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Let F be an algebraically closed field of prime characteristic p, and let

$$1 \longrightarrow K \longrightarrow H \longrightarrow G \longrightarrow 1$$

be an extension of finite groups. Let B be a block of FK (considered as a subalgebra of FK), and let A be a block of FH covering B (i. e. $1_A 1_B \neq 0$). Following a suggestion by J. L. Alperin [1] we consider the following

QUESTION. *When are A and B Morita equivalent?*

Our main results concerning this question are given by theorems 1, 7, 8 and proposition 10 below. Special cases of this question are dealt with in [2] and [7].

THEOREM 1. *With notation as above, the map $B \longrightarrow 1_A B \subset A$, $b \longmapsto 1_A b$, is an isomorphism of F-algebras.*

Before proving theorem 1 we introduce some notation and state some preliminary results. Obviously K is contained in $H(B) := \{h \in H\colon hBh^{-1} = B\}$, the *stabilizer* of B in H, and we set $G(B) := H(B)/K$. The following facts are well-known (see [8; theorem I], for example).

PROPOSITION 2. (i) *$FH1_BFH$ is the sum of all blocks of FH covering B.*
(ii) *If $h_1, \ldots, h_t$ denote a transversal for $H(B)$ in H then the map*

$$\mathrm{Mat}(t, 1_B FH(B)) \longrightarrow FH1_BFH,\ [a_{ij}]_{i,j=1}^{t} \longrightarrow \textstyle\sum_{i,j=1}^{t} h_i a_{ij} h_j^{-1},$$

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is an isomorphism of F-algebras.

(iii) *The maps*

$$Z(1_B FH(B)) \longrightarrow Z(FH1_B FH),\ z \longmapsto \sum_{i,j=1}^{t} h_i z h_i^{-1},$$

and

$$Z(FH1_B FH) \longrightarrow Z(1_B FH(B)),\ z \longmapsto 1_B z,$$

are isomorphisms of F-algebras and inverse to each other.

For $h \in H(B)$, the map $B \longrightarrow B$, $b \longmapsto hbh^{-1}$, is an F-algebra automorphism of B. It is easy to see that the elements $h \in H$ for which the map $B \longrightarrow B$, $b \longmapsto hbh^{-1}$, is an inner automorphism of B form a normal subgroup $H[B]$ of $H(B)$ containing K (cf. [3; proposition 2.7]). Define $G[B] := H[B]/K$.

Setting $C := 1_B C_{FH}(K)$ and $C_g := C \cap hFK$ for $g = hK \in G$ we obtain $C = \bigoplus_{g \in G} C_g$ and $C_g C_{g'} \subset C_{gg'}$ for $g, g' \in G$, i. e. C is a G-graded F-algebra in the sense of [4]. It is easy to see that $C_g = 0$ for $g \in G \setminus G(B)$. Thus $C = \bigoplus_{g \in G(B)} C_g$ can also be viewed as a $G(B)$-graded F-algebra.

PROPOSITION 3. ([3; lemma 3.3]) $I := \bigoplus_{g \in G[B]} (JZB)C_g \oplus \bigoplus_{g \in G(B) \setminus G[B]} C_g$ *is an ideal of C contained in the radical JC of C.*

Setting $C[B] := \bigoplus_{g \in G[B]} C_g$ we thus have $C = C[B] + JC$. By lifting theorems for idempotents one obtains the following result.

COROLLARY 4. ([3; theorem 3.5]) *All idempotents of ZC are contained in C[B].*

It is easy to see that $C[B]$ is a crossed product of $G[B]$ with ZB, in the sense of [4]; in particular, $C[B]$ is free as a ZB-module, and $\overline{C[B]} := C[B]/(JZB)C[B]$ is a crossed product of $G[B]$ with $ZB/(JZB) \cong F$, i. e. a twisted group algebra of $G[B]$ over F. Our next result is [8; theorem C].

PROPOSITION 5. *If $G = G[B]$ then the map $B \otimes_{ZB} C \longrightarrow 1_B FH$, $b \otimes c \longmapsto bc$, is an isomorphism of F-algebras.*

We are now in a position to prove theorem 1.

Proof of theorem 1. Obviously the map $B \longrightarrow 1_A B$, $b \longmapsto 1_A b$, is an epimorphism of F-algebras. Hence it suffices to prove injectivity. By proposition 2, $1_A 1_B$ is the block idempotent of a block of $FH(B)$ covering B. Hence we may replace H by $H(B)$ and assume $H = H(B)$. By corollary 4, 1_A is contained in $FH[B]$. Replacing A by a block of $1_A FH[B]$ we may assume that $H = H[B]$. In this case the map $B \otimes_{ZB} C \longrightarrow 1_B FH$, $b \otimes c \longmapsto bc$, is an isomorphism of F-algebras by proposition 5. Moreover, C is free over ZB. This isomorphism maps $B \otimes_{ZB} 1_A C$ onto A. Since $C = 1_A C \oplus (1_B - 1_A)C$, $1_A C$ is projective over ZB. Since ZB is local, $1_A C$ is even free over ZB. Thus A is free over B, and the result follows. ☒

In order to prove our next theorem we need a result on the behaviour of defect groups.

PROPOSITION 6. ([3; theorem 7.7]) *$1_A + (JZB)C[B]$ is a primitive idempotent in $C_{\overline{C[B]}}(G(\bar{B}))$, and A has a defect group P such that $P \cap K$ is a defect group of B and PK/K is a defect group of $1_A + (JZB)C[B]$ in $G(B)$.*

Part of proposition 6 has also been proved in [6; 4.2]. We will say that A and B are *"naturally" Morita equivalent of degree n* if there exists a simple F-subalgebra S of A of dimension n^2 such that the map $1_A B \otimes_F S \longrightarrow A$, $b \otimes s \longmapsto bs$, is an isomorphism of F-algebras. In this case A and B are Morita equivalent since $1_A B$ is isomorphic to B by theorem 1 and S is a complete matrix algebra of degree n over F.

THEOREM 7. *A and B are "naturally" Morita equivalent if and only if $G = G[B]$ and A and B have the same defect.*

Proof. Suppose first that $G = G[B]$ and that A and B have the same defect. By proposition 6, the block $1_A C + (JZB)C/(JZB)C$ of the twisted group algebra $C/(JZB)C$ of $G[B] = G$ over F has defect 0 in $G(B) = G$. It is well-known that this implies that the block $1_A C + (JZB)C/(JZB)C$ of $C/(JZB)C$ is a simple F-algebra; in particular, $1_A JC = (JZB)1_A C$. By the Wedderburn-Malcev theorem there is a simple F-subalgebra S of $1_A C$ such that $1_A C = S \oplus 1_A JC = S \oplus (JZB)1_A C$. Then $1_A C = (ZB)S + (JZB)1_A C$, and Nakayama's lemma implies that $1_A C = (ZB)S$. In the proof of theorem 1 we had shown that $1_A C$ is free over ZB. Thus $1_A C/1_A JC$ is free of the same rank over $ZB/JZB \cong F$. Therefore the rank of $1_A C$ over ZB equals the dimension of S over F. Comparing dimensions we see that the map $ZB \otimes_F S \longrightarrow 1_A C$, $z \otimes s \longmapsto zs$, is an isomorphism of F-algebras. By proposition 5, the map $B \otimes_F S \longrightarrow A$, $b \otimes s \longmapsto bs$, is an isomorphism as well.

Suppose now conversely that A and B are "naturally" Morita equivalent, and let S be a simple F-subalgebra of A such that the map $1_A B \otimes_F S \longrightarrow A$, $b \otimes s \longmapsto bs$, is an isomorphism of F-algebras. Then $1_A = 1_S = 1_A 1_B$. On the other hand, it follows from proposition 2 that $1_A = \sum_{i=1}^{t} 1_A(h_i 1_B h_i^{-1})$ with pairwise orthogonal idempotents $1_A(h_i 1_B h_i^{-1})$ where $t = |H:H(B)|$. Thus $H(B) = H$ and $G(B) = G$.

We know from proposition 3 that $C = C[B] + JC$; in particular, $1_A C = 1_A C[B] + 1_A JC$. On the other hand, since A and B are "naturally" Morita equivalent the map

$$1_A ZB \otimes_F S \longrightarrow 1_A ZB \cdot S = C_A(B) = 1_A C, \ z \otimes s \longmapsto zs,$$

is an isomorphism of F-algebras. By the Wedderburn-Malcev theorem we may find a unit u in $1_A C$ *such that* S^u is contained in $1_A C[B]$. Then the map $1_A B \otimes_F S^u \longrightarrow A$, $b \otimes s \longmapsto bs$, is an isomorphism of F-algebras as well. Hence we may assume that S is contained in $FH[B]$. Since also $1_A \in FH[B]$ by corollary 4 we obtain $A \subset FH[B]$ which clearly implies that $H[B] = H$.

Since $1_A C$ is isomorphic to $ZB \otimes_F S$. $1_A C + (JZB)C/(JZB)C$ is a simple F-algebra. It is well-known that this implies that the block $1_A C + (JZB)C/(JZB)C$ of $\overline{C[B]}$ has defect 0 in $G[B] = G$. By proposition 6, A and B have the same defect. ☒

In the following we assume that $G(B) = G$; in view of proposition 2, this is not an important restriction. In this case we can reduce the question of whether A and B are "naturally" Morita equivalent to their Brauer correspondents. Let Q be a defect group of B, and let B' be the Brauer correspondent of B in $N_K(Q)$. Since $G(B) = G$ the Frattini argument shows that $H = N_H(Q)K$, and we obtain a finite group extension

$$1 \longrightarrow N_K(Q) \longrightarrow N_H(Q) \longrightarrow N_H(Q)/N_K(Q) \longrightarrow 1$$

with $N_H(Q)/N_K(Q)$ naturally isomorphic to G. By proposition 6, A has a defect group P such that $Q = P \cap K$; in particular, $N_H(P) \subset N_H(Q)$. By Brauer's First Main Theorem, A has a unique Brauer correspondent A' in $N_H(Q)$. By [5; theorem], A' covers B'.

THEOREM 8. *With notation as above, A and B are "naturally" Morita equivalent if and only if A' and B' are "naturally" Morita equivalent.*

In order to prove theorem 8, we need the following result which is a consequence of [3; corollary 12.6].

PROPOSITION 9. *In the situation above,* $H[B] = (N_H(Q)[B'])K$.

Proof of theorem 8. Suppose first that A and B are "naturally" Morita equivalent. By theorem 7, $G = G[B]$, and A and B have the same defect. By proposition 6, Q is a defect group of A. By Brauer's First Main Theorem, Q is a defect group of A' and B' as well. Moreover, since $H = H[B] = (N_H(Q)[B'])K$ by proposition 9, we have

$$N_H(Q) = (N_H(Q)[B'])N_K(Q) = N_H(Q)[B'].$$

By theorem 7, A' and B' are "naturally" Morita equivalent.

Suppose now conversely that A' and B' are "naturally" Morita equivalent. By theorem 7, $N_H(Q) = N_H(Q)[B']$, and A' and B' have the same defect. By Brauer's First Main Theorem, B' has defect group Q. By proposition 6, A' has defect group Q as well. Again by Brauer's First Main Theorem, A has defect group Q. Moreover, proposition 9 implies that $H[B] = (N_H(Q)[B'])K = N_H(Q)K = H$. By theorem 7, A and B are "naturally" Morita equivalent. ☒

This result can be strengthened by using additional information from [2]. Suppose that A and B are "naturally" Morita equivalent. Let S be a simple F-subalgebra of A such that the map $1_A B \otimes_F S \longrightarrow A$, $b \otimes s \longmapsto bs$, is an isomorphism of F-algebras. By theorem 8, A' and B' are "naturally" Morita equivalent. Thus there is a simple F-subalgebra S' of A' such that the map $1_{A'} B' \otimes_F S' \longrightarrow A'$, $b' \otimes s' \longmapsto b's'$, is an isomorphism of F-algebras.

We have seen above that B determines a twisted group algebra $\overline{C[B]}$ of $G[B] = G$ over F. In the same way, B' determines a twisted group algebra $\overline{C'[B']}$ of $N_H(Q)/N_K(Q)$ over F. Since $N_H(Q)/N_K(Q)$ and G are naturally isomorphic we can view $\overline{C'[B']}$ as a twisted group algebra of G over F. Then [3; corollary 12.6] (which is the main result of [3]) tells us that the Brauer homomorphism induces a natural isomorphism between $\overline{C[B]}$ and $\overline{C'[B']}$. This isomorphism maps the block of defect 0 in $\overline{C[B]}$ determined by A onto the block of defect 0 in $\overline{C'[B']}$ determined by A'. Now the proof of theorem 7 shows that S and S' are isomorphic. Hence we may add the following result to theorem 8.

PROPOSITION 10. *If, in the situation of theorem 8, A and B are "naturally" Morita equivalent of degree n then so are A' and B'.*

Finally, let us interpret our results in the language of [9]. The block B of FK corresponds to a pointed group K_β over FK, and the block A of FH corresponds to a pointed group H_α over FH. Let Q_δ be a maximal local pointed subgroup of K_β. Suppose that A and B are "naturally" Morita equivalent, and let S be a simple F-subalgebra of A such that the map $1_A B \otimes_F S \longrightarrow A$, $b \otimes s \longmapsto bs$, is an isomorphism of F-algebras. Then S and B centralize each other; in particular, every element of S is fixed by Q. It follows easily that the source algebras of H_α and K_β are isomorphic (as interior Q-algebras).

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