 \cap I_{\mathbf{s}} \mid$, $\mid J_4 \cap I_{\mathbf{s}} \mid$ are not changed. So we may assume that $J = \emptyset, H_0 \cap I_{\mathbf{s}} = \emptyset$. Then $\sigma_i = e$ if $i \in J_3 \cup J_4$ and $\mathbf{t} = (\sigma_1,\dots,\sigma_r)$ is a sequence in $I \cup \{e\}$ such that $\sigma_i = v_{i-1}s_i v_i^{-1}$ for $i \in [1,r]$, whence

$$\begin{aligned} \sigma_r \dots \sigma_{i+1}\sigma_i\sigma_{i+1}\dots\sigma_r &= v_0 t_i v_0^{-1} \quad \text{if} \quad i \in H_0,\\ &= e \qquad\qquad \text{if} \quad i \notin H_0. \end{aligned}$$

Moreover $\sigma_1\dots\sigma_r = v_0 s_1 \dots s_r v_0^{-1}$. We put $w = s_1\dots s_r$, so $w \in W_\xi^!$.

Since $v_0 t_i v_0^{-1} \notin W_{v_0\xi}$ if $i \in H_0$ we have by 3.2.5 that $v_0 w v_o^{-1} \in W_{v_0\xi}^*$. Also, by 3.2.7(i), $\sigma_i \dots \sigma_r \in W_{v_0\xi}^*$ for $i \in [1,r]$.

Let α_i be the simple root such that $s_i = s_{\alpha_i}$. If $v_{i-1}s_i > v_{i-1}$ then $v_{i-1}(\alpha_i) \in R^+$ and conversely. Now $v_{i-1}(\alpha_i) = \sigma_i \dots \sigma_r v_0 s_r \dots s_i(\alpha_i)$. If $i \in I_{\mathbf{s}}$ then $v_0 s_r \dots s_i(\alpha_i) \in R_{v_0\xi}$ and since $\sigma_1 \dots \sigma_r \in W_{v_0\xi}^*$ we see that $v_{i-1}(\alpha_i)$ and $v_0 s_r \dots s_i(\alpha_i)$ have the same sign.

To prove the lemma it suffices to show:

(*) The sets of $i \in I_{\mathbf{s}}$ such that $v_0 s_r \dots s_i \alpha_i$ is positive resp. negative have the same number of elements.

We need an auxiliary result about decompositions in Weyl groups.

9.2.14. **Lemma.** *Let $\beta \in R^+$ and put $K_\beta = \{i \in [1,r] \mid t_i = s_\beta\}$.*

(i) $w\beta \in R^+$ *if and only if* K_β *has an even number of elements.*
(ii) *The sequence of roots* $s_r \dots s_i\alpha_i (i \in K_\beta)$ *is of the form*
$\pm\beta, \dots, \beta, -\beta, \beta - \beta$.
(i) follows from the results of [B, p.13-14]. To prove (ii) it suffices to consider the case that K_β has at most two elements, in which case (ii) follows from (i). (One could also use [loc.cit.,p.157]).

Fix $\beta \in R_\xi^+$. To prove (*) it suffices to prove a similar assertion for the i with $t_i = s_\beta$. Using 9.2.14 (ii) one sees that this assertion is true if K_β has an even number of elements, i.e. if $w\beta > 0$ (9.2.14(i)).
Now (*) will follow from :
(**) The set $\{v_0\beta \mid \beta \in R_\xi^+, w\beta \in -R^+\}$ has as many positive roots as negative ones.
Put $y = v_0 w v_0^{-1}, \eta = v_0\xi$, so $y \in W'_\eta \cap W^*_\eta$. Then (**) is equivalent to: there are as many positive as negative roots $\alpha \in R_\eta$ such that $v_0^{-1}\alpha$ is positive and $v_0^{-1}y\alpha$ negative. Now y permutes the roots of R_η^+. It is clear that in each orbit in R_η^+ of the group generated by y there are as many elements α with $v_0^{-1}\alpha, -v_0^{-1}y\alpha \in R^+$ as there are with $-v_0^{-1}\alpha, v_0^{-1}y\alpha \in R^+$. Hence (**) and (*) follow, and we have proved 9.2.13, and also 9.2.2.
Remark. 9.2.2 is a conjunction of [CS, 3.9] and [CS, 6.7]. The proof of the latter result is not given completely in [loc.cit]. Also, the proof of the auxiliary result [CS, 3.5] (corresponding to lemmas 9.2.8 and 9.2.9) is incomplete.
9.2.15. **Example.** Assume that $G = SL_n$ with n prime to char k and consider $C_{\xi,w}$ with w a Coxeter element and $W_\xi = e$. So $S = \{s_1, \dots, s_r\}$ and $w = s_1 \dots s_r$. We claim that now

$$c_{\xi,w} = c_{s_2 \dots, s_r\xi, s_1} \dots c_{s_r\xi, s_{r-1}} c_{\xi, s_r}.$$

In fact, since all s_i are distinct it follows that the right-hand side has the properties required in 3.3.4, all polynomials $P_{\xi x w}$ being the constant 1.
The formulas of 9.2.11 now show that $\mathrm{tr}_I c_{\xi,w} = 0$ for all proper subsets I of S. It follows from 9.2.2 (ii) and 5.4.11 that $C_{\xi,w}$ is a cuspidal character sheaf.

9.3 Induction and restriction of character sheaves

We shall first give the - much easier - counterpart of 9.2.2 for induction.
9.3.1. **Proposition.** *Let* $\xi \in \hat{X}, w \in W_I \cap W_\xi^!$. *Then* $\mathrm{ind}_P^G C_{\xi,w}^L = C_{\xi,w}^G$.
We have a diagram

$$\begin{array}{ccccccccc} L \times_{B_I} \bar{L}_w & \overset{\alpha_1}{\leftarrow} & G \times (P \times_B \bar{G}_w) & \overset{\beta_1}{\rightarrow} & G \times_B \bar{G}_w & & \\ \downarrow \gamma^L & & \downarrow \mathrm{id}\times\gamma^G & & \downarrow_\epsilon & \searrow^{\gamma^G} & \\ L & \overset{\alpha}{\leftarrow} & G \times P & \overset{\beta}{\rightarrow} & G \times_P P & \overset{\delta}{\rightarrow} & G. \end{array}$$

The lower row is the one used to define induction in 7.1.1 and γ^G is as in 5.1.1. The map α_1 is given by

$$\alpha_1(g, x * h) = \pi_P(x) * \pi_P(h),$$

where $g \in G, x \in P, h \in \bar{G}_w$ and $x * h$ denotes the image of (x, h) etc. The left-hand square is Cartesian. The map β_1 is defined by

$$\beta_1(g, x * h) = gx * h,$$

and the second square is also Cartesian. Further, ε is the canonical map. The triangle is commutative.
Let $\tilde{A}^G_{\xi,w}$ be as in 5.1.1. The projection $\pi_P : P \to L$ induces a fibration $\bar{G}_w \to \bar{L}_w$ with fibers $U(P)$, and it follows from the definitions, using 1.2.6(b) that $\pi_P^* A^L_{\xi,w}[\dim U(P)] = A^G_{\xi,w}$. This implies that

$$\alpha_1^* \tilde{A}^L_{\xi,w}[2\dim U(P)] = \beta_1^* \tilde{A}^G_{\xi,w}.$$

Now $C^G_{\xi,w} = \gamma^G_*(\tilde{A}^G_{\xi,w}) = \delta_*(\epsilon_* \tilde{A}^G_{\xi,w})$ and

$$\beta^* \epsilon_*(\tilde{A}^G_{\xi,w}) = (\mathrm{id} \times \gamma_G)_*(\beta_1^* \tilde{A}^G_{\xi,w}) =$$

$$(\mathrm{id} \times \gamma_G)_* \alpha_1^*(\tilde{A}^L_{\xi,w})[2\dim U(P)] = \alpha^*(\gamma^L_*(\tilde{A}^L_{\xi,w})[2\dim U(P)] = \alpha^* C^L_{\xi,w},$$

by the definition of $C^L_{\xi,w}$ (5.1.1). The definition of induction (7.1.1) shows that $C^G_{\xi,w} = \mathrm{ind}^G_P C^L_{\xi,w}$, as asserted.
Remark. 9.3.1. is contained in [CS, 15.7]. Our proof is somewhat different from the one of [loc.cit].
Let A be a character sheaf on G. The parabolic subgroup P is as before (9.1.1).
9.3.2. **Theorem.**
(i) *A is admissible*;
(ii) $\mathrm{res}^G_P A$ *is isomorphic to a direct sum of character sheaves.*
We may assume that P is a proper subgroup and that character sheaves on reductive groups of dimension $< \dim G$ are admissible. Admissibility was discussed in no.8.
Let $\xi \in \hat{X}, w \in W^!_\xi$ be such that A is a constituent of $C^G_{\xi,w}$. We use the notations of 5.1.6. Then

$$\chi^G(C_{\xi,w}) = \Sigma_i f_i(t)[A_i],$$

where the A_i are character sheaves on G and the f_i are Laurent polynomials with non-negative integral coefficients, such that $f_i(t) = f_i(t^{-1})$ (by 5.1.7(i)). It follows from 9.2.2(i) (using the induction assumption made above) that $\mathrm{res} A_i$ satisfies the condition of 7.2.5, hence $\mathrm{res}\, A_i \in \mathcal{D}L^{\leq 0}$.
So

$$\chi^L(\mathrm{res} A_i) = \Sigma_j g_{ij}(t)[B_j],$$

where $B_j \in \hat{L}$, the g_{ij} being polynomials in t^{-1} with non-negative integral coefficients. Hence

$$\chi^L(\mathrm{res} C_{\xi,w}) = \Sigma_j g_j(t)[B_j],$$

with

$$g_j(t) = \Sigma_i f_i(t) g_{ij}(t).$$

By 5.1.7(i), 9.1.5 and 9.2.2(ii) we know that $g_j(t) = g_j(t^{-1})$. The following elementary lemma (whose proof we leave to the reader) now shows that the g_{ij} are constant, which will establish (ii).
Lemma. *Let F_i* resp. *$G_i (i \leq i \leq a)$ be polynomials resp. Laurent polynomials with non-negative real coefficients. Put $F = \sum_i F_i G_i$. If $F(t) = F(t^{-1}), G_i(t) = G_i(t^{-1}) (1 \leq i \leq a)$ then all F_i are constant.*
We now prove (i). The statement is trivial if A is cuspidal. Otherwise there is a proper parabolic subgroup Q with Levi group M such that $\mathrm{res}^G_Q A \notin \mathcal{D}M^{<0}$ We have already seen that $\mathrm{res}^G_Q A \in \mathcal{D}M^{\leq 0}$, and we now conclude from (ii) that there exists a character sheaf

C on M with Hom $(\mathrm{res}_Q^G A, C) \neq 0$ (one can also see this without the use of (ii)). By Frobenius duality (7.1.3) we have Hom $(A, \mathrm{ind}_Q^G C) \neq 0$. By induction we know that C is admissible. Using transitivity of induction (7.1.2) we conclude that A is admissible.

We give a number of corollaries.

9.3.3. **Corollary.** *If B is a character sheaf on L then* $\mathrm{ind}_P^G B$ *is a direct sum of character sheaves.*

This follows from 9.3.2(i), 9.3.1 and the results of no.8.

9.3.4. **Corollary.** *A is cuspidal if and only if it is strongly cuspidal.*

This is immediate from the definitions, using 9.3.2(ii).

The next corollary is proved by an adaptation of the argument used to prove 9.3.2(i) (using 9.3.4).

9.3.5. **Corollary.** *There exists a parabolic subgroup Q of G with Levi group M and a strongly cuspidal character sheaf C on M such that A is a constituent of* $\mathrm{ind}_Q^G C$.

We finally record the following consequence, involving the stratification of G introduced in 6.2.8.

9.3.6. **Corollary.** *The restriction of the cohomology sheaves $H^i A$ to a stratum are locally constant with finite monodromy.*

This is a consequence of 8.3.1 and admissibility of A.

Remarks.

(a) 9.3.2(ii) and 9.3.4 are established in [CS, 6.9], 9.3.3 in [CS, 4.8(b)], 9.3.1 is contained in [CS, 7.1.14].

(b) For $k = \mathbb{C}$, a different proof of 9.3.2(ii) has been given by V. Ginzburg (private communication). It uses $\mathcal{D}$-module theory.

10 Further properties of induction and restriction, duality

Notations are as in no.9.

10.1 Mackey's formula

10.1.1. The next proposition holds in a more general situation, as the proof will show. To avoid cumbersome assumptions we formulate it only for the case of character sheaves. Let P and Q be parabolic subgroups of G with Levi groups L and M, respectively. Choose a set $\sum$ of coset representatives for $Q\backslash G/P$ such that L and all conjugates $x^{-1}Mx (x \in \sum)$ have a common maximal torus. We write ${}^xP, \dots$ for xPx^{-1} and $P^x \dots$ for $x^{-1}Px$. For any $x \in \sum$, the group ${}^xP \cap M$ is a parabolic subgroup of M with Levi group ${}^xL \cap M$ and $L \cap Q^x$ is a parabolic subgroup of L with Levi group $L \cap M^x$.

10.1.2. **Proposition.** *For any character sheaf A on L we have*

$$\mathrm{res}_Q^G(\mathrm{ind}_P^G A) = \bigoplus_{x\in\Sigma} \mathrm{ind}_{M\cap {}^xP}^M {}^x(\mathrm{res}_{L\cap Q^x}^L A)$$

Consider the diagram

$$\begin{array}{ccccccc} & & & & & & M \\ & & & & & \overset{\varphi}{\nearrow} & \uparrow \pi_Q \\ & & & & V & \xrightarrow{\delta} & Q \\ & & & & \curvearrowleft & & \curvearrowleft \\ L & \xleftarrow{\alpha} & G \times P & \xrightarrow{\beta} & G \times_P P & \xrightarrow{\delta} & G \end{array}$$

where the row is as before, with $V = \delta^{-1}Q$, and $\varphi = \pi_Q \circ \delta$.
If $A \in \hat{L}$ we have a perverse sheaf K_1 on $G \times_P P$ with

$$\alpha^* A[2 \dim U(P)] = \beta^* K_1.$$

Let K be the restriction of K_1 to V. Then

(1) $$\mathrm{res}_Q^G(\mathrm{ind}_P^G A) = \varphi_! K.$$

We have a partition $V = \coprod_{x\in\sum} V_x$ into locally closed subvarieties, with

$$V_x = \{g * h \in G \times_P P \mid g \in QxP, ghg^{-1} \in Q\}.$$

Let K_x be the restriction of K to V_x. Put $P_x^* = M \cap {}^xP$. We have the following diagram

$$\begin{array}{ccccccccc} L \cap Q^x & \xrightarrow{\sim} & {}^xL \cap Q & & & & V_x & & \\ \downarrow & & \downarrow \pi_Q & & & & \downarrow \rho & \searrow^{\varphi_x} & \\ L \cap M^x & \xrightarrow{\sim} & {}^xL \cap M & \xleftarrow{\alpha_1} & M \times P_x^* & \xrightarrow{\beta_1} & M \underset{P_x^*}{\times} P_x^* & \xrightarrow{\delta_1} & M. \end{array}$$

Here $\alpha_1, \beta_1, \delta_1$ are the analogues of α, β, δ for M and P_x^* and ρ is defined by

$$\rho(g * h) = \pi_Q(q) * \pi_Q(xh'h(h')^{-1}x^{-1}),$$

if $g * h \in V_x, g = qxh'$, where $h, h' \in P, q \in Q$. Then $\delta_1 \circ \rho$ is the restriction φ_x of φ to V_x. The map ρ is a locally trivial fibration by affine spaces of dimension $\dim U(Q)$.

Let A_x be the complex on ${}^xL \cap Q$ corresponding to the restriction of A to $L \cap Q^x$. Then

$$^x(\mathrm{res}^L_{L\cap Q^x} A) = (\pi_Q)_! A_x\ (\dim\ {}^xL \cap U(Q)).$$

By 9.3.2 (ii) this is a perverse sheaf.

10.1.3. Lemma. $\alpha_1^*((\pi_Q)_! A_x)[2\dim U(P_x^*)] = \beta_1^* \rho_! K_x$.

To prove this consider the variety $X = Q \times ({}^xP \cap Q)$ and define maps

$$\begin{array}{lllll} f_1 & : & X \to {}^xL \cap Q & , & (g,h) \mapsto {}^xL \cap Q-\text{ part of } h, \\ f_2 & : & X \to V_x & , & (g,h) \mapsto gx * x^{-1}hx, \\ f & : & X \to M \times P_x^* & , & (g,h) \mapsto (\pi_Q(g), \pi_Q(h)). \end{array}$$

We have $\pi_Q \circ f_1 = \alpha_1 \circ f$ and $\rho \circ f_2 = \beta_1 \circ f$. Obviously

$$(2) \qquad f_1^* A_x = f_2^* K_x[-\dim G + \dim L]$$

(both sides are equal to the inverse image of A under the map $X \to L$ sending (g,h) to $\pi_p(x^{-1}gx)$).

Let Y be the fibre product of ${}^xL \cap Q$ and $M \times P_x^*$ over ${}^xL \cap M$ and Z that of V_x and $M \times P_x^*$ over $M \underset{P_x^*}{\times} P_x^*$. Then f_1 and f_2 induce $\varphi_1 : X \to Y$ and $\varphi_2 : X \to Z$. We have a commutative diagram

$$\begin{array}{ccccccccc} & & & & X & & & & \\ & & & \swarrow^{\varphi_1} & & \searrow^{\varphi_2} & & & \\ {}^xL \cap Q & \overset{\mu}{\leftarrow} & Y & & & & Z & \overset{\nu}{\rightarrow} & V_x \\ \downarrow \pi_Q & & & \searrow^{\sigma} & & \swarrow_{\tau} & & & \downarrow \rho \\ {}^xL \cap M & \overset{\alpha_1}{\leftarrow} & & & M \times P_x^* & & & \overset{\beta_1}{\rightarrow} & M \underset{P_x^*}{\times} P_x^* \end{array},$$

with $\mu \circ \varphi_1 = f_1, \nu \circ \varphi_2 = f_2$. Then φ_1 and φ_2 are locally trivial fibrations by affine spaces of dimensions $d_1 = \dim U(Q) + \dim U({}^xP) \cap U(Q), d_2 = \dim\ {}^xP \cap U(Q)$, respectively. We have $(\varphi_1)_! \varphi_1^! C = C$ for any complex C on Y, and similarly for φ_2. We can write (2) as

$$\varphi_1^! \mu^* A_x[2\dim U(P) - 2d_1] = \varphi_2^! \nu^* K_x[-2d_2].$$

Since $\sigma_!(\varphi_1)_! = \tau_!(\varphi_2)_!$ we obtain

$$\sigma_! \mu^* A_x[2\dim U(P) - 2d_1] = \tau_! \nu^* K_x[-2d_2].$$

Since $\sigma_! \mu^* = \alpha_1^*(\pi_Q)_!, \tau_! \nu^* = \beta_1^* \rho_!$, we find

$$\alpha_1^*(\pi_Q)_! A_x[2\dim U(P) - 2d_1 + 2d_2] = \beta_1^* \rho_! K_x.$$

Now

$$\dim U(P) - d_1 + d_2 = \dim U(P) - \dim U(Q) + \dim\ {}^xL \cap U(Q).$$

As is well-known $({}^xP \cap Q)U({}^xP)$ is a parabolic subgroup with unipotent radical $({}^xL \cap U(Q))U({}^xP)$. Likewise, $({}^xP \cap Q)U(Q)$ is a parabolic subgroup with unipotent radical

$(M \cap U({}^xP))U(Q)$. Since both parabolic subgroups have Levi group ${}^xL \cap M$, their unipotent radicals have the same dimension, i.e.

$$\dim {}^xL \cap U(Q) + \dim U(P) = \dim M \cap U({}^xP) + \dim U(Q).$$

It follows that

$$\dim U(P) - d_1 + d_2 = \dim M \cap U({}^xP) = \dim U(P_x^*),$$

and the assertion of 10.1.3 follows.
It follows from 10.1.3. that

$$(3) \qquad (\varphi_x)_! K_x = \operatorname{ind}_{P_x^*}^M((\pi_Q)_! A_x).$$

To prove 10.1.2 we may assume that k is the algebraic closure of the finite field F_q, that all varieties in question are defined over F_q and that A is pure of weight a, say. Then $\operatorname{res}_P^G A$ is pure of weight $a - 2\dim U(P)$. Similarly, if A' is a character sheaf on L which is pure of weight a' then $\operatorname{ind}_P^G A'$ is pure of weight $a' + 2\dim U(P)$. These remarks imply that $(\varphi_x)_! K_x$ is pure of weight $a + 2\dim U(P)$, which is independent of x. We can now apply the lemma of 1.3.3, for the decomposition $V = \coprod V_x$. It follows that

$$\varphi_! K = \bigoplus_{x \in \Sigma} (\varphi_x)_! K_x.$$

(2) and (3) show that this is the equality of the proposition.
Example. Take $Q = B, M = T$. From example (b) in 7.1.1 and 10.1.2 we conclude that

$$\operatorname{res}_P^G C_{\xi,e}^G = \bigoplus_{y \in W_I^*} C_{y\xi,e}^L$$

(notations of 9.1.1).
The result of 10.1.2 is established in [CS, 15.2], under a more restrictive assumption. Our proof is different from the one of [loc.cit].

10.1.4. We now have available properties of parabolic induction and restriction similar to the ones known in the theory of finite groups of Lie type. These properties have a number of formal consequences, to be discussed presently.
We denote by $\mathbf{C}G$ the subgroup of the Grothendieck group of perverse sheaves on G spanned by the character sheaves. A semi-simple perverse sheaf on G which is a direct sum of character sheaves is completely determined by its image in $\mathbf{C}G$, we shall identify it with this image. Likewise, a semi-simple perverse sheaf on G whose irreducible constituents are character sheaves is identified with the element $\chi(A)$ of $\mathsf{Z}[t, t^{-1}] \otimes \mathbf{C}G$ which it determines according to 5.1.6. We denote by $(,)$ or $(,)_G$ the symmetric bilinear form on $\mathbf{C}G$ (or on modules like $\mathsf{Z}[t, t^{-1}] \otimes \mathbf{C}G$) for which the character sheaves form an orthonormal basis. If P is a parabolic subgroup with Levi group L it follows from 9.3.1 and 9.3.2 that we have homomorphisms $\operatorname{res}_P^G : \mathbf{C}G \to \mathbf{C}L$ and $\operatorname{ind}_P^G : \mathbf{C}L \to \mathbf{C}G$, induced by restriction and induction of perverse sheaves. Moreover, by 7.1.3 we have for $A \in \mathbf{C}G, A' \in \mathbf{C}L$

$$(4) \qquad (A, \operatorname{ind}_P^G A')_G = (\operatorname{res}_P^G A, A')_L,$$

and in the situation of 10.1.2 we have for $A \in CL, A' \in CM$

(5) $$(\mathrm{ind}_P^G A,\ \mathrm{ind}_Q^G A')_G = \sum_{x \in \Sigma} ({}^x(\mathrm{res}^L_{L \cap Q^x} A),\ \mathrm{res}^M_{M \cap {}^x P} A')_{{}^x L \cap M}.$$

We notice the following consequence of these formulas.

10.1.5. **Lemma.** *Let P' be a second parabolic subgroup with Levi group L.*

(i) *For $A \in CG$ we have* $\mathrm{res}_P^G A = \mathrm{res}_{P'}^G A$;

(ii) *For $B \in CL$ we have* $\mathrm{ind}_P^G B = \mathrm{ind}_{P'}^G B$.

Using induction on $\dim G$ one deduces from (5) that

$$(\mathrm{res}_P^G A - \mathrm{res}_{P'}^G A,\ \mathrm{res}_P^G A - \mathrm{res}_{P'}^G A) = 0,$$

whence (i). Then (ii) follows from (4).

10.2 Duality

10.2.1. If $I \subset S$ we denote by P_I the parabolic subgroup defined by I (as in 9.1.1), with Levi group L_I. We have for $I \subset J \subset S$ restriction and induction maps

$$\begin{aligned} i_I^J &= \mathrm{ind}^{L_J}_{P_I \cap L_J} &: \mathrm{C}(L_I) \to \mathrm{C}(L_J) \\ r_I^J &= \mathrm{res}^{L_J}_{P_I \cap L_J} &: \mathrm{C}(L_J) \to \mathrm{C}(L_I). \end{aligned}$$

The notation is legitimate by 10.1.5.

Let $I \subset J \subset K$. By 7.1.2. we have

(6) $$i_J^K i_I^J = i_I^K.$$

If $A \in \mathrm{C}(L_J), A' \in \mathrm{C}(L_I)$ then by (4)

(7) $$(A, i_I^J A') = (r_I^J A, A'),$$

whence

(8) $$r_I^J r_J^K = r_I^K.$$

It follows from (5) that for I, J arbitrary subsets in S

(9) $$r_J^S i_I^S = \sum_x i^J_{I^x \cap J} \circ \gamma_x \circ r^I_{I \cap {}^x J},$$

where x runs through the elements of maximal length in the cosets $W_I w W_J$ and γ_x : $\mathrm{C}(L_{I \cap {}^x J}) \xrightarrow{\sim} \mathrm{C}(L_{I^x \cap J})$ is induced by conjugation.

(notations similar to those of 10.1.4).

10.2.2. Now define $d_S = d : CG \to CG$ by

$$d = \sum_{I \subset S} (-1)^{|I|} i_I^S r_I^S.$$

This is a homomorphism analogous to the duality map of class functions on a finite group of Lie type, studied by Curtis, Alvis and Kawanaka (see for example [Ca, 8.2, p.266-278]). Our map d has properties to be stated below, analogous to the ones of the duality map. These properties are formal consequences of (6), (7), (8), (9) and the following identity: if $I, K \subset S$ then

$$\sum_{J \subset S} (-1)^{|J|} \operatorname{card} \{x \in W \mid I \cap {}^x J = K \text{ and } x \text{ is minimal in } W_I x W_J\} =$$

$$(-1)^{|K|}.$$

The properties of d are :

(a) $d_S^2 = \text{id}$,
(b) $d_S i_I^S = i_I^S d_I$, $d_I r_I^S = r_I^S d_S$,
If $A, A' \in \mathrm{C}G$ then
(c) $(dA, A') = (A, dA')$,
(d) $(dA, dA') = (A, A')$.
It follows from (d) that if $A \in \hat{G}$ (i.e. A is a character sheaf) we have $\pm dA \in \hat{G}$.
10.2.3. Proposition. *Let $A \in \hat{G}$.*
(i) *If A is cuspidal then $dA = (-1)^{|S|}A$;*
(ii) *$dA = (-1)^{\operatorname{codim} \operatorname{supp} A} A'$ where A' is a character sheaf with the same support as A.*
If A is cuspidal it is strongly cuspidal (9.3.4) so $r_I^s A = 0$ for $I \neq S$, whence (i).
Otherwise, let I be minimal such that $r_I^S A \neq 0$ and let $A' \in \hat{L}_I$ be a constituent of $r_I^S A$ (see 9.3.5). Then A is a constituent of $i_I^S A'$ by (7). By property (b)

$$d i_I^S A' = i_I^S dA' = (-1)^{|I|} i_I^S A'.$$

It follows that

$$dA = (-1)^{|I|} A_1,$$

where A_1 is a constituent of $i_I^S A'$. It follows from 7.2.2 that all such constituents have the same support, which is a variety $\bar{Y}_{(L,\sum)}$ as in 6.2.4. We have

$$\operatorname{codim} \operatorname{supp} A = \dim L - \dim \textstyle\sum \equiv |\, I \,| \mod 2,$$

the congruence following from the fact that a conjugacy class in $L/Z(L)^0$ has even dimension. (ii) is proved.
We have followed here [CS, 15.3, 15.4].

11 Further analysis of character sheaves

We keep the notations used before.

11.1 Properties of Hecke algebras

Let $\xi \in \hat{X}$ and denote by $\mathcal{H}'_\xi$ the Hecke algebra of the group W'_ξ (see 3.3.7(ii)). We review a number of results which will be needed. See also Curtis' contribution [Cu].

11.1.1. **Lemma.** *$E(t) \otimes_{\mathbf{Z}[t,t^{-1}]} \mathcal{H}'_\xi$ is a semi-simple algebra over $E(t)$.*

As usual, E is our fixed coefficient field. The lemma is well-known for ordinary Hecke algebras (see e.g. [B, p.56]). The proof of [loc.cit] carries over.

Denote by $\hat{W}'_\xi$ the set of irreducible characters of W'_ξ. We identity $\hat{W}'_\xi$ with the set of isomorphism classes of irreducible $E[W'_\xi]$-modules, or the set of classes of irreducible representations of W'_ξ.

If $\lambda \in E^*$ we define the E-algebra $\mathcal{H}'_\xi(\lambda)$ to be the quotient of $E[t,t^{-1}] \otimes \mathcal{H}'_\xi$ by the ideal generated by $(t-\lambda)$. This is the algebra obtained by *specializing* t to λ. In particular, $\mathcal{H}'_\xi(1) = E[W'_\xi]$. One knows that $\mathcal{H}'_\xi(\lambda)$ is semi-simple if λ^2 is a non-negative power of a prime number (cf. [loc.cit]).

11.1.2. **Theorem.** *To any $M \in \hat{W}'_\xi$ one can associate canonically an $E[t,t^{-1}] \otimes \mathcal{H}'_\xi$- module $M(t)$, which is free as an $E[t,t^{-1}]$-module, such that*

(a) *the $E(t) \otimes \mathcal{H}'_\xi$-modules $E(t) \otimes M(t)$ represent the isomorphism classes of absolutely irreducible $E(t) \otimes \mathcal{H}'_\xi$-modules.*

For $\lambda \in E^*$ denote by $M(\lambda)$ the $\mathcal{H}'_\xi(\lambda)$-modules $\mathcal{H}'_\xi(\lambda) \otimes_{\mathcal{H}'_\xi} M(t)$.

(b) *$M(1)$ is an irreducible $E[W'_\xi]$-module in the class M;*

(c) *If q is a non-negative power of a prime number then the modules $M(q^{\frac{1}{2}})$ represent the isomorphism classes of irreducible $\mathcal{H}'_\xi(q^{\frac{1}{2}})$-modules.*

For the case of ordinary Hecke algebras the construction of $M(t)$ was given by Lusztig. See [Cu] for more details and references. The proof carries over to the case of $\mathcal{H}'_\xi$.

Denote now by $(e_w)_{w \in W'_\xi}$ (and not by $(e_{\xi,w})$) the basis of $\mathcal{H}'_\xi$ of 3.3.7(ii). If $M \in \hat{W}'_\xi$ we denote by M^* the class of representations of W'_ξ dual to those of M. The character $w \mapsto (-1)^{\ell(w)}$ of the Weyl group W induces a character of W'_ξ, denoted by ε.

We denote by $\mathfrak{o}$ (resp. C) the ring of cyclotomic integers in E (resp. its quotient field) and by conj the automorphism of C sending each root of unity to its inverse and its extension to $C(t)$ fixing t (E is assumed to be sufficiently large, as always.)

11.1.3. **Lemma.** *Let $M \in \hat{W}'_\xi$, $w \in W'_\xi$.*

(i) $\mathrm{Tr}(e_w, M(t)) \in \mathfrak{o}[t]$;

(ii) $\mathrm{Tr}(e^{-1}_{w^{-1}}, M(t) = \overline{\mathrm{Tr}(e_w, M(t))}$;

(iii) $\mathrm{Tr}(e_{w^{-1}}, M(t)) = \mathrm{Tr}(e_w, M^*(t)) = \mathrm{Tr}(e_w, M(t))^{\mathrm{conj}}$;

(iv) $\mathrm{Tr}(e_w, \varepsilon \otimes M(t)) = \varepsilon(w) t^{2\ell_\xi(w)}\, \overline{\mathrm{Tr}(e_w, M(t))}$.

The bar denotes, as before, the automorphism of $E[t,t^{-1}]$ sending t to t^{-1} (and fixing the elements of E).

To prove (i) one shows that the elements e_w can be represented on a suitable basis of $M(t)$ by matrices with entries in $\mathfrak{o}[t]$. This follows from the construction of $M(t)$.

To prove (ii) we use the automorphism $u \mapsto \bar{u}$ of $\mathcal{H}'_\xi$ with $\bar{e}_w = e^{-1}_{w^{-1}}, \bar{t} = t^{-1}$ (see 3.3.2).

The construction of $M(t)$ gives that there exists an automorphism $m \mapsto \bar{m}$ of $M(t)$ such that for $u \in \mathcal{H}'_\xi, m \in M(t)$ we have $\overline{u.m} = \bar{u}.\bar{m}$. It follows that there is $\bar{M} \in \hat{W}'_\xi$ with

$$\mathrm{Tr}(e_w, \bar{M}(t)) = \overline{\mathrm{Tr}\,(e_{w^{-1}}^{-1}, M(t))}.$$

The bar automorphism of $\mathcal{H}'_\xi$ induces the trivial automorphism of the specialized algebra $\mathcal{H}'_\xi(1) = E[W'_\xi]$. Using 11.1.2(b) we can conclude that $\bar{M} = M$, proving (ii).
A similar argument proves (iii) and (iv), using respectively the involution $e_w \mapsto e_{w^{-1}}$, the automorphism conj of $C(t)$ and the automorphism sending t to t^{-1} and e_w to $\varepsilon(w)t^{2\ell_\xi(w)}e_{w^{-1}}^{-1}$.

The results of 11.1.3 are also due to Lusztig. They are used in [L1, Ch.5]. There one also finds further references.
We next recall the orthogonality relations for $\mathcal{H}'_\xi$.
11.1.4. **Lemma.** *For $M, M' \in \hat{W}'_\xi$ we have*

$$\sum_{w \in W'_\xi} t^{-2\ell_\xi(w)}\,\mathrm{Tr}\,(e_w, M(t))\,\mathrm{Tr}\,(e_{w^{-1}}, M'(t) = \qquad (1)$$
$$\begin{cases} 0 & \text{if } M' \neq M, \\ (\dim M) F_M & \text{if } M' = M. \end{cases}$$

Here $F_M \in \mathbf{Q}(t^2)$ is such that if

$$P = \sum_{w \in W'_\xi} t^{2\ell_\xi(w)}$$

we have $PF_M^{-1} \in \mathbf{Q}[t^2]$.
We sketch a proof of (1). There exists a linear function ℓ on $\mathcal{H}'_\xi$ such that

$$\ell(e_x e_{y^{-1}}) = t^{2\ell_\xi(w)}\delta_{x,y}$$

Let (f_i) be a basis of $\mathcal{H}'_\xi$ and (f'_i) the dual basis relative to ℓ, i.e. such that $\ell(f_i f'_j) = \delta_{ij}$.
If ρ and ρ' are two absolutely irreducible matrix representations of $E(t) \otimes \mathcal{H}'_\xi$ one shows that, putting

$$A = \sum_i \rho(f_i)\rho(f'_i),$$

we have for all $u \in \mathcal{H}'_\xi$

$$\rho(u)A = A\rho(u).$$

Application of Schur's lemma leads to orthogonality relations for matrix elements, as in the case of finite groups. Then (1) follows.
In the case of ordinary Hecke algebras, the polynomial PF_M^{-1} is a generic degree polynomial from the theory of finite groups of Lie type. See [Cu].

11.2 The function c

In 5.1.6 we have defined a linear map τ of the algebra $\mathcal{K}$ to $\mathbf{Z}[t, t^{-1}] \otimes \mathbf{C}G$. Since $\mathcal{H}'_\xi$ is a subring of $\mathcal{K}$, we obtain a map of $\mathcal{H}'_\xi$, also denoted by τ.
11.2.1. **Proposition.** *There is a unique function $c_G = c : \hat{G}(\xi) \times \hat{W}'_\xi \to E(t)$ such that for $u \in \mathcal{H}'_\xi$*

$$\tau(u) = \sum_{\substack{A \in \hat{G}(\xi) \\ M \in \hat{W}'_\xi}} c(A, M)\,\mathrm{Tr}\,(u, M(t))A.$$

If necessary we write $c = c_\xi$.
A linear function on the semi-simple algebra $E(t) \otimes \mathcal{H}'_\xi$ with the property of 5.1.7 (ii) is a linear combination of trace functions defined by the irreducible representations of that algebra. The proposition now follows from 11.1.2.
Recall that, $C_{\xi,w}$ being the semi-simple complex of 5.1.1, we have ($c_w = c_{\xi,w}$ as in 3.3.4)

$$\tau(c_w) = \sum_i t^i \, {}^pH^i(C_{\xi,w}).$$

The following properties of c are rather direct consequences of the definition.
11.2.2. **Corollary.** *Let $A \in \hat{G}(\xi), M \in \hat{W}'_\xi$.*
(i) *$Pc(A,M) \in C[t,t^{-1}]$, in particular $c(A,M) \in C(t)$;*
(ii) *$c_{-\xi}(DA,M) = c_\xi(A,M)$;*
(iii) *$c(A,M^*) = c(A,M)^{\mathrm{conj}}$, where* conj *denotes the extension to $C(t)$ fixing t.*
From the orthogonality relations of 11.1.4 we see that

$$\sum_A (F_M \dim M) c(A,M) A = \sum_{w \in W'_\xi} t^{-2\ell_\xi(w)} \tau(e_w) \operatorname{Tr}(e_{w^{-1}}, M(t)).$$

Now (i) follows from 11.1.3(i).
(ii) follows from 5.1.10 observing that $W'_{-\xi} = W'_\xi$ and (iii) follows from 11.1.3(iii).
The function c, introduced in [CS, no.12], is a crucial object for the study of character sheaves.
We next introduce a notion to be needed below.
11.2.3. **Definition.** *G is clean if for any Levi subgroup L of G we have that the cuspidal character sheaves on L are clean in the sense of* 6.3.4.
We come now to the first main result of this section.
11.2.4. **Theorem.** *Assume that G is clean. Then the values of c are constants.*
By 11.2.2, the values of c are cyclotomic numbers. We shall review the proof of the theorem given in [CS, nos.13, 14]. By an argument of the kind discussed in [BBD, no.6] one sees that it suffices to prove the theorem in the case that k is an algebraic closure of a finite field F_q.

11.3 Working over finite fields

We assume that G is defined over F_q. We assume that the maximal torus T and the Borel group $B \supset T$ are also defined over F_q and that T is split over F_q.
11.3.1. Fix $\xi \in \hat{X}$, then $F\xi = \xi$. Replacing F_q by a suitable finite extension we may assume that for all $A \in \hat{G}(\xi)$ we have $F^*A \xrightarrow{\sim} A$.
If $A \in \hat{G}(\xi)$ there exists by 9.3.5 and 7.2.2 a Levi group L together with a subset $\sum$ of L as in 6.2.4 such that, putting

$$Y = Y_{(L,\sum)} = \bigcup_{g \in G} g \Sigma_{\mathrm{reg}} g^{-1},$$

we have supp $A = \bar{Y}$.
For each class in $\hat{G}(\xi)$ we choose a representative A and an isomorphism $\varphi_A : F^*A \xrightarrow{\sim} A$ such that (A, φ_A) is pure of weight zero.
Let $C_{\xi,\dot{w}}$ be as in 5.1.1 (with a suitable set of representatives $(\dot{w})$). We have

$${}^pH^i(C_{\xi,\dot{w}}) = \bigoplus_{A \in \hat{G}(\xi)} A \otimes V_{A,i,\dot{w}},$$

where $V_{A,i,\dot{w}} =$ Hom$(A, {}^pH^i(C_{\xi,\dot{w}}))$ is a finite dimensional vector space, zero for all but finitely many i. Then the isomorphism

$$F : F^* \, {}^pH^i(C_{\xi,\dot{w}}) \tilde{\to} {}^pH^i(C_{\xi,\dot{w}})$$

corresponds to an isomorphism $\oplus\varphi_A \otimes \psi_{A,i,\dot{w}}$, where $\psi_{A,i,\dot{w}}$ is an invertible linear map of $V_{A,i,\dot{w}}$.

11.3.2. **Lemma.** *The eigenvalues of $\psi_{A,i,\dot{w}}$ are algebraic integers all whose complex conjugates have absolute value $q^{\frac{1}{2}(\dim G+\ell(w)+i)}$.*

$C_{\xi,\dot{w}}$ is pure of weight $\dim G + \ell(w)$ (5.3.1). The assertion now follows from [BBD, 5.3.4]. From 5.1.10 we see that

$${}^pH^i(C_{-\xi,\dot{w}}) = \bigoplus_{A\in\hat{G}(\xi)} DA \otimes V_{DA,i,\dot{w}}.$$

We take φ_{DA} to be the contragradient $\varphi_A^\vee$ of φ_A. Then (DA, φ_{DA}) is pure of weight zero. Moreover

$$V_{DA,i,\dot{w}} = V^*_{A,-i,\dot{w}}(-\dim G - \ell(w)),$$

the star denoting vector space dual, the Tate twist having the obvious meaning.

11.3.3. **Theorem.** *Assume G to be clean. Let $A \in \hat{G}(\xi)$, $w \in W'_\xi$ and $i \in \mathbb{Z}$ be such that A is a constituent of ${}^pH^i(C_{\xi,w})$.*

(i) *The parity of $i + \ell(w)$ is an invariant of A;*

(ii) *There exists an algebraic number ξ_A all whose complex conjugates have absolute value one, such that the eigenvalues of $\psi_{A,i,\dot{w}}$ (resp. $\psi_{DA,i,\dot{w}}$) are all equal to $\xi_A q^{\frac{1}{2}(\dim G+\ell(w)+i)}$ (resp. $\xi_A^{-1} q^{\frac{1}{2}(\dim G+\ell(w)+i)}$).*

Denote by χ_{A,φ_A} (resp. $\chi_{DA,\varphi_{DA}}$) the corresponding characteristic functions (1.3.4). We have the following orthogonality relation, see [CS, 10.7]. A variant is discussed in [Sh].

11.3.4. **Lemma.** *Assume G to be clean. Then for $A, A' \in \hat{G}(\xi)$*

$$| G^F |^{-1} \sum_{g\in G^F} \chi_{A,\varphi_A}(g)\chi_{DA',\varphi_{DA'}}(g) = \begin{cases} 0 \text{ if } A \neq A' \\ q^{-\dim G} \text{ if } A = A'. \end{cases}$$

As in 5.3.1 we denote by $\gamma_{\xi,\dot{w}}$ the characteristic function of $C_{\xi,\dot{w}}$.

11.3.5. **Lemma.** *Let $x, w \in W'_\xi$. The sum*

$$(2) \qquad | G^F |^{-1} \sum_{g\in G^F} \gamma_{\xi,\dot{w}}(g)\gamma_{-\xi,\dot{x}}(g)$$

equals the trace of the linear map of the specialized Hecke algebra $\mathcal{H}'_\xi(q^{\frac{1}{2}})$ induced by the map

$$u \mapsto (-t)^{\ell(w)+\ell(x)} c_{x^{-1}} u c_w$$

of $\mathcal{H}'_\xi$.

(We have written $c_w = c_{\xi,w} \ldots$).

See [CS, 13.7]. Using 5.3.3 the proof is reduced to a similar statement for the sum

$$| G^F |^{-1} \sum_{g\in G^F} \text{Tr}\,(g\theta_{\dot{w}}, V_\varphi)\ \text{Tr}\,(g\theta_{\dot{x}}, V_{\varphi^{-1}}),$$

notations being as in 5.3.2. This statement is proved in a straightforward manner. We refer to [loc.cit] for the details.

11.3.6. **Lemma.** *There exists a Laurent polynomial $F \in \mathbb{Z}[T, T^{-1}]$ independent of the base field $\mathbb{F}_q$ such that* (2) *equals* $F(q)$.
This follows by using the form of the multiplication rules in $\mathcal{H}'_\xi$, taking into account the corollary of 3.2.6 and 5.2.5.
11.3.7. We can now prove 11.3.3. Using 11.3.4 one finds that (2) equals

$$(3) \qquad q^{-\dim G} \sum_{A\in\hat{G}(\xi)} \sum_{i,j} (-1)^{i+j} \operatorname{Tr}(\psi_{A,i,\dot{w}}, V_{A,i,\dot{w}}) \operatorname{Tr}(\psi_{DA,j,\dot{x}}, V_{DA,j,\dot{x}}).$$

Let $(\alpha_{ih,A})$ resp. $(\beta_{j\ell,A})$ be the set of eigenvalues of $\psi_{A,i,\dot{w}}$ resp. $\psi_{DA,j,\dot{x}}$. Applying (3) for the ground field $\mathbb{F}_{q^n}$ we conclude from 11.3.6 that for all integers $n \geq 1$

$$\sum_{A\in\hat{G}(\xi)} \sum_{i,j,h,\ell} (-1)^{i+j} (\alpha_{ih,A}\beta_{j\ell,A} q^{-\dim G})^n = F(q^n),$$

F being as in 11.3.6. Now use the following elementary fact: if $(z_1, \ldots, z_N)$ is a set of distinct elements of E^* and $(a_1, \ldots, a_N)$ a set of elements of E such that $a_1 z_1^n + \ldots + a_N z_N^n = 0$ for infinitely many positive integers n then all a_i are zero.
We conclude from 11.3.2 that

$$\begin{cases} \alpha_{ih,A}\beta_{j\ell,A} &= 0 \text{ if } i+j+\ell(w)+\ell(x) \text{ is odd} \\ &= q^{\frac{1}{2}(\ell(w)+\ell(x)+i+j+2\dim G)} \text{ if it is non-zero.} \end{cases}$$

Now fix $A \in \hat{G}(\xi)$ and choose j, x such that $V_{DA,j,x} \neq 0$.
Let β be an eigenvalue of $\psi_{DA,j,x}$. It follows that if $V_{A,i,\dot{w}} \neq 0$ for some i, w we have

$$i + \ell(w) \equiv j + \ell(x) \pmod 2,$$

proving 11.3.3(i). In that case we also have that for all h

$$\alpha_{ih} = \beta^{-1} q^{\frac{1}{2}(i+j+\ell(w)+\ell(x)+2\dim G)},$$

from which 11.3.3 (ii) follows.
11.3.8. **Corollary.** *Assume G to be clean. If $M, M' \in \hat{W}'_\xi$ then*

$$\sum_{A\in\hat{G}(\xi)} c(A,M)c(A,M') = \begin{cases} 0 & \text{if } M^* \neq M' \\ 1 & \text{if } M^* = M'. \end{cases}$$

By 11.3.3(ii) we can rewrite (3) as

$$(4) \qquad \sum_{A\in\hat{G}(\xi)} (-1)^{i+j} q^{\frac{1}{2}(i+j+\ell(w)+\ell(x))} \dim V_{A,i,\dot{w}} . \dim V_{DA,j,\dot{x}}.$$

From the definition of the function c we see, using 11.3.3(i) that there is a sign $\varepsilon_A = \pm 1$ such that

$$(5) \qquad \sum_i (-1)^i q^{\frac{1}{2}i} \dim V_{A,i,\dot{w}} = \varepsilon_A (-1)^{\ell(w)} \sum_M c(A,M)(q^{\frac{1}{2}}) \operatorname{Tr}(\bar{c}_w, M(q^{\frac{1}{2}})),$$

where $\bar{c}_w$ denotes the image of c_w in $\mathcal{H}'_\xi(q^{\frac{1}{2}})$. Notice that $c(A,M)(q^{\frac{1}{2}})$ makes sense because of 11.2.2(i). On the other hand, using 11.3.5 and 11.1.2 we see that (5) also equals

$$(-q^{\frac{1}{2}})^{\ell(w)+\ell(x)} \sum_{M\in\hat{W}'_\xi} \operatorname{Tr}(\bar{c}_w, M(q^{\frac{1}{2}})) \operatorname{Tr}(\bar{c}_x, M^*(q^{\frac{1}{2}})).$$

Since the functions $(w,x) \mapsto \operatorname{Tr}(\bar{c}_x, M(q^{\frac{1}{2}}) \operatorname{Tr}(\bar{c}_w, M'(q^{\frac{1}{2}}))$ on $W'_\xi \times W'_\xi$ (M, M' running through $\hat{W}'_\xi$) are linearly independent (as a consequence of 11.1.2) we can conclude that

$$\sum_{A\in\hat{G}(\xi)} c(A,M)(q^{\frac{1}{2}})c(A,M')(q^{\frac{1}{2}}) = \delta_{M^*,M'}.$$

(One uses that $\varepsilon_A = \varepsilon_{DA}$, which follows from the proof of 11.3.3.)
Working over the base field $\mathbf{F}_{q^n}$ one obtains a similar result, with $q^{\frac{n}{2}}$ instead of $q^{\frac{1}{2}}$. This implies the asserted identity of rational functions.

We next state some results complementing those of 5.3. We use the notations of that section.
Let $\operatorname{End}_{G^F}(V_\varphi)$ be the commuting algebra of the representation of G^F in V_φ. We choose an algebra homomorphism $h : \operatorname{End}_{G^F}(V_\varphi) \to E$. Such a homomorphism exists, as there are irreducible representations of G^F occurring in V_φ with multiplitity one. θ_n and the decomposition $w = w^* w_1$ are as in 5.3.2.
11.3.9. **Lemma.** *There exists a choice of representatives $(\dot{w})_{w\in W'_\xi}$ lying in G^F such that the linear map $\varsigma : \mathcal{H}'_\xi(q^{\frac{1}{2}}) \to \operatorname{End}_{G^F}(V_\varphi)$ defined by*

$$\varsigma(\bar{e}_w) = q^{-\frac{1}{2}(\ell(w)-\ell(w^*)-\ell_\xi(w_1))} h(\theta_{\dot{w}^*})^{-1}\theta_{\dot{w}^*\dot{w}_1} (w \in W'_\xi)$$

is an isomorphism of algebras. Moreover, $q^{-\frac{1}{2}\ell(w^*)}h(\theta_{\dot{w}^*})$ is a root of unity.
Here $\bar{e}_w$ denotes the image of e_w in $\mathcal{H}'_\xi(q^{\frac{1}{2}})$.
This is purely a result about finite groups of Lie type. For more details see [CS, no.13] or [HK].
By the previous lemma V_φ becomes a module over the algebra $\mathcal{H}'_\xi(q^{\frac{1}{2}}) \otimes E[G^F]$. It can be decomposed as

$$V_\varphi = \bigoplus_{M\in\hat{W}'_\xi} M(q^{\frac{1}{2}}) \otimes V_{\varphi,M},$$

where $V_{\varphi,M}$ is an irreducible $E[G^F]$- module.
11.3.10. **Lemma.** *To any $A \in \hat{G}(\xi)$ one can associate a root of unity ν_A such that for $g \in G^F$*

$$\operatorname{Tr}(g, V_{\varphi,M}) = \sum_{A\in\hat{G}(\xi)} q^{\frac{1}{2}\dim G} \xi_A \nu_A c(A,M)(q^{\frac{1}{2}}) \chi_{A,\varphi_A}(g).$$

Here ξ_A is as in 11.3.3 (ii). Using 11.3.9 we find

$$\operatorname{Tr}(g\theta_{\dot{w}}, V_\varphi) = \sum_{M\in\hat{W}'_\xi} q^{\frac{1}{2}(\ell(w)-\ell(w^*)-\ell_\xi(w_1))} h(\theta_{\dot{w}^*}) \operatorname{Tr}(\bar{e}_w, M(q^{\frac{1}{2}})) \operatorname{Tr}(g, V_{\varphi,M}),$$

whence by 5.3.3

$$\gamma_{\xi,\dot{w}}(g) = (-1)^{\dim G + \ell(w)} \sum_{M\in\hat{W}'_\xi} q^{\frac{1}{2}(\ell(w)-\ell(w^*))} h(\theta_{\dot{w}^*}) \operatorname{Tr}(\bar{c}_w, M(q^{\frac{1}{2}})) \operatorname{Tr}(g, V_{\varphi,M}). \tag{6}$$

On the other hand, an application of 11.3.3 gives

$$\gamma_{\xi,\dot{w}}(g) = \sum_{A\in\hat{G}(\xi)} \xi_A \chi_{A,\varphi_A}(g) \Big(\sum_i (-1)^i q^{\frac{1}{2}(i+\ell(w)+\dim G)} \dim V_{A,i,\dot{w}}\Big).$$

By (5) there is a sign $\varepsilon_A = \pm 1$ such that this can be written as

(7)
$$\gamma_{\xi,\dot{w}}(g) = (-q^{\frac{1}{2}})^{\ell(w)} \sum_{\substack{A \in \hat{G}(\xi) \\ M \in \hat{W}'_\xi}} \varepsilon_A \xi_A q^{\frac{1}{2}\dim G} \chi_{A,\varphi_A}(g) c_{A,M}(q^{\frac{1}{2}}) \operatorname{Tr}(\bar{c}_w, M(q^{\frac{1}{2}})).$$

Since the functions $w \mapsto \operatorname{Tr}(\bar{c}_w, M(q^{\frac{1}{2}}))$ are linearly independent, the assertion follows from (6) and (7), taking into account the last point of 11.3.9.

We need some results of a general nature. In 11.3.11 and 11.3.12 the assumption that k is an algebraic closure of a finite field is not needed.

11.3.11. Lemma. *Let $K = I(\sum, \mathcal{L})$ be an irreducible cuspidal perverse sheaf on G such that DK is strongly cuspidal and clean. Put $d = \dim \sum$. Let K' be a perverse sheaf on G obtained by induction of a strongly cuspidal clean irreducible perverse sheaf on a Levi group such that* supp $K' \neq \bar{\sum}$. *Then the local system $H^i K' |_{\sum}$ has no direct summands isomorphic to $\mathcal{L}(i \in \mathbf{Z})$.*

We first establish that $H_c^\cdot(G, DK \otimes K') = 0$. If K' is cuspidal this follows directly from the assumptions. Otherwise there is a proper parabolic subgroup P with Levi group L, and a perverse sheaf $\tilde{K}$ on $G \times_P P$ such that $K' = \delta_* \tilde{K}$ (notations of 7.1.1). Let $\Gamma \subset (G \times_P P) \times G$ be the graph of δ. Then

$$H_c^\cdot(G, DK \otimes K') = H_c^\cdot(\Gamma, DK \boxtimes \tilde{K}).$$

The projection $G \times_P P \to G/P$ induces a morphism $\Gamma \to G/P$ and it suffices to prove that for any of its fibers F we have

(8)
$$H_c^\cdot(F, DK \boxtimes \tilde{K}) = 0.$$

Now F is isomorphic to $P = L \times U(P)$ and under an appropriate isomorphism $DK \boxtimes \tilde{K} |_F$ corresponds to

$$(DK) |_P \otimes (C \boxtimes E),$$

where C is a complex on L and E is the constant sheaf on $U(P)$. Since DK is strongly cuspidal we have for all $\ell \in L$

$$H_c^\cdot(\ell U(P), DK) = 0,$$

and (8) follows.

To prove 11.3.11 it suffices to show that $H^i(\mathcal{L}^* \otimes K' |_{\sum})$ has no direct summand isomorphic to $E(\mathcal{L}^*$ denoting the dual; notice that $H^i(K' |_{\sum})$ is a local system by 8.3.1). Assume this is not the case and take i maximal such that $H^i(\mathcal{L}^* \otimes K' |_{\sum})$ has such a direct summand. There is a spectral sequence

$$E_2^{p,q} = H_c^p(\Sigma, H^q(\mathcal{L}^* \otimes K' |_{\sum})) \Rightarrow H_c^\cdot(\Sigma, \mathcal{L}^* \otimes K' |_{\sum}).$$

We have $E_2^{2d,i} \neq 0$ and $E_2^{p,q} = 0$ for $p > 2d$ and for $q > i$ (by 6.3.5). It follows that $E_2^{2d,i} = E_\infty^{2d,i}$, whence $H_c^{2d+i}(\sum, \mathcal{L}^* \otimes K' |_{\sum}) = H_c^{d+i}(G, DK \otimes K') \neq 0$, contradicting the statement at the beginning of the proof.

11.3.12. **Lemma.** *Assume G to be clean. Let A, A' be character sheaves. Let $Y = Y_{(L,\Sigma)}$ be as in 6.2.2 such that* supp $A = \bar{Y}$ *and put* $d = \dim Y$. *If A' is not isomorphic to A then no restriction $H^i(A') |_Y$ has an irreducible constituent isomorphic to $H^{-d}(A) |_Y$.*

A is an irreducible constituent of a complex $K = \mathrm{ind}(\bar{\Sigma}, \mathcal{L})$, where Σ is a subset of a Levi group L of G as in 6.2.2 and $\mathcal{L}$ is an irreducible local system on Σ (so $Y = Y_{(L,\Sigma)}$). Let s be the semi-simple part of an element of Σ_{reg}. Put $H = Z_G(s)^0$. Likewise, A' is a constituent of a complex K' of the same kind. We may assume that supp $K' \supset$ supp K and that these sets are distinct (otherwise the assertion is obvious).

We now apply 8.2.3. This shows that there exists an open neighbourhood U of s in H such that

$$s^*(K |_U) \simeq \oplus A_i |_{s^{-1}U} [\dim G - \dim H]$$

$$s^*(K' |_U) \simeq \oplus A'_j |_{s^{-1}U} [\dim G - \dim H]$$

where s^* denotes the inverse image for $x \mapsto sx$, the A_i are cuspidal perverse sheaves on H and A'_j is induced from a cuspidal perverse sheaf on a Levi subgroup of H. The A_i are irreducible, strongly cuspidal and clean (see 8.1.3, 8.1.6), and the same holds for the DA_i. The A_i are obtained as follows: choose representatives g_i for the double cosets HgL of elements $g \in G$ with $g^{-1}sg \in \Sigma_s$ (the set of semi-simple parts of the elements of Σ), with $g_1 = 1$. Let C be the set of unipotent elements u in H with $su \in \Sigma$, this is a conjugacy class in H (8.1.4). Let $\Sigma_i = (g_i C g_i^{-1}) Z(H)^0$ and let $\mathcal{L}_i$ the inverse image of $\mathcal{L}$ for the map $\Sigma_i \to \Sigma$ with $x \mapsto g_i^{-1} s x g_i$. Then $A_i = I(\bar{\Sigma}_i, \mathcal{L}_i)$.

Put $U' = \{y \in \Sigma_{\mathrm{reg}} \mid y_s \in sZ(L)^0\}$. Then U' is an irreducible subset of H which contains s. It suffices to show that $H^p(A_i) |_{s^{-1}(U \cap U')}$ and $H^q(A'_j) |_{s^{-1}(U \cap U')}$ have no common irreducible constituents (i, j, p, q arbitrary).

Now $s^{-1}(U \cap U')$ is open and dense in Σ_1. So it suffices to prove the previous assertion, with $s^{-1}(U \cap U')$ replaced by Σ_1. But $H^p(A_i) |_{\Sigma_1} = 0$ unless $g_i C g_i^{-1} = C$. So we can assume that

supp $A_i = \overline{\Sigma_1}$.

We now show that supp $A_i \neq$ suppA'_j for all i, j. In fact, assume that supp $A'_j = \bar{\Sigma}_1 = Z(H)^0 \bar{C}$. There is a Levi group L' of G such that A' is induced from a cuspidal perverse sheaf on the Levi group $M = L' \cap H$ of H. We then have $Z(M)^0 \subset Z(H)^0$, whence $M = H$ and $L' \supset H$. Since L is the smallest Levi group containing H (see 6.2.2) we have $L \subset L'$. It also follows that the unipotent class D contained in Σ' coincides with C.

Since $\bar{Y}_{(L,\Sigma)} =$ supp A is contained in $\bar{Y}_{(L',\Sigma')} =$ supp A' we have by 6.2.7 that L' is conjugate to a subgroup of L. It follows that $L = L'$ and supp $A =$ supp A', contradicting the assumption that supp $K \neq$ supp K'. We now know that supp $A_i \neq$ supp A'_j for all i, j. Application of 11.3.11 then proves 11.3.12.

We can now continue the proof of 11.2.4. Again assume that we are working over a finite field F_q and that G is clean.

11.3.13. **Lemma.** *Let $\xi \in \hat{X}$ and $A \in \hat{G}(\xi)$. There exist natural numbers $a \neq 0$ and b such that for infinitely many natural numbers N*

$$a \xi_A^N c(A, M)(q^{\frac{1}{2}N}) q^{\frac{1}{2}Nb}$$

is an algebraic integer for all $A \in \hat{G}(\xi), M \in \hat{W}^!_\xi$.

Here ξ_A is as in 11.3.3.

By 11.3.10 there exists for $A \in \hat{G}(\xi)$ and any integer $n \geq 1$ a root of unity $\nu_{A,n}$ such that for all $g \in G^{F^n}$

$$(9) \qquad \sum_{A \in \hat{G}(\xi)} q^{\frac{1}{2}n \dim G} \xi_A^n \nu_{A,n} c(A,M)(q^{\frac{1}{2}n}) \chi_{A,\phi_A}(g) \text{ is an algebraic integer.}$$

We shall deduce 11.3.13 by using this for suitable g.

Fix $A \in \hat{G}(\xi)$. There is a variety $Y = Y_{(L,\Sigma)}$ as in 6.2.2 such that supp $A = \bar{Y}$. Choose a finite Galois covering $\pi : \tilde{Y} \to Y$, with group Γ, such that for any $A' \in \hat{G}(\xi)$ the restrictions $H^i(A') \mid_Y$ are local systems whose pull-back to $\tilde{Y}$ is trivial (this is possible by 8.3.1). We may assume that $Y, \tilde{Y}$ and all elements of Γ are defined over F_q. The local system $H^i(A') \mid_Y$ is associated to a class of representations of Γ denoted by $[H^i(A')]$.

Let $\gamma \in \Gamma$. There exist infinitely many natural numbers N such that $F^N z = \gamma z$ for some $z \in \tilde{Y}$. In fact, $\gamma^{-1} F^N$ is a Frobenius map for some F_{q^N} structure on $\tilde{Y}$, which must have fixed points for large N. Put $\pi z = y_{N,\gamma}$. If $i \in \mathsf{Z}$ and $A' \in \hat{G}(\xi)$ there exists $\alpha_{A',N,i} \in E^*$, independent of γ , such that

$$\mathrm{Tr}(\varphi_{A'}^N, H^i(A')_{y_{N,\gamma}}) = \alpha_{A',N,i} \,\mathrm{Tr}(\gamma, [H^i(A')]),$$

where φ'_A is as in 11.3.1. If $A' = A$ then $\alpha_{A,N,-\dim Y}$ is a root of unity times $q^{-\frac{1}{2}N \dim Y}$ (as a consequence of 2.3.2) and

$$\mathrm{Tr}\,(\varphi_{DA}^N, H^{-d}(DA) y_{N,\gamma}) = \alpha_{A,N,-d}^{-1} q^{-Nd} \,\mathrm{Tr}(\gamma^{-1}, [H^{-d}(A)]),$$

where $d = \dim Y$ (see 11.3.1).

By 11.3.12, $[H^i(A')]$ contains no irreducible representation of Γ isomorphic to $[H^{-\dim Y}(A)]$, if $A' \neq A$. Using the orthogonality relations for the group characters of Γ we conclude that

$$\mid \Gamma \mid^{-1} \sum_{\gamma \in \Gamma} \chi_{A',\varphi_{A'}^N}(y_{N,\gamma}) \chi_{DA,\varphi_A^N}(y_{N,\gamma}) = \begin{cases} 0 & \text{if } A' \neq A \\ q^{-N \dim Y} & \text{if } A' = A \end{cases}.$$

Using (9) with $n = N$ as before and $g = y_{N,\gamma} (\gamma \in \Gamma)$ the assertion of 11.3.13 readily follows.

11.3.14. **End of the proof of 11.2.4.** Fix $A \in \hat{G}(\xi), M \in \hat{W}'_\xi$. We first show that $c(A,M)$ is a Laurent polynomial. Let a and b be as in 11.3.13 and write

$$t^b c(A,M)(t) = f + g,$$

where f is a polynomial and g a rational function which is zero at infinity. From 11.3.13 we see that for infinitely many natural numbers N

$$a_N = a \xi_A^N g(q^{\frac{1}{2}N})$$

is an algebraic number with bounded denominator. As ξ_A is an algebraic number all whose complex conjugates have absolute value one, the norm of ξ_A (relative to Q) equals one. Moreover, all complex conjugates of $ag(q^{\frac{1}{2}N})$ are small if N is large. It follows that the norm of a_N is less than one if N is large. Since a_N has bounded denominator it must be zero, whence $g = 0$.

We can now write for all $A \in \hat{G}(\xi), M \in \hat{W}'_\xi$

$$c(A,M) = \gamma(A,M) t^m + \text{ lower powers of } t,$$

From 11.3.8 and 11.2.2(iii) we conclude that if $m > 0$ we have

$$\sum_{A\in\hat{G}(\xi)} \gamma(A,M)\gamma(A,M)^{\text{conj}} = 0,$$

which can only be if all $\gamma(A,M)$ are zero. Hence $m \leq 0$ and a similar argument gives $m \geq 0$. It follows that all $c(A,M)$ are constant, finishing the proof of 11.2.4.
We have followed here rather faithfully the proof of 11.2.4 contained in [CS, nos.13, 14].
11.3.15. We discuss some consequences of the proof of 11.2.4, assuming G to be clean. Now k is arbitrary.
By the definition of c we have for $w \in W'_\xi$ (notations of 11.1)

$$\tau(e_w) = \sum_{A,M} c(A,M) \operatorname{Tr}(e_w, M(t))A.$$

We specialize t to 1 and indicate this in the left-hand side by a suffix 1. Using the orthogonality relations for the group characters of W'_ξ we see that

$$(10)\qquad \sum_{A\in\hat{G}(\xi)} c_\xi(A,M)A = | W'_\xi |^{-1} \sum_{w\in W'_\xi} \operatorname{Tr}(w^{-1},M)\tau(e_{\xi,w})_1.$$

Introduce the symmetric bilinear form (,) of 10.1.4 on the Grothendieck group $\mathbf{C}G$ and extend it to $E \otimes \mathbf{C}G$. Write $R_\xi(M) = R(M)$ for the left-hand side of (10). Then $R_\xi M \in E \otimes \mathbf{C}G$ and

$$c_\xi(A,M) = (A, R_\xi(M)).$$

11.3.16. **Corollary.**
(i) *For any $A \in \hat{G}(\xi)$ there exists $M \in \hat{W}'_\xi$ such that $(A, R_\xi(M)) \neq 0$;*
(ii) $(R_\xi(M), R_\xi(M')) = \begin{cases} 0 & \text{if } M' \neq M^* \\ 1 & \text{if } M' = M^*. \end{cases}$
(i) is clear and (ii) follows from 11.3.8.
11.3.17. **Corollary.** *Let $A \in \hat{G}$. If $\xi \in \hat{X}, w \in W'_\xi$, $i \in \mathbf{Z}$ are such that A is a constituent of ${}^pH^i(C_{\xi,w})$ then the parity of $i + \ell(w)$ depends only on A.*
If (η, x, j) is another triple with the same property the assertion is true if $\eta = \xi$ by 11.3.3(i). If we have $\eta \in W\xi$ (5.2.2) and $\eta \neq \xi$ it suffices to deal with the case that $\eta = s\xi$, with $s \in S$. One can then apply 5.1.8 and 5.1.9.
11.3.18. **Definition.** *G satisfies the parity condition if for all $A \in \hat{G}$ we have that if A is a constituent of ${}^pH^i(C_{\xi,w})$ then*

$$i \equiv \ell(w) + \operatorname{codim} \operatorname{supp} A \pmod 2.$$

11.4 Induction, restriction, duality

11.4.1. We assume G to be clean. Fix $\xi \in \hat{X}$. We extend the function $c : \hat{G}(\xi) \times \hat{W}'_\xi \to$ to a bilinear map $(A, M) \mapsto c(A, M)$ where $A \in E \otimes \mathbf{C}G$ and M is a virtual representation of W'_ξ, such that $c(A,M) = 0$ if $A \notin \hat{G}(\xi)$. If necessary we write $c(A,M) = c_G(A,M) = c_G(A,M)_\xi$.
Let P be a parabolic subgroup of G with Levi group L containing our maximal torus T. We now denote by W_1 the Weyl group of (L,T). Let $L = L_I$ and put $W^* = W_I^*$, as in 9.1. We put $W'_{\xi,1} = W_1 \cap W'_\xi$. Denote by res : $E \otimes \mathbf{C}G \to E \otimes \mathbf{C}L$ resp. ind: $E \otimes \mathbf{C}L \to E \otimes \mathbf{C}G$

the linear maps defined by restriction resp. induction. (Notice that here one needs 9.3.1 and 9.3.2). We also denote by res and ind the usual maps of virtual characters for $\hat{W}'_\xi$ and $\hat{W}'_{\xi,1}$.

11.4.2. Proposition.

(i) *If* $A \in \hat{G}(\xi), M_1 \in \hat{W}'_{\xi,1}$ *then*

$$c_G(A, \text{ ind } M_1) = c_L(\text{res } A, M_1);$$

(ii) *If* $A_1 \in \hat{L}(\xi), M \in \hat{W}'_\xi$ *then*

$$c_G(\text{ind } A_1, M) = c_L(A_1, \text{ res}M).$$

By 9.3.1 we have for $w \in W'_{\xi,1}$

$$\sum_{\substack{A_1 \in \hat{L}(\xi) \\ M_1 \in \hat{W}'_{\xi,1}}} c_L(A_1, M_1)\text{Tr}(e_w, M_1(t)) \text{ ind}A_1 =$$

$$= \sum_{\substack{A \in \hat{G}(\xi) \\ M \in \hat{W}'_\xi}} c_G(A, M)\text{Tr}(e_w, M(t))A$$

whence, with obvious notations,

$$\sum_{A, A_1, M_1} c_L(A_1, M_1)(\text{ind}A_1, A)\text{Tr}(w, M_1)A = \sum_{A,M} c_G(A, M)\text{Tr}(w, M)A$$

and

$$\sum_{A_1} c_L(A_1, M_1)(\text{ind}A_1, A) = \sum_{M} c_G(A, M)(\text{res } M, M_1).$$

By Frobenius duality we have $(\text{ind}A_1, A) = (A_1, \text{res}A), (\text{res } M, M_1) = (M, \text{ind}M_1)$ and the formula of (i) follows.

To prove (ii) we observe that 9.2.2 (ii) gives for

$$\sum_{\substack{A \in \hat{G}(\xi) \\ M \in \hat{W}'_\xi}} c_G(A, M) \text{ Tr}(e_w, M(t)) \text{ res}A$$

a formula involving the function tr_I on the algebra $\mathcal{K}$ of loc.cit. We specialize t to 1. From the definition of tr_I given in 9.1 one finds (the suffix 1 denoting specialization)

$$tr_I(e_{\xi,w})_1 = \sum_{\substack{x \in W^\bullet \\ xwx^{-1} \in W_1}} (e_{x\xi, xwx^{-1}})_1.$$

Using the formula of 9.2.2(ii) we obtain for $A_1 \in \hat{L}(\xi), w \in W'_\xi$

$$\sum_{\substack{A \in \hat{G}(\xi) \\ M \in \hat{W}'_\xi}} c_G(A, M) \text{ Tr}(w, M)(\text{res}A, A_1) =$$

$$= \sum_{\substack{x \in W^* \\ xwx^{-1} \in W_1}} \left(\sum_{M_1 \in \hat{W}'_{x\xi,1}} c_L(A_1, M_1)_{x\xi} \operatorname{Tr}(xwx^{-1}, M_1) \right).$$

From 5.2.2 we see that $c_L(A_1, M_1)_{x\xi} = 0$ unless $x\xi \in W_1\xi$. Suppose this is so and choose $y \in W_1$ with $x\xi = y\xi$. We have a decomposition $W_1 = W^*_{\xi,1} W_{\xi,1}$ as in 3.2. Writing $y = y^* y_1$ accordingly, we have $x\xi = y^*\xi$. Now observe that if $M_1 \in \hat{W}'_{x\xi,1}$ we have $c_L(A_1, M_1)_{x\xi} = c_L(A_1, M_2)_\xi$, where $M_2 \in \hat{W}'_{\xi,1}$ corresponds to M_1 via the isomorphism $W'_{\xi,1} \xrightarrow{\sim} W'_{x\xi,1}$ defined by conjugation with y^*. It follows that the sum in brackets equals

$$\sum_{M_1 \in \hat{W}'_{\xi,1}} c_L(A_1, M_1) \operatorname{Tr}(y^{-1}xwx^{-1}y, M_1).$$

Hence the right-hand side of the equality equals

$$\sum_{M_1 \in \hat{W}'_{\xi,1}} c_L(A_1, M_1)(M, \operatorname{ind} M_1) \operatorname{Tr}(w, M),$$

whence for $M \in \hat{W}'_\xi$

$$\sum_{A \in \hat{G}(\xi)} c_G(A, M)(\operatorname{res} A, A_1) = \sum_{M_1 \in \hat{W}'_{\xi,1}} c_L(A_1, M_1)(M, \operatorname{ind} M_1).$$

The formula of (ii) follows by using Frobenius duality.

We shall need a result which is somewhat more general than 11.4.2 (ii). We skip the proof.

11.4.3. Corollary.
(i) *If* $A_1 \in \hat{L}(\eta), M \in \hat{W}'_\xi$ *and* $\eta = v\xi \in W\xi$ *then*

$$c_G(\operatorname{ind} A_1, M)_\xi = c_L(A_1, \operatorname{res}^{W'_\eta}_{W_1 \cap W'_\eta} {}^vM),$$

where ${}^vM \in \hat{W}'_\eta$ *corresponds to* $M \in \hat{W}'_\xi$ *via the isomorphism* $W'_\xi \to W'_\eta$ *induced by conjugation by* v;
(ii) *If* $A_1 \in \hat{L}(\eta), M \in \hat{W}'_\xi$ *and* $\eta \notin W\xi$ *then* $c_G(\operatorname{ind} A_1, M)_\xi = 0$.
In 10.2 we have defined the duality map d of character sheaves. It extends to a linear map d of $E \otimes \mathbf{C}G$ to itself.
As before, we denote by $\varepsilon : W'_\xi \to \{\pm 1\}$ the restriction of the sign character of W (so $\varepsilon(w) = (-1)^{\ell(w)}$).
11.4.4. Theorem. *If* $A \in \hat{G}(\xi), M \in \hat{W}'_\xi$ *then* $c(dA, M) = c(A, \varepsilon \otimes M)$.
With the notations of 10.2 we have

$$c_G(dA, M)_\xi = \sum_{I \subset S} (-1)^{|I|} c_G(i^S_I r^S_I A, M)_\xi.$$

Now

$$c_G(i^S_I r^S_I A, M)_\xi = \sum_{A_1 \in \hat{L}_I} (r^S_I A, A_1) c_G(i^S_I A_1, M)_\xi.$$

Using 11.4.3 this can be rewritten as

$$\sum_{v \in W_I \backslash W / W'_\xi} \sum_{A_1 \in \hat{L}_I(v\xi)} (r^S_I A, A_1) c_{L_I}(A_1, \operatorname{res}^{W'_{v\xi}}_{W_I \cap W'_{v\xi}} ({}^vM)) =$$

$$= \Sigma_v c_{L_I}(r_I^S A, \mathrm{res}^{W_{v\xi}}_{W_I \cap W'_{v\xi}}({}^v M)).$$

Using 11.4.2 (i) this is seen to be equal to

$$\Sigma_v c_G(A, \mathrm{ind}^{W'_\xi}_{v^{-1}W_I v \cap W'_\xi}(\mathrm{res}^{W'_\xi}_{v^{-1}W_I v \cap W'_\xi} M)).$$

Putting

$$dM = \Sigma_{I \subset S}(-1)^{|I|} \Sigma_{v \in W_I \backslash W / W'_\xi} \mathrm{ind}^{W'_\xi}_{v^{-1}W_I v \cap W'_\xi}(\mathrm{res}^{W'_\xi}_{v^{-1}W_I v \cap W'_\xi} M)$$

we conclude that

$$c(dA, M) = c(A, dM).$$

Hence it suffices to show that $dM = \varepsilon \otimes M$. Now it is an easy consequence of the definitions that

$$dM = d1 \otimes M$$

(1 denoting the trivial character), so we are reduced to proving $d1 = \varepsilon$.
Let $w \in W'_\xi$. The definition of $d1$ shows that

$$(d1)(w) = \Sigma_{I \subset S}(-1)^{|I|} \mid W_I \mid^{-1} \Sigma_{v \in W_I \backslash W / W'_\xi} \mathrm{card}\{z \in W_I v W'_\xi \mid w \in z^{-1} W_I z\} =$$

$$= \Sigma_{I \subset S}(-1)^{|I|} \mid W_I \mid^{-1} \mathrm{card}\{z \in W \mid w \in z^{-1} W_I z\} =$$

$$= \Sigma_{I \subset S}(-1)^{|I|} (\mathrm{ind}^W_{W_I} 1)(w) = \varepsilon(w).$$

For the last equality see for example [Ca, p.188]. This proves that $d1 = \varepsilon$ and the theorem follows.
11.4.5. **Corollary.** *If* $A \in \hat{G}(\xi)$ *then* $\pm dA \in \hat{G}(\xi)$.
This is a consequence of 10.2.3.

Most of the results of 11.4 are contained in [CS, no.15], in a somewhat different formulation.

11.5 Cells in W'_ξ

We shall need some facts about cells in groups W'_ξ. For Weyl groups most of them have been established by Lusztig. (See [Cu] for a review). The extension to W'_ξ presents no problem.
11.5.1. Consider the Hecke algebra $\mathcal{H}'_\xi$ of W'_ξ (see 11.1) and let $c_x (x \in W'_\xi)$ be the Kazhdan-Lusztig basis (so $c_x = c_{x,\xi}$ as in 3.3.4). If $x, y \in W'_\xi$ write

$$c_x c_y = \Sigma_{z \in W'_\xi} h_{xyz} c_z.$$

The h_{xyz} are Laurent polynomials with non-negative integral coefficients (4.2.6).
We write $x \leq_L y$ (resp. $x \leq_R y$) if $h_{zyx} \neq 0$ (resp. $h_{yzx} \neq 0$) for some $z \in W'_\xi$. Then $\leq_L$ and $\leq_R$ are preorder relations on W'_ξ. The corresponding equivalence relations are denoted by $\sim_L$ and $\sim_R$. The preorder relation generated by $\leq_L$ and $\leq_R$ is denoted by $\leq_{LR}$ and the corresponding equivalence relation by $\sim_{LR}$.
x and y are in the same left (resp. right, resp. two-sided) cell if $x \sim_L y$ (resp. $x \sim_R y$, resp. $x \sim_{LR} y$).
Notice that $x \leq_L y$ if and only if $x^{-1} \leq_R y^{-1}$. Also, if $x = x^* x_1, y = x^* y_1$ (as in 3.2.6)

then $x \leq_L y$ if and only if $x_1 \leq_L y_1$ (by 3.3.4). This reduces questions about cells in W'_ξ to similar questions in W_ξ.
If $z \in W'_\xi$ define

$$a(z) = \max_{x,y} \deg h_{xyz}.$$

Let $P_{x,y}(t^2)$ denote the Kazhdan-Lusztig polynomials for $\mathcal{H}'_\xi$ (see 3.3.4) and put for $x = x^*x_1 \in W'_\xi$

$$2\delta(x) = \deg P_{e,x}(t^2).$$

The $\delta(x) = \delta(x_1)$. We collect a number of properties. (Compare with [Cu, Ch.II].)
11.5.2. **Theorem.**
(i) *Let $x \in W'_\xi$. Then $a(x) \leq \ell_\xi(x) - 2\delta(x)$. If $x \in W_\xi$ and equality holds, then x is an involution, called Duflo involution;*
(ii) *Any left cell of W_ξ contains a unique Duflo involution;*
(iii) *If $y \in W'_\xi$ and $x \leq_L y$* (resp. $x \leq_R y$) *then $a(y) \leq a(x)$;*
(iv) *If $y \in W'_\xi$ then $x \sim_L y$ if and only if $x \leq_L y$ and $a(x) = a(y)$. Similarly for $\sim_R$ and $\sim_{LR}$.*
We denote by $\mathcal{D}$ the set of Duflo involutions of W_ξ and we put $\mathcal{D}' = (W^* \cap W'_\xi)\ \mathcal{D}$. This is the set of elements for which we have equality in the inequality of (ii).
Let γ_{xyz} be the coefficient of $t^{a(z)}$ in $h_{xyz} (x, y, z \in W'_\xi)$. Define an E-algebra J with basis $(\theta_x)_{x \in W'_\xi}$ by the multiplication rules

$$\theta_x \theta_y = \Sigma_{z \in W'_\xi} \gamma_{xyz} \theta_z.$$

11.5.3. **Theorem.**
(i) *J is an associative algebra, isomorphic to $E[W'_\xi]$;*
(ii) *$E(t) \otimes \mathcal{H}'_\xi$ is isomorphic to $E(t) \otimes J$, an isomorphism φ being given by*

$$\varphi(c_x) = \sum_{\substack{d \in \mathcal{D},\, a(d)=a(z) \\ z \in W'_\xi}} h_{xdz} \theta_z.$$

(See for example [Cu, Ch.III].)

11.6 Cells in $\hat{W}'_\xi$

The material of this section is discussed in [L1, Ch. 5] for the case of Weyl groups.
11.6.1. If $x \in W'_\xi$ the elements $c_y (= 1 \otimes c_y)$ of $E(t) \otimes \mathcal{H}'_\xi$ with $y \leq_{LR} x$ span a two-sided ideal I'_x of $E(t) \otimes \mathcal{H}'_\xi$. Similarly, $x \in W_\xi$ defines a two-sided ideal I_x of $E(t) \otimes H_\xi$. If $x = x^*x_1$ (as usual) then $I'_x = E[W'_\xi \cap W^*] \otimes I_x$ (recall that $E[W'_\xi] = E[W'_\xi \cap W^*] \otimes E[W_\xi]$).
A two-sided ideal in the semi-simple algebra $E(t) \otimes \mathcal{H}'_\xi$ is a direct sum of simple ideals, each of which is defined by an absolutely irreducible representation of $E(t) \otimes \mathcal{H}'_\xi$, hence by an element $M \in \hat{W}'_\xi$ (11.1.2). Let I'_M be the two-sided ideal in $E(t) \otimes \mathcal{H}'_\xi$ defined by M. We use similar notations, without a prime, for $E(t) \otimes \mathcal{H}_\xi$.
If $M \in \hat{W}'_\xi$ and $x \in W'_\xi$ we write $M \leq_{LR} x$ if $I'_M \subset I'_x$. We write $M \sim_{LR} x$ if $M \leq_{LR} x$ and $M\ _{LR}\ y$ for all $y \in W'_\xi$ with $y <_{LR} x$.
If $M, M' \in \hat{W}'_\xi$ we write $M \sim_{LR} M'$ if there exists $x \in W'_\xi$ with $M \sim_{LR} x, M' \sim_{LR} x$. It follows from the definitions that for any $M \in \hat{W}'_\xi$ there exists x with $M \sim_{LR} x$. We then

write $a(M) = a(x)$. This is independent of the choice of x (by 11.5.2(iii)).
11.6.2. **Lemma.** *Let* $x = x^*x_1 \in W_\xi^!$ *(where* $x_1 \in W_\xi$ *and* $x^* \in W^*$*). If* $M \in \hat{W}_\xi^!$ *then* $M \leq_{LR} x$ *(resp.* $M <_{LR} x$*, resp.* $M \sim_{LR} x$*) if and only if there is* $M_1 \in \hat{W}_\xi$ *with* $(\mathrm{res}\, M, M_1) \neq 0$ *such that* $M_1 \leq_{LR} x_1$ *(resp.* $M_1 <_{LR} x_1$*, resp.* $M_1 \sim_{LR} x_1$*).*
Here res and (,) have the obvious meanings. The proof uses the fact that $I_x^!$ is the ideal induced by I_x and Frobenius duality.
11.6.2 shows that our definition of the relations $M \leq_{LR^*}, \ldots$ is equivalent to the one given in [CS, 16.2].

To $M \in \hat{W}_\xi^!$ we associate a class of representations of the algebra J of 11.5, also denoted by M, such that

$$\mathrm{Tr}\,(c_x, M(t)) = \sum_{\substack{d \in \mathcal{D},\, a(d)=a(z) \\ z \in W_\xi^!}} h_{xdz}\, \mathrm{Tr}\,(\theta_z, M) \tag{11}$$

(compare with 11.5.3(ii)). We put $\mathrm{Tr}\,(\theta_z, M) = \theta(z, M)$.
11.6.3. **Proposition.** *Let* $x \in W_\xi^!; M, M' \in \hat{W}_\xi^!$.
(i) $\theta(x, M) \neq 0$ *implies* $M \sim_{LR} x$;
(ii) $\mathrm{Tr}\,(c_x, M(t)) = \theta(x, M)t^{a(M)} +$ *lower powers,* $\mathrm{Tr}\,(e_x, M(t)) = \theta(x, M)t^{a(M)+\ell_\xi(x)} +$ *lower powers* ;
(iii) *The rational function* F_M *of* 11.1.4 *has a development at infinity*

$$F_M(t^2) = a_\xi f_M t^{2a(M)} + \textit{lower powers},$$

where f_M *is strictly positive*;
(iv)

$$\Sigma_{x \in W_\xi^!} \theta(x, M)\theta(x^{-1}, M') = \begin{cases} 0 & \textit{if} \quad M' \neq M \\ a_\xi f_M \dim M & \textit{if} \quad M' = M \end{cases},$$

where $a_\xi = \mid W^* \cap W_\xi^! \mid$.
Moreover $\theta(x^{-1}, M) = \theta(x, M^*) = \theta(x, M)^{\mathrm{conj}}$.

For (i) and (ii) see [Cu, no.10]. (iii) and (iv) then follow from the orthogonality relations of 11.1.4.
We record an auxiliary result.
11.6.4. **Lemma.** *Let* $x \in W_\xi^!, M \in \hat{W}_\xi^!$.
(i) *If* $M \sim_{LR} x$ *then* $M \sim_{LR} x^{-1}$, *in particular* $x \sim_{LR} x^{-1}$;
(ii) *Let* w_0 *be the longest element of* W_ξ. *If* $M \sim_{LR} x$ *then* $\varepsilon \otimes M \sim_{LR} w_0 x$.
As before, ε is induced by the sign character of W. The assertion of (i) follows from 11.6.3(iv). For (ii) see [BV, p.358-359].

11.7 The cell associated to a character sheaf

Using (11), the definition of the function c (11.2.1) shows that

$$\Sigma_i t^i\, {}^pH^i(C_{\xi,w}) = \sum_{\substack{d \in \mathcal{D},\, a(d)=a(x) \\ x \in W_\xi^!, A \in \hat{G}(\xi)}} h_{wdx}\gamma(x, A)A, \tag{12}$$

where

$$\gamma(x,A) = \Sigma_{M\in \hat{W}'_\xi} c(A,M)\theta(x,M)$$

Then $\gamma(x,A)$ is a cyclotomic number.

11.7.1. **Lemma.** *If $x, y \in W'_\xi$ and $x \not\sim_{LR} y$ then*

$$\Sigma_{A\in\hat{G}(\xi)}\gamma(x,A)\gamma(y,A)^{\text{conj}} = 0.$$

This follows from 11.3.8 and 11.6.3 (iv).

The second main result of this section is the following.

11.7.2. **Theorem.** *Assume that G is clean and satisfies the parity condition* 11.3.18.

(i) *If $x \in W'_\xi, A \in \hat{G}(\xi)$ then $\gamma(x,A)$ is a non-negative integer;*

(ii) *Let $A \in \hat{G}(\xi)$ and $M, M' \in \hat{W}'_\xi$ be such that $c(A,M) \neq 0, c(A,M') \neq 0$. Then $M \sim_{LR} M'$.*

The proof of the theorem uses an elementary result, to be described now.

Let V be a finite dimensional vector space over E, with a distinguished basis $(e_i)_{i\le i\le n}$ and a hermitian form $<,>$ relative to an automorphism of order 2 of E such that $< e_i, e_j >= \delta_{ij}$. Denote by P the set of non-negative integral linear combinations of the e_i.

Assume given a finite set I with a preorder relation $\le$. Denote by $\sim$ the corresponding equivalence relation.

Assume given two families $(v_i)_{i\in I}$, $(\tilde{v}_j)_{j\in I}$ of elements of $\bar{V}$ such that

(a) $< v_i, v_j >= 0$ whenever $i \not\sim j$,

(b) when i runs through a fixed equivalence class the v_i and $\tilde{v}_j$ span the same subspace of V;

(c) for any $i \in I$ there exists a linear combination $v_i + \Sigma_{j<i} d_{ij} v_j$ which lies in $P(d_{ij} \in E)$;

(d) for any $i \in I$ there exists a linear combination $\tilde{v}_i + \sum_{j>i} d_{ij}\tilde{v}_j$ which lies in $P(d_{ij} \in E)$.

11.7.3. **Lemma.** *In this situation v_i and $\tilde{v}_i$ lie in P, for all $i \in I$.*

A very similar result is proved in [L1, p.197-199]. The proof (which is elementary and self-contained) carries over to the present situation.

To prove part (i) of 11.7.2. we shall apply the lemma to the following situation: V is the subspace of $E \otimes CG$ spanned by the $A \in \hat{G}(\xi)$. They define the distinguished basis. Furthermore, σ is an automorphism of E extending the automorphism conj of the field of cyclotomic numbers. I is the set W'_ξ with preorder relation the opposite of $\le_{LR}$. For the v_i we take the elements

$$\sum_A \gamma(x,A)A.$$

It follows from (12) that (c) holds. By 11.7.1 we have (a). From $x \in W'_\xi, A \in \hat{G}(\xi)$ we put

$$\tilde{\gamma}(x,A) = (-1)^{\ell(w_0x)+a(w_0x)} \sum_{M\in\hat{W}'_\xi} c(A,M)\theta(w_0x, \varepsilon\otimes M).$$

We take for the $\tilde{v}_i$ the elements $\sum_A \tilde{\gamma}(x,A)A$.

Denote by d the duality map of 10.2. Using 10.2.3 (ii) the parity condition implies that (with obvious notations)

$$\sum_i (-1)^{i+\ell(w)} t^i \, {}^pH^i(dC_{\xi,w})$$

has the form $\sum_{A\in\hat{G}(\xi)} f_A(t)A$, where f_i is a Laurent polynomial with positive integral coefficients. Using 11.4.3 and 11.6.4 (ii) we deduce (d). That (b) holds follows from 11.3.8. So we can apply 11.7.3 to conclude that assertion (i) of 11.7.2 holds.

To prove (ii) we observe that by 11.6.3

$$\sum_{x\in W_\xi^!} \gamma(x,A)\theta(x^{-1},M) = ac(A,M),$$

with $a > 0$. So if $c(A,M) \neq 0$ there is $x \in W_\xi^!$ with $x^{-1} \sim_{LR} M$ and $\gamma(x,A) \neq 0$. Similarly, if $c(A,M') \neq 0$ there is $y \in W_\xi^!$ with $y^{-1} \sim_{LR} M'$ and $\delta(y,A) \neq 0$. Using 11.7.1 we see that $x \sim_{LR} y$, whence $x^{-1} \sim_{LR} y^{-1}$ (11.6.4(i)) and $M \sim_{LR} M'$. This proves (ii).

11.7.4. **Corollary.** *There is a surjective map Γ of $\hat{G}(\xi)$ onto the set of two-sided cells of $\hat{W}_\xi^!$ such that $c(A,M) \neq 0$ if and only if $M \in \Gamma(A)(A \in \hat{G}(\xi), M \in \hat{W}_\xi^!)$.*

This follows from (ii). We shall view $\Gamma(A)$ also as a cell in $W_\xi^!$.

11.7.5. **Corollary.** *Let Γ be as above. If $A \in \hat{G}(\xi)$ there is $x \in \Gamma(A)$ such that A is a constituent of ${}^pH^{a(x)}(C_{\xi,x})$.*

We saw in the proof of 11.7.2 that there is $x \in \Gamma(A)$ with $\gamma(x,A) \neq 0$. Using the fact that h_{xdx} has degree $a(x)$, where $d \in \mathcal{D}$ lies in the left cell of x, we see that the right-hand side of (12) with $w = x$ contains a term $t^{a(x)}\gamma(x,A)A$, whence 11.7.5.

The above results are established in [CS, no.16].

12 Résumé of the classification of character sheaves

So far, we have reviewed the contents of parts 1-16 of [CS]. The subsequent parts of [CS] have a rather different flavour. The results of these parts are proved by a case by case analysis. They rely on detailed explicit information about, for example, cells of representations of Weyl groups.
We shall not go into details here, and we content ourselves with stating the main results.

12.1 General results

G will have the usual meaning. For simplicity we *assume that the characteristic p of k is good for G*, i.e.. that p does not divide a highest root coefficient of any of the irreducible pieces of the root system of G (in [CS] the assumption is slightly less restrictive).
12.1.1. **Theorem.** *G is clean and satisfies the parity condition*
See 11.2.3 and 11.3.18 for the notions involved in the statement of the theorem. The theorem implies that the main results of no.11 (11.2.4 and 11.7.2) hold unconditionally. It is not hard to see that it suffices to establish the theorem in the case that G is semi-simple and simply connected (see [CS, 17.10, 17.11, 17.16.4]).
12.1.2. **Theorem.** *Any irreducible cuspidal perverse sheaf on G is a character sheaf.*
To deal with this theorem the explicit classification of irreducible cuspidal perverse sheaves given in [L2] is used.
12.1.3. **Corollary.** *Any irreducible admissible perverse sheaf is a character sheaf.*
12.1.4. **Example.** $G = SL_n$ and $p \nmid n$. The classification results of [L2] show that in this case the only irreducible cuspidal perverse sheaves on G are the ones of 5.4.11 [loc.cit, p.247]. We see that 12.1.2 is true in the present case. The assertions of 12.1.1 follow from the properties of 5.4.11. In fact, cleanness has been established there. Using 11.3.3(i) (and induction on n) the parity condition readily follows.
If $G = SL_n$ and $p \mid n$ there are no cuspidal perverse sheaves on G by [loc.cit].
It follws that 12.1.1 and 12.1.2 hold whenever the simple constituents of G are all of type A.

We mention another general result, proved in [CS, 24.11], using properties of generalized Green functions (these are discussed in [Sh. no.iv]).
12.1.5. **Theorem.** *A character sheaf is even.*
Recall that this means that $H^iA = 0$ if $i \not\equiv \dim \operatorname{supp} A (\mathrm{mod}\ 2)$.

12.2 Classification results

Lusztig's main classification result gives a description of the function c of no.11. We begin with an auxiliary construction.
12.2.1. Let Φ be a finite group. Denote by S the set

$$S = \{(x, y) \in \Phi \times \Phi \mid xy = yx\},$$

Φ acts on it by $x(y, z) = (xyx^{-1}, xzx^{-1})(x, y, z \in \Phi)$. Let $\mathcal{C}\Phi$ be the vector space of Φ-invariant functions on S, with values in the field C of cyclotomic numbers.
If $x \in \Phi$ and if χ is an irreducible character of the centralizer $Z_\Phi(x)$ we define $e_{(x,\chi)} \in \mathcal{C}\Phi$

by

$$e_{(x,\chi)}(y,z) = \begin{cases} 0 & \text{if } y \text{ is not conjugate to } x, \\ \chi(z) & \text{if } y = z. \end{cases}$$

Then $\mathcal{M}\Phi = (e_{(x,\chi)}$ is a basis of $\mathcal{C}\Phi$.
We define a hermitian form $\{,\}$ on $\mathcal{C}\Phi$ by

$$\{f,g\} = \mid A \mid^{-1} \sum_{x,y \in S} f(x,y)g(y,x)^{\text{conj}}.$$

(It is easy to see that this induces the pairing on $\mathcal{M}\Phi$ described in [CS, 17.8].)
12.2.2. Let $\xi \in \hat{X}$ and let W'_ξ be as usual. In [CS, no.17] Lusztig associates to any two-sided cell $\sum$ in $\hat{W}'_\xi$ a finite group $\Phi_\sum$.
In the case that W'_ξ is an irreducible Weyl group this is done in [L1, Ch.4] in a case by case manner. The finite groups occurring there are elementary abelian 2-groups, or symmetric groups S_3, S_4, S_5.
Moreover, an imbedding $\sum \hookrightarrow \mathcal{M}(\Phi_\sum)$ is defined, from which one obtains an imbedding

$$\sigma : \hat{W}'_\xi \to \coprod_\sum \mathcal{M}(\Phi_\sum)$$

The main classification result now is as follows ([CS, 17.8.3]).
12.2.3. **Theorem.** *There is a bijection $\rho : \hat{G}(\xi) \to \coprod_\sum \mathcal{M}(\Phi_\sum)$ such that for $A \in \hat{G}(\xi), M \in \hat{W}'_\xi$*

$$c(A,M) = (-1)^{\dim \operatorname{supp} A}\{\rho(A), \sigma(M)\}$$

There is a more refined version, given in [CS, 23.1(c)].
The theorem shows that the character sheaves in $\hat{G}(\xi)$ are described by combinatorial data, determined by the two-sided cells in $\hat{W}'_\xi$.
With the statement of this theorem we conclude these notes.

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